Approximate Bi-Additive Mappings in Intuitionistic Fuzzy Normed Spaces

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Abstract : In this paper, we determine some stability results concerning a 2-dimensional vector variable bi-additive functional equation in intuitionistic fuzzy normed spaces (IFNS). We generalize the intuitionistic fuzzy continuity to the bi-additive mappings and we prove that the existence of a solution for any approximately bi-additive mapping implies the completeness of IFNS.

Keywords : intuitionistic fuzzy normed spaces; generalized Ulam-Rassias stability; functional equations.

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1 Introduction

In recent years, the fuzzy theory has emerged as the most active area of research in many branches of mathematics and engineering. This new theory was introduced by Zadeh [1], in 1965 and since then a large number of research papers have appeared by using the concept of fuzzy set/numbers and fuzzification of many classical theories has also been made. It has also very useful application in various fields, e.g. population dynamics [2], chaos control [3], computer programming [4], nonlinear dynamical systems [5], fuzzy physics [6], fuzzy topology [7], fuzzy stability [8][12], nonlinear operators [13], statistical convergence [14][15], etc.

The concept of intuitionistic fuzzy normed spaces, initially has been introduced by Saadati and Park [16]. In [17], by modifying the separation condition and strengthening some conditions in the definition of Saadati and Park, Saadati et al. have obtained a modified case of intuitionistic fuzzy normed spaces. Many authors have
considered the intuitionistic fuzzy normed linear spaces, and intuitionistic fuzzy 2-normed spaces (see [18–21]).

Let $X$ be a real linear space. A function $N : X \times \mathbb{R} \to [0, 1]$ (the so-called fuzzy subset) is said to be a fuzzy norm on $X$ if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,

(N1) $N(x, c) = 0$ for $c \leq 0$;
(N2) $x = 0$ if and only if $N(x, c) = 1$ for all $c > 0$;
(N3) $N(cx, t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;
(N4) $N(x + y, s + t) \geq \min\{N(x, s), N(y, t)\}$;
(N5) $N(x, \cdot)$ is a non-decreasing function on $\mathbb{R}$ and $\lim_{t \to \infty} N(x, t) = 1$;
(N6) For $x \neq 0$, $N(x, \cdot)$ is continuous on $\mathbb{R}$.

The pair $(X, N)$ is called a fuzzy normed linear space. One may regard $N(x, t)$ as the truth value of the statement the norm of $x$ is less than or equal to the real number $t$.

The concept of stability of a functional equation arises when one replaces a functional equation by an inequality which acts as a perturbation of the equation. The first stability problem concerning group homomorphisms was raised by Ulam [22] in 1940 and affirmatively solved by Hyers [23]. The result of Hyers was generalized by Aoki [24] for approximate additive function and by Rassias [25] for approximate linear functions by allowing the difference Cauchy equation $\|f(x_1 + x_2) - f(x_1) - f(x_2)\|$ to be controlled by $\varepsilon(\|x_1\|^p + \|x_2\|^p)$. Taking into consideration a lot of influence of Ulam, Hyers and Rassias on the development of stability problems of functional equations, the stability phenomenon that was proved by Rassias is called the generalized Ulam-Rassias stability or Hyers-Ulam-Rassias stability (see [26–28]). In 1994, a generalization of Rassias theorem was obtained by Gavruta [29], who replaced $\varepsilon(\|x_1\|^p + \|x_2\|^p)$ by a general control function $\varphi(x_1, x_2)$.

The stability problem for the 2-dimensional vector variable bi-additive functional equation was proved by the authors [30] for mappings $f : X \times X \to Y$, where $X$ is a real normed space and $Y$ is a Banach space. In this paper, we determine some stability results concerning the 2-dimensional vector variable bi-additive functional equation

$$f(x + y, z - w) + f(x - y, z + w) = 2f(x, z) - 2f(y, w) \quad (1.1)$$

in intuitionistic fuzzy normed spaces. We apply the intuitionistic fuzzy continuity of the 2-dimensional vector variable bi-additive mappings and prove that the existence of a solution for any approximately 2-dimensional vector variable bi-additive mapping implies the completeness of intuitionistic fuzzy normed spaces (IFNS). It has shown that each mapping satisfies in (1.1) is C-bilinear (see [31]).

In the following section, we recall some notations and basic definitions used in this paper.
2 Preliminaries

We use the definition of intuitionistic fuzzy normed spaces given in [16, 32, 33] to investigate some stability results for the functional equation (1.1) in the intuitionistic fuzzy normed vector space setting.

Definition 2.1 ([34]). A binary operation \( \ast : [0, 1] \times [0, 1] \rightarrow [0, 1] \) is said to be a \textit{continuous t-norm} if it satisfies the following conditions:

(a) is commutative and associative;
(b) is continuous;
(c) \( a \ast 1 = a \) for all \( a \in [0, 1] \);
(d) \( a \ast b \leq c \ast d \) whenever \( a \leq c \) and \( b \leq d \) for all \( a, b, c, d \in [0, 1] \).

Definition 2.2 ([34]). A binary operation \( \circ : [0, 1] \times [0, 1] \rightarrow [0, 1] \) is said to be a \textit{continuous t-conorm} if it satisfies the following conditions:

(a) is commutative and associative;
(b) is continuous;
(c) \( a \circ 0 = a \) for all \( a \in [0, 1] \);
(d) \( a \circ b \leq c \circ d \) whenever \( a \leq c \) and \( b \leq d \) for all \( a, b, c, d \in [0, 1] \).

Using the continuous t-norm and t-conorm, Saadati and Park [16], have introduced the concept of intuitionistic fuzzy normed space.

Definition 2.3 ([10, 32]). The five-tuple \( (X, \mu, \nu, \ast, \circ) \) is said to be an \textit{intuitionistic fuzzy normed space} (for short, IFNS) if \( X \) is a vector space, \( \ast \) is a continuous t-norm, \( \circ \) is a continuous t-conorm, and \( \mu, \nu \) fuzzy sets on \( X \times (0, \infty) \) satisfying the following conditions: For every \( x, y \in X \) and \( s, t > 0 \),

\[
\begin{align*}
(\text{IF}_1) \quad & \mu(x, t) + \nu(x, t) \leq 1; \\
(\text{IF}_2) \quad & \mu(x, t) > 0; \\
(\text{IF}_3) \quad & \mu(x, t) = 1 \text{ if and only if } x = 0; \\
(\text{IF}_4) \quad & \mu(\alpha x, t) = \mu(x, \frac{t}{|\alpha|}) \text{ for each } \alpha \neq 0; \\
(\text{IF}_5) \quad & \mu(x, t) \ast \mu(y, s) \leq \mu(x + y, t + s); \\
(\text{IF}_6) \quad & \mu(x, .) : (0, \infty) \rightarrow [0, 1] \text{ is continuous}; \\
(\text{IF}_7) \quad & \lim_{t \to \infty} \mu(x, t) = 1 \text{ and } \lim_{t \to 0} \mu(x, t) = 0; \\
(\text{IF}_8) \quad & \nu(x, t) < 1; \\
(\text{IF}_9) \quad & \nu(x, t) = 0 \text{ if and only if } x = 0; \\
(\text{IF}_{10}) \quad & \nu(\alpha x, t) = \nu(x, \frac{t}{|\alpha|}) \text{ for each } \alpha \neq 0; \\
(\text{IF}_{11}) \quad & \nu(x, t) \circ \nu(y, s) \geq \nu(x + y, t + s); \\
(\text{IF}_{12}) \quad & \nu(x, .) : (0, 1) \rightarrow [0, 1] \text{ is continuous}; \\
(\text{IF}_{13}) \quad & \lim_{t \to \infty} \nu(x, t) = 0 \text{ and } \lim_{t \to 0} \nu(x, t) = 1.
\end{align*}
\]

Example 2.4. Let \( (X, \|\|) \) be a normed space, \( a \ast b = ab \) and \( a \circ b = \min\{a + b, 1\} \) for all \( a, b \in [0, 1] \). For all \( x \in X \) and every \( t > 0 \), consider

\[
\mu(x, t) = \begin{cases} 
\frac{t}{\|x\|} & \text{if } t > 0 \\
0 & \text{if } t \leq 0
\end{cases}
\]

and

\[
\nu(x, t) = \begin{cases} 
\frac{\|x\|}{t} & \text{if } t > 0 \\
0 & \text{if } t \leq 0.
\end{cases}
\]
Then \((X, \mu, \nu, *, \circ)\) is an IFNS.

**Remark 2.5.** In intuitionistic fuzzy normed space \((X, \mu, \nu, *, \circ)\), \(\mu(x, .)\) is non-decreasing and \(\nu(x, .)\) is non-increasing for all \(x \in X\) (see [16]).

**Definition 2.6.** Let \((X, \mu, \nu, *, \circ)\) be an IFNS. A sequence \(\{x_n\}\) is said to be intuitionistic fuzzy convergent to \(L \in X\) if \(\lim_{k \to \infty} \mu(x_k - L, t) = 1\) and \(\lim_{k \to \infty} \nu(x_k - L, t) = 0\) for all \(t > 0\). In this case we write \(x_k \to L\) as \(k \to \infty\). A sequence \(\{x_n\}\) is said to be intuitionistic fuzzy Cauchy sequence if \(\lim_{k \to \infty} \mu(x_k + p - x_k, t) = 1\) and \(\lim_{k \to \infty} \nu(x_k + p - x_k, t) = 0\) for all \(p \in \mathbb{N}\) and all \(t > 0\). Then IFNS \((X, \mu, \nu, *, \circ)\) is said to be complete if every intuitionistic fuzzy Cauchy sequence in \((X, \mu, \nu, *, \circ)\) intuitionistic fuzzy convergent in \((X, \mu, \nu, *, \circ)\) and \((X, \mu, \nu, *, \circ)\) is also called an intuitionistic fuzzy Banach space.

The concepts of convergence and Cauchy sequences in an intuitionistic fuzzy normed space are studied in [16].

# 3 Intuitionistic Fuzzy Stability

For notational convenience, given a function \(f : X \times X \to Y\), we define the difference operator

\[
D_b f(x, y, z, w) = f(x + y, z - w) + f(x - y, z + w) - 2f(x, z) + 2f(y, w).
\]

We begin with a generalized Hyers-Ulam type theorem in IFNS for the functional equation (1.1).

**Theorem 3.1.** Let \(X\) be a linear space and let \((Z, \mu', \nu, \cdot)\) be an IFNS. Let \(\varphi : X \times X \times X \times X \to Z\) be a mapping such that, for some \(0 < \alpha < 4\).

\[
\begin{align*}
\mu'(\varphi(2x, 2y, 2z, 2w), t) & \geq \mu'(\alpha \varphi(x, y, z, w), t), \\
\nu'(\varphi(2x, 2y, 2z, 2w), t) & \leq \nu'(\alpha \varphi(x, y, z, w), t),
\end{align*}
\]

for all \(x, y, z, w \in X\) and all \(t > 0\). Let \((Y, \mu, \nu)\) be an intuitionistic fuzzy Banach space and let \(f : X \times X \to Y\) be a mapping such that

\[
\begin{align*}
\mu(D_b f(x, y, z, w), t) & \geq \mu'(\varphi(x, y, z, w), t), \\
\nu(D_b f(x, y, z, w), t) & \leq \nu'(\varphi(x, y, z, w), t)
\end{align*}
\]

for all \(x, y, z, w \in X\) and all \(t > 0\). Then there exists a unique mapping \(F :\)
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\( X \times X \to Y \) satisfying (4.4) such that

\[
\begin{align*}
\mu \left( F(x, y) - f(x, y) + \frac{1}{3} f(0, 0), t \right) \\
\geq *\mu' \left( \varphi(x, x, y, y), \frac{(4-\alpha)}{8} t \right), \\
\mu(0, 0, y, y), \frac{(4-\alpha)}{8} t), \\
\nu \left( F(x, y) - f(x, y) + \frac{1}{3} f(0, 0), t \right) \\
\leq o\nu' \left( \varphi(x, x, y, y), \frac{(4-\alpha)}{8} t \right), \\
o\nu'(0, 0, y, y), \frac{(4-\alpha)}{8} t)
\end{align*}
\]

for all \( x, y, z, w \in X \) and all \( t > 0 \), where \( *\mu := a * a * \cdots \) and \( o\nu := a \circ a \circ \cdots \)
for all \( a \in [0, 1] \).

**Proof.** Put \( y = -x \) and \( w = z \) in (3.2) to obtain

\[
\begin{align*}
\mu(f(2x, 2z) - 2f(x, z) - 2f(0, 0), t) & \geq \mu'(\varphi(x, x, z, z), t), \\
\nu(f(2x, 2z) - 2f(x, z) - 2f(0, 0), t) & \leq \nu'(\varphi(x, x, z, z), t)
\end{align*}
\]

for all \( x, z \in X \) and all \( t > 0 \). Let \( z = z = 0 \) in (3.2), we get

\[
\begin{align*}
\mu(f(y, -w) + f(0, 0), t) & \geq \mu'\left( \varphi(0, y, 0, w), t \right), \\
\nu(f(y, -w) + f(0, 0), t) & \leq \nu'(\varphi(0, y, 0, w), t)
\end{align*}
\]

for all \( y, w \in X \) and all \( t > 0 \). Replacing \( y \) by \( x \) and \( w \) by \( z \) in (3.3), we get

\[
\begin{align*}
\mu(f(x, -z) + f(0, 0), t) & \geq \mu'\left( \varphi(0, x, 0, z), t \right), \\
\nu(f(x, -z) + f(0, 0), t) & \leq \nu'(\varphi(0, x, 0, z), t)
\end{align*}
\]

for all \( x, z \in X \) and all \( t > 0 \). Putting \( x = y \) and \( w = -z \) in (3.2), we obtain

\[
\begin{align*}
\mu(f(2x, 2z) - 2f(x, -z) + f(0, 0), t) & \geq \mu'(\varphi(x, x, z, -z), t), \\
\nu(f(2x, 2z) - 2f(x, -z) + f(0, 0), t) & \leq \nu'(\varphi(x, x, z, -z), t)
\end{align*}
\]

for all \( x, z \in X \) and all \( t > 0 \). By inequalities (4.4) and (4.7), we get

\[
\begin{align*}
\mu(2f(-x, z) - 2f(x, z) + f(0, 0), t) & \geq \mu'(\varphi(x, x, z, -z), t), \\
\nu(2f(-x, z) - 2f(x, z) + f(0, 0), t) & \leq \nu'(\varphi(x, x, z, -z), t)
\end{align*}
\]

for all \( x, z \in X \) and all \( t > 0 \). And from (3.8), we can write

\[
\begin{align*}
\mu(f(-x, z) - f(x, -z) + f(0, 0), t) & \geq \mu'(\varphi(x, x, z, -z), t), \\
\nu(f(-x, z) - f(x, -z) + f(0, 0), t) & \leq \nu'(\varphi(x, x, z, -z), t)
\end{align*}
\]
for all \( x, z \in X \) and all \( t > 0 \). By \((3.9)\) and \((3.10)\), we have

\[
\begin{align*}
\mu(f(2x, 2z) - 4f(x, z) + f(x, -z) - f(-x, z) + 3f(0, 0), t) & \geq \mu'((\varphi(x, x, z, -z), \frac{t}{8})) \circ \mu'((\varphi(0, x, 0, z), \frac{t}{8})), \\
\nu(f(2x, 2z) - 4f(x, z) + f(x, -z) - f(-x, z) + 3f(0, 0), t) & \leq \nu'((\varphi(x, x, z, -z), \frac{t}{8})) \circ \nu'((\varphi(0, x, 0, z), \frac{t}{8})).
\end{align*}
\]  

(3.10)

for all \( x, z \in X \) and all \( t > 0 \). From \((3.9)\) and \((3.10)\), we get

\[
\begin{align*}
\mu(f(2x, 2z) - 4f(x, z) + 4f(0, 0), t) & \geq \mu'(2\varphi(x, x, z, -z), \frac{t}{4}) \circ \mu'(\varphi(x, x, z, -z), \frac{t}{4}) \\
& \circ \mu'(\varphi(x, -x, x, z), \frac{t}{4}) \circ \mu'(\varphi(0, x, 0, z), t) \\
& \geq \mu'(\varphi(x, x, z, -z), \frac{t}{8}) \circ \mu'(\varphi(x, x, z, -z), \frac{t}{8}) \\
& \circ \mu'(\varphi(x, -x, z, z), \frac{t}{8}) \circ \mu'(\varphi(0, x, 0, z), \frac{t}{8}) \\
& = \mu'(\varphi(x, x, z, -z), \frac{t}{8}) \circ \mu'(\varphi(x, -x, z, z), \frac{t}{8}) \circ \mu'(\varphi(0, x, 0, z), \frac{t}{8}),
\end{align*}
\]

and also

\[
\begin{align*}
\nu(f(2x, 2z) - 4f(x, z) + 4f(0, 0), t) & \leq \nu'(\varphi(x, x, z, -z), \frac{t}{2}) \circ \nu'((\varphi(x, x, z, z), \frac{t}{2}) \circ \nu'((\varphi(0, x, 0, z), \frac{t}{2})
\end{align*}
\]

for all \( x, z \in X \) and all \( t > 0 \). We can write above inequalities as following

\[
\begin{align*}
\mu\left(\frac{f(2x, 2z) + f(0, 0)}{2} - f(x, z), \frac{t}{4}\right) & \geq \mu'(\varphi(x, x, z, -z), \frac{t}{8}) \circ \mu'(\varphi(0, x, 0, z), \frac{t}{8}), \\
\nu\left(\frac{f(2x, 2z) + f(0, 0)}{2} - f(x, z), \frac{t}{4}\right) & \leq \nu'(\varphi(x, x, z, -z), \frac{t}{8}) \circ \nu'(\varphi(x, x, z, z), \frac{t}{8}) \circ \nu'(\varphi(0, x, 0, z), \frac{t}{8}).
\end{align*}
\]

(3.11)

for all \( x, z \in X \) and all \( t > 0 \). Replacing \( x \) by \( 2^n x \) and \( z \) by \( 2^n z \) in \((3.11)\) and
using (3.11), we get
\[
\begin{align*}
& \left\{ \begin{array}{l}
\mu \left( \frac{f(2^{n+1}x, 2^{n+1}z) + f(0, 0)}{4^{n+1}} - \frac{f(2^n x, 2^n z)}{4^n}, \frac{a^n t}{4^{n+1}} \right) \\
\geq *^2 \mu' \left( \varphi(2^n x, 2^n x, 2^n x, -2^n z), \frac{t}{8^n} \right) *^2 \mu' \left( \varphi(2^n x, 2^n x, 2^n z, 2^n z), \frac{t}{8^n} \right) \\
+ \mu' \left( \varphi(0, 2^n x, 0, 2^n z), \frac{t}{8^n} \right), \\
\geq *^2 \mu' \left( \varphi(x, x, z, -z), \frac{t}{8^n} \right) *^2 \mu' \left( \varphi(-x, x, z, 2^n z), \frac{t}{8^n} \right) *^2 \mu' \left( \varphi(0, 0, z), \frac{t}{8^n} \right), \end{array} \right.
\end{align*}
\]
for all \( x, z \in X \), all \( n \in \mathbb{N} \) and all \( t > 0 \). By replacing \( t \) by \( a^n t \) in above inequalities, we have
\[
\begin{align*}
& \left\{ \begin{array}{l}
\mu \left( \frac{f(2^{n+1}x, 2^{n+1}z) + f(0, 0)}{4^{n+1}} - \frac{f(2^n x, 2^n z)}{4^n}, \frac{a^n t}{4^{n+1}} \right) \\
\geq *^2 \mu' \left( \varphi(x, x, z, -z), \frac{t}{8^n} \right) *^2 \mu' \left( \varphi(-x, x, z, 2^n z), \frac{t}{8^n} \right) *^2 \mu' \left( \varphi(0, 0, z), \frac{t}{8^n} \right), \\
\geq *^2 \mu' \left( \varphi(x, x, z, -z), \frac{t}{8^n} \right) *^2 \mu' \left( \varphi(-x, x, z, 2^n z), \frac{t}{8^n} \right) *^2 \mu' \left( \varphi(0, 0, z), \frac{t}{8^n} \right), \end{array} \right.
\end{align*}
\]
(3.12)
for all \( x, z \in X \), all \( n \in \mathbb{N} \) and all \( t > 0 \). It follows from
\[
\sum_{k=0}^{n-1} \left[ \frac{f(2^{k+1}x, 2^{k+1}z) + f(0, 0) - f(2^k x, 2^k z)}{4^{k+1}} \right] = \frac{f(2^n x, 2^n z)}{4^n} - f(x, z) + \frac{1}{3} \left( 1 - \frac{1}{4^n} \right) f(0, 0)
\]
and (3.12),
\[
\begin{align*}
& \left\{ \begin{array}{l}
\mu \left( \frac{f(2^n x, 2^n z)}{4^n} - f(x, z) + \frac{1}{3} \left( 1 - \frac{1}{4^n} \right) f(0, 0), \sum_{k=0}^{n-1} \frac{a^n t}{4^{k+1}} \right) \\
\geq *^n \mu' \left( \varphi(x, x, z, -z), \frac{t}{8^n} \right) *^n \mu' \left( \varphi(-x, x, z, 2^n z), \frac{t}{8^n} \right) *^n \mu' \left( \varphi(0, 0, z), \frac{t}{8^n} \right), \\
\geq *^n \mu' \left( \varphi(x, x, z, -z), \frac{t}{8^n} \right) *^n \mu' \left( \varphi(-x, x, z, 2^n z), \frac{t}{8^n} \right) *^n \mu' \left( \varphi(0, 0, z), \frac{t}{8^n} \right), \end{array} \right.
\end{align*}
\]
(3.13)
for all \( x, z \in X \), all \( n \in \mathbb{N} \) and all \( t > 0 \), where \( \prod_{j=1}^n a_j := a_1 * a_2 * \cdots * a_n \), \( \prod_{j=1}^n a := a_1 \circ a_2 \circ \cdots \circ a_n \), \( *^n a := \prod_{j=1}^n a \) and \( \circ^n a := \prod_{j=1}^n a = \)
for all \( a, a_1, a_2, \ldots, a_n \in [0, 1] \). By replacing \( x \) with \( 2^m x \) and \( z \) with \( 2^m z \) in (3.13), we have

\[
\begin{align*}
\mu\left( \frac{f(2^{m+n} x, 2^{m+n} z)}{4^{m+n}} - \frac{f(2^m x, 2^m z)}{4^m} + \frac{1}{3.4^m} \left( 1 - \frac{1}{4^m} \right) f(0,0), \sum_{k=0}^{n-1} \frac{\alpha^k t}{4^{m+k+r}} \right) \\
\geq \ast 2^n \nu'(\varphi(2^m x, 2^m z, -2^m z), \frac{t}{8^m}) \ast n \mu'(\varphi(0, 2^m x, 0, 2^m z), \frac{t}{8^m}),
\end{align*}
\]

\[
\begin{align*}
\nu\left( \frac{f(2^{m+n} x, 2^{m+n} z)}{4^{m+n}} - \frac{f(2^m x, 2^m z)}{4^m} + \frac{1}{3.4^m} \left( 1 - \frac{1}{4^m} \right) f(0,0), \sum_{k=0}^{n-1} \frac{\alpha^k t}{4^{m+k+r}} \right) \\
\leq \ast 2^n \nu'(\varphi(0, x, 0, z), \frac{t}{8^m}) \ast n \nu'(\varphi(x, x, z, z), \frac{t}{8^m}) \ast n \nu'(\varphi(0, 0, x, z), \frac{t}{8^m}),
\end{align*}
\]

for all \( x, z \in X \), all \( m, n \in \mathbb{N} \) and all \( t > 0 \). So we have gotten that

\[
\begin{align*}
\mu\left( \frac{f(2^{m+n} x, 2^{m+n} z)}{4^{m+n}} - \frac{f(2^m x, 2^m z)}{4^m} + \frac{1}{3.4^m} \left( 1 - \frac{1}{4^m} \right) f(0,0), \sum_{k=0}^{n+m-1} \frac{\alpha^k t}{4^{m+k+r}} \right) \\
\geq \ast 2^n \nu'(\varphi(x, x, z, -z), \frac{t}{8^m}) \ast n \mu'(\varphi(0, 0, x, z), \frac{t}{8^m}),
\end{align*}
\]

\[
\begin{align*}
\nu\left( \frac{f(2^{m+n} x, 2^{m+n} z)}{4^{m+n}} - \frac{f(2^m x, 2^m z)}{4^m} + \frac{1}{3.4^m} \left( 1 - \frac{1}{4^m} \right) f(0,0), \sum_{k=0}^{n+m-1} \frac{\alpha^k t}{4^{m+k+r}} \right) \\
\leq \ast 2^n \nu'(\varphi(x, x, z, -z), \frac{t}{8^m}) \ast n \nu'(\varphi(0, x, z, z), \frac{t}{8^m}) \ast n \nu'(\varphi(0, 0, x, z), \frac{t}{8^m}),
\end{align*}
\]

for all \( x, z \in X \), all \( m, n \in \mathbb{N} \) and all \( t > 0 \). Replacing \( t \) by \( \frac{t}{\sum_{k=m}^{n+m-1} \frac{\alpha^k}{4^k}} \), we obtain

\[
\begin{align*}
\mu\left( \frac{f(2^{m+n} x, 2^{m+n} z)}{4^{m+n}} - \frac{f(2^m x, 2^m z)}{4^m} + \frac{1}{3.4^m} \left( 1 - \frac{1}{4^m} \right) f(0,0), \frac{t}{\sum_{k=m}^{n+m-1} \frac{\alpha^k}{4^k}} \right) \\
\geq \ast 2^n \nu'(\varphi(x, x, z, -z), \frac{t}{8 \sum_{k=m}^{n+m-1} \frac{\alpha^k}{4^k}}) \ast n \mu'(\varphi(0, x, z, z), \frac{t}{8 \sum_{k=m}^{n+m-1} \frac{\alpha^k}{4^k}}),
\end{align*}
\]

\[
\begin{align*}
\nu\left( \frac{f(2^{m+n} x, 2^{m+n} z)}{4^{m+n}} - \frac{f(2^m x, 2^m z)}{4^m} + \frac{1}{3.4^m} \left( 1 - \frac{1}{4^m} \right) f(0,0), \frac{t}{\sum_{k=m}^{n+m-1} \frac{\alpha^k}{4^k}} \right) \\
\leq \ast 2^n \nu'(\varphi(x, x, z, -z), \frac{t}{8 \sum_{k=m}^{n+m-1} \frac{\alpha^k}{4^k}}) \ast n \nu'(\varphi(0, 0, x, z), \frac{t}{8 \sum_{k=m}^{n+m-1} \frac{\alpha^k}{4^k}}) \ast n \nu'(\varphi(0, 0, x, z), \frac{t}{8 \sum_{k=m}^{n+m-1} \frac{\alpha^k}{4^k}})
\end{align*}
\]

for all \( x, z \in X \), all \( m, n \in \mathbb{N} \) and all \( t > 0 \). Since \( 0 < \alpha < 4 \), \( \sum_{k=0}^{\infty} \frac{\alpha^k}{4^k} < \infty \) and \( \sum_{k=m}^{n+m-1} \frac{\alpha^k}{4^k} \to 0 \) as \( m \to \infty \) for all \( n \in \mathbb{N} \). Thus \( \frac{t}{\sum_{k=m}^{n+m-1} \frac{\alpha^k}{4^k}} \to \infty \) and

\[
\begin{align*}
\ast 2^n \nu'(\varphi(x, x, z, -z), \frac{t}{8 \sum_{k=m}^{n+m-1} \frac{\alpha^k}{4^k}}) \ast n \nu'(\varphi(0, x, z, z), \frac{t}{8 \sum_{k=m}^{n+m-1} \frac{\alpha^k}{4^k}}) \\
\geq \ast 2^n \nu'(\varphi(x, x, z, -z), \frac{t}{8 \sum_{k=m}^{n+m-1} \frac{\alpha^k}{4^k}}) \ast n \nu'(\varphi(0, 0, x, z), \frac{t}{8 \sum_{k=m}^{n+m-1} \frac{\alpha^k}{4^k}}) \to 1
\end{align*}
\]
as \( m \to \infty \) for all \( x, z \in X \), all \( m, n \in \mathbb{N} \) and all \( t > 0 \). Hence the Cauchy criterion for convergence in IFNS shows that \( \left( \frac{f(x, y, z, w)}{4^n} \right) \) is a Cauchy sequence in \((Y, \mu, \nu)\) for all \( x, z \in X \). Since \((Y, \mu, \nu)\) is complete, then this sequence converges to some point \( F(x, y) \in Y \) defined by \( F(x, y) = \lim_{n \to \infty} \frac{f(x, y, z, w)}{4^n} \) for all \( x, z \in X \). Now by putting \( m = 0 \) in (3.14), we obtain

\[
\left\{ \begin{array}{l}
\mu \left( \frac{f(x, y, z, w)}{4^n} - f(x, z) + \frac{1}{3} \left( 1 - \frac{1}{4^n} \right) f(0, 0, t) \right) \\
\nu \left( \frac{f(x, y, z, w)}{4^n} - f(x, z) + \frac{1}{3} \left( 1 - \frac{1}{4^n} \right) f(0, 0, t) \right)
\end{array} \right.
\]

for all \( x, z \in X \), all \( n \in \mathbb{N} \) and all \( t > 0 \). By taking limit from above inequalities as \( n \to \infty \) and using the definition of IFNS, we get

\[
\left\{ \begin{array}{l}
\mu \left( \frac{f(x, y) - f(x, z) + \frac{1}{3} f(0, 0, t)}{4^n} \right) \geq \frac{\mu'}{\mu} \left( \varphi(x, x, z, -z), \frac{(4^n - 1)}{4^n} \right) \\
\nu \left( \frac{f(x, y) - f(x, z) + \frac{1}{3} f(0, 0, t)}{4^n} \right) \leq \frac{\nu'}{\nu} \left( \varphi(x, x, z, -z), \frac{(4^n - 1)}{4^n} \right)
\end{array} \right.
\]

for all \( x, z \in X \) and all \( t > 0 \), which are the desired inequalities (3.3).

Now we show that \( F \) satisfies in (1.1). Replacing \( x, y, z, w \) and \( t \) in (3.2) respectively by \( 2^n x, 2^n y, 2^n z, 2^n w \) and \( 4^n t \), we get

\[
\left\{ \begin{array}{l}
\mu \left( \frac{2^{2n} f(x, y, z, w) + 2 f(x, y, z, w)}{4^n} + 2 f(x, y, z, w) \right) \\
\nu \left( \frac{2^{2n} f(x, y, z, w) + 2 f(x, y, z, w)}{4^n} - 2 f(x, y, z, w) \right)
\end{array} \right.
\]

for all \( x, y, z, w \in X \) all \( n \in \mathbb{N} \) and all \( t > 0 \). Since \( \frac{4^n t}{\alpha^n} \to \infty \) as \( n \to \infty \), then

\[
\lim_{n \to \infty} \frac{\mu'(\varphi(x, n y, z, n w), \frac{4^n t}{\alpha^n})}{\alpha^n} = 1
\]

and

\[
\lim_{n \to \infty} \frac{\nu'(\varphi(x, n y, z, n w), \frac{4^n t}{\alpha^n})}{\alpha^n} = 0
\]
for all \( x, y, z, w \in X \) and all \( t > 0 \).

To prove the uniqueness of the mapping \( F \), assume that there exists a mapping \( G : X \times X \to Y \) which satisfies (1.1) and (3.3). For fix \( x, y \in X \), we know that \( F(2^n x, 2^n y) = 4^n F(x, y) \) and \( G(2^n x, 2^n y) = 4^n G(x, y) \) for all \( n \in \mathbb{N} \). It follows from (3.3) that

\[
\mu(F(x, y) - G(x, y), t) = \mu\left( \frac{F(2^n x, 2^n y)}{4^n} - \frac{G(2^n x, 2^n y)}{4^n}, t \right) \\
\geq *\mu\left( - \frac{G(2^n x, 2^n y)}{4^n} + \frac{f(2^n x, 2^n y)}{4^n} - \frac{1}{3.4^n f(0, 0)}, t \right) \\
= *^2 *^\infty \mu'\left( \varphi(2^n x, 2^n y, 2^n y), \frac{4^n(4 - \alpha)t}{16} \right) \\
= *^2 *^\infty \mu'\left( \varphi(0, 2^n x, 2^n y), \frac{4^n(4 - \alpha)t}{16} \right) \\
\geq *^2 *^\infty \mu'\left( \varphi(x, y), \frac{4^n(4 - \alpha)t}{16n} \right) \\
= *^2 *^\infty \mu'\left( \varphi(x, y), \frac{4^n(4 - \alpha)t}{16n} \right) \\
= *^2 *^\infty \mu'\left( \varphi(0, x, 0), \frac{4^n(4 - \alpha)t}{16n} \right)
\]

for all \( x, y \in X \), all \( n \in \mathbb{N} \) and all \( t > 0 \), and similarly

\[
\nu(F(x, y) - G(x, y), t) \leq o^2 o^\infty \nu'\left( \varphi(x, y), \frac{4^n(4 - \alpha)t}{16n} \right) \\
o^2 o^\infty \nu'\left( \varphi(x, y), \frac{4^n(4 - \alpha)t}{16n} \right) \\
o^2 o^\infty \nu'\left( \varphi(0, x, 0), \frac{4^n(4 - \alpha)t}{16n} \right)
\]

for all \( x, y \in X \), all \( n \in \mathbb{N} \) and all \( t > 0 \). Since \( \lim_{n \to \infty} \frac{4^n(4 - \alpha)t}{16n} = \infty \) for all \( t > 0 \), we get

\[
\lim_{n \to \infty} \mu'\left( \varphi(x, y), \frac{4^n(4 - \alpha)t}{16n} \right) = 1
\]

and

\[
\lim_{n \to \infty} \nu'\left( \varphi(x, y), \frac{4^n(4 - \alpha)t}{16n} \right) = 0
\]
for all \( x, y \in X \) and all \( t > 0 \). Therefore \( \mu(F(x, y) - G(x, y), t) = 1 \) and \( \nu(F(x, y) - G(x, y), t) = 0 \) for all \( t > 0 \). Thus it is concluded that \( F(x, y) = G(x, y) \).

**Example 3.2.** Let \( X \) be a Hilbert space with inner product \( \langle \cdot, \cdot \rangle \) and \( Z \) be a normed spaced. Denote by \((\mu, \nu)\) and \((\mu', \nu')\) the intuitionistic fuzzy norms given as in Example 2.4 on \( X \) and \( Z \), respectively. Let \( \|\cdot\| \) be induced norm on \( X \) by the inner product \( \langle \cdot, \cdot \rangle \) on \( X \). Let \( \varphi : X \times X \times X \times X \to Z \) be a mapping defined by \( \varphi(x, y, z, w) = 2(\|x\| + \|y\| + \|z\| + \|w\|)z_0 \) for all \( x, y, z, w \in X \), where \( z_0 \) is a fixed unit vector in \( Z \). Define a mapping \( f : X \times X \to X \) by

\[
f(x, y) := (x, y + x_0)x_0 \quad \text{for all } x, y \in X, \text{ where } x_0 \text{ is a fixed unit vector in } X.
\]

Then

\[
\mu(f(x + y, z - w) + f(x - y, z + w) - 2f(x, z) + 2f(y, w), t) = \mu(2(y, x_0)x_0, t)
\]

and

\[
\nu(f(x + y, z - w) + f(x - y, z + w) - 2f(x, z) + 2f(y, w), t) = \nu(2(y, x_0)x_0, t)
\]

for all \( x, y, z, w \in X \) and all \( t > 0 \). Also we can get

\[
\mu'(\varphi(2x, 2y, 2z, 2w), t) = \frac{t}{t + 2(\|x\| + \|y\| + \|z\| + \|w\|)} = \mu'(2\varphi(x, y, z, w), t)
\]

and

\[
\nu'(\varphi(2x, 2y, 2z, 2w), t) = \frac{4(\|x\| + \|y\| + \|z\| + \|w\|)}{t + 4(\|x\| + \|y\| + \|z\| + \|w\|)} = \nu'(2\varphi(x, y, z, w), t)
\]

for all \( x, y, z, w \in X \) and all \( t > 0 \). Therefore

\[
\lim_{n \to \infty} \mu'(\varphi(2x, 2y, 2z, 2w), 4^n t) = \lim_{n \to \infty} \frac{4^n t}{4^n t + 2^{n+1}(\|x\| + \|y\| + \|z\| + \|w\|)} = 1
\]

and

\[
\lim_{n \to \infty} \nu'(\varphi(2x, 2y, 2z, 2w), 4^n t) = \lim_{n \to \infty} \frac{2^{n+1}(\|x\| + \|y\| + \|z\| + \|w\|)}{4^n t + 2^{n+1}(\|x\| + \|y\| + \|z\| + \|w\|)} = 0
\]

for all \( x, y, z, w \in X \) and all \( t > 0 \). Hence the assumptions of Theorem 3.1 for \( \alpha = 2 \) are fulfilled. Therefore, there exist a unique bi-additive mapping \( F : X \times X \to X \) such that

\[
\mu(F(x, y) - f(x, y), t) \geq 2^t \mu'(4(\|x\| + \|y\|)z_0, t) \ast \mu'(2(\|x\| + \|y\|)z_0, t)
\]

and

\[
\nu(F(x, y) - f(x, y), t) \leq c^2 \nu'(4(\|x\| + \|y\|)z_0, t) \circ \nu'(2(\|x\| + \|y\|)z_0, t)
\]

for all \( x, y \in X \) and all \( t > 0 \).
The following theorem will be proved the case \( \alpha > 4 \).

**Theorem 3.3.** Let \( X \) be a linear space and let \((Z, \mu', \nu')\) be an IFNS. Let \( \varphi : X \times X \times X \times \rightarrow Z \) be a mapping such that, for some \( \alpha > 4 \),

\[
\mu'(\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}, \frac{w}{2}\right), t) \geq \mu'(\varphi(x, y, z, w), ct),
\]

\[
\nu'(\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}, \frac{w}{2}\right), t) \leq \nu'(\varphi(x, y, z, w), ct),
\]

for all \( x, y, z, w \in X \) and all \( t > 0 \). Let \((Y, \mu, \nu)\) be an intuitionistic fuzzy Banach space and let \( f : X \times X \rightarrow Y \) be a \( \varphi \)-approximately bi-additive mapping in the sense of \ref{3.2} with \( f(0, 0) = 0 \). Then there exists a unique mapping \( F : X \times X \rightarrow Y \) such that

\[
\mu(F(x, y) - f(x, y), t) \geq \ast \mu'(\varphi(x, x, y, -y), \frac{(\alpha - 4)}{8}t)
\]

\[
\ast \mu'(\varphi(x, -x, y, y), \frac{(\alpha - 4)}{8}t) \ast \mu(\varphi(0, x, 0, y), \frac{(\alpha - 4)}{8}t)
\]

and

\[
\mu(F(x, y) - f(x, y), t) \leq \circ \nu'(\varphi(x, x, y, -y), \frac{(\alpha - 4)}{8}t)
\]

\[
\circ \nu'(\varphi(x, -x, y, y), \frac{(\alpha - 4)}{8}t) \circ \nu'(\varphi(0, x, 0, y), \frac{(\alpha - 4)}{8}t)
\]

for all \( x, y \in X \) and all \( t > 0 \).

**Proof.** The proof is similar to the proof of Theorem 3.1. Then we present a summary proof. From \ref{3.11}, we have

\[
\mu(f(2x, 2z) - 4f(x, z), t) \geq \ast \mu'(\varphi(x, x, z, -z), \frac{1}{8}) \ast \mu'(\varphi(x, -x, z, z), \frac{1}{8})
\]

\[
\ast \mu'(\varphi(0, x, 0, z), \frac{1}{8}),
\]

\[
\nu(f(2x, 2z) - 4f(x, z), t) \leq \circ \nu'(\varphi(x, x, z, -z), \frac{1}{8}) \circ \nu'(\varphi(x, -x, z, z), \frac{1}{8})
\]

\[
\circ \nu'(\varphi(0, x, 0, z), \frac{1}{8})
\]

for all \( x, z \in X \) and all \( t > 0 \). Thus we get

\[
\mu\left(f(x, z) - 4f\left(\frac{x}{2}, \frac{z}{2}\right), t\right) \geq \ast \mu'(\varphi(x, x, z, -z), \frac{\alpha t}{8})
\]

\[
\ast \mu'(\varphi(x, -x, z, z), \frac{\alpha t}{8}) \ast \mu'(\varphi(0, x, 0, z), \frac{\alpha t}{8}),
\]

\[
\nu\left(f(x, z) - 4f\left(\frac{x}{2}, \frac{z}{2}\right), t\right) \leq \circ \nu'(\varphi(x, x, z, -z), \frac{\alpha t}{8}) \circ \nu'(\varphi(x, -x, z, z), \frac{\alpha t}{8})
\]

\[
\circ \nu'(\varphi(0, x, 0, z), \frac{\alpha t}{8})
\]
for all \(x, z \in X\) and all \(t > 0\). Similar in (3.13), for all \(x, z \in X\), all \(m, n \in \mathbb{N}\) and \(t > 0\), we can conclude

\[
\begin{array}{l}
\mu \left( 4^m f\left( \frac{x}{2^m}, \frac{z}{2^m} \right) - 4^n f\left( \frac{x}{2^m}, \frac{z}{2^n} \right), t \right) \\
\geq *^2 \mu' \left( \varphi(x, x, z), \frac{t}{8 \sum_{k=0}^{n+m-1} \alpha^k t} \right) *^n \mu' \left( \varphi(x, x, z), \frac{t}{8 \sum_{k=0}^{n+m-1} \alpha^k t} \right) \\
*^n \mu' \left( \varphi(0, x, 0), \frac{t}{8 \sum_{k=0}^{n+m-1} \alpha^k t} \right)
\end{array}
\]

(3.15)

for all \(x, z \in X\), all \(m, n \in \mathbb{N}\) and all \(t > 0\). Since \(\alpha > 4\), \(\sum_{k=0}^{\infty} \frac{1}{\alpha^k} \) is Cauchy and \(\sum_{k=m}^{n+m-1} \frac{1}{\alpha^k} \rightarrow 0\) as \(m \rightarrow \infty\) for all \(n \in \mathbb{N}\). Thus \(\frac{t}{\sum_{k=m}^{n+m-1} \frac{1}{\alpha^k}} \rightarrow \infty\), then we have

\[
*^2 \mu' \left( \varphi(x, x, z), \frac{t}{8 \sum_{k=m}^{n+m-1} \alpha^k} \right) = \mu' \left( \varphi(x, x, z), \frac{t}{8 \sum_{k=m}^{n+m-1} \alpha^k} \right) \quad \rightarrow 0
\]

and

\[
*^2 \nu' \left( \varphi(x, x, z), \frac{t}{8 \sum_{k=m}^{n+m-1} \alpha^k} \right) = \nu' \left( \varphi(x, x, z), \frac{t}{8 \sum_{k=m}^{n+m-1} \alpha^k} \right) \quad \rightarrow 0
\]

as \(m \rightarrow \infty\) for all \(x, z \in X\), all \(m, n \in \mathbb{N}\) and all \(t > 0\). Hence the Cauchy criterion for convergence in IFNS shows that \(4^n f\left( \frac{x}{2^n}, \frac{z}{2^n} \right) \) is a Cauchy sequence in \((Y, \mu, \nu)\) for all \(x, z \in X\). Since \((Y, \mu, \nu)\) is complete, then this sequence converges to some point \(F(x, z) \in Y\) defined by \(F(x, y) = \lim_{n \rightarrow \infty} 4^n f\left( \frac{x}{2^n}, \frac{y}{2^n} \right)\) for all \(x, z \in X\). By putting \(m = 0\) in (3.13), we can deduce

\[
\mu(F(x, y) - f(x, y), t) \geq *^\infty \mu' \left( \varphi(x, x, y, -y), \frac{(\alpha - 4)}{8} t \right) *^\infty \mu' \left( \varphi(x, x, y, -y), \frac{(\alpha - 4)}{8} t \right)
\]

and

\[
\nu(F(x, y) - f(x, y), t) \leq *^\infty \nu' \left( \varphi(x, x, y, -y), \frac{(\alpha - 4)}{8} t \right) *^\infty \nu' \left( \varphi(x, x, y, -y), \frac{(\alpha - 4)}{8} t \right)
\]
for all \( x, y \in X \) and all \( t > 0 \). The remainder of the proof is similar to the proof of Theorem 3.1. 

\[
\text{Definition 4.1. Let } g : \mathbb{R} \to X \text{ be a mapping, where } \mathbb{R} \text{ is endowed with the Euclidean topology and } X \text{ is an intuitionistic fuzzy normed space equipped with intuitionistic fuzzy norm } (\mu, \nu). \text{ Then } L \in X \text{ is said to be intuitionistic fuzzy limit of } g \text{ at some } r_0 \in \mathbb{R} \text{ if and only if for every } \varepsilon > 0 \text{ and } \alpha, \beta \in (0, 1) \text{ there exists some } \delta = \delta(\varepsilon, \alpha, \beta) > 0 \text{ such that } \mu(g(r) - L, \varepsilon) \geq \alpha \text{ and } \mu(g(r) - L, \varepsilon) \leq 1 - \beta \text{ whenever } 0 < |r - r_0| < \delta. \text{ In this case, we write } \lim_{r \to r_0} g(r) = L, \text{ which also means that } \lim_{r \to r_0} \mu(g(r) - L, t) = 1 \text{ and } \lim_{r \to r_0} \nu(g(r) - L, t) = 0 \text{ or } \mu(g(r) - L, t) = 1 \text{ and } \nu(g(r) - L, t) = 0 \text{ as } r \to r_0 \text{ for all } t > 0. \]

\[
\text{Theorem 4.2. Let } X \text{ be a normed space and } (Y, \mu, \nu) \text{ be an intuitionistic fuzzy Banach space. Let } (Z, \mu', \nu') \text{ be an IFNS and let } 0 < p < 2 \text{ and } z_0 \in Z. \text{ Let } f : X \times X \to Y \text{ be a mapping such that}
\]

\[
\begin{align*}
\begin{cases}
\mu(D_0 f(x, y, z, w), t) & \geq \mu'((\|x\| + \|y\| + \|z\| + \|w\|)z_0, t), \\
\nu(D_0 f(x, y, z, w), t) & \leq \nu'((\|x\| + \|y\| + \|z\| + \|w\|)z_0, t)
\end{cases}
\end{align*}
\]

\[
(4.1)
\]

\[
\text{for all } x, y, z, w \in X \text{ and all } t > 0. \text{ Then there exists a unique mapping } F : X \times X \to Y \text{ satisfies } (1.1) \text{ such that}
\]

\[
\begin{align*}
\begin{cases}
\mu(F(x, y) - f(x, y), t) & \geq \mu'(2(\|x\| + \|y\|)z_0, \frac{(4-2^p)}{8}t), \\
\nu(F(x, y) - f(x, y), t) & \leq \nu'(2(\|x\| + \|y\|)z_0, \frac{(4-2^p)}{8}t)
\end{cases}
\end{align*}
\]

\[
(4.2)
\]

\[
\text{for all } x, y, z, w \in X \text{ and all } t > 0. \text{ Furthermore, if the mapping } g : \mathbb{R} \to Y \text{ defined by } g(r) := \frac{1}{4n} \text{ is intuitionistic fuzzy continuous for some } x, y \in X \text{ and all } n \in \mathbb{N}, \text{ then the mapping } r \to F(rx, ry) \text{ from } \mathbb{R} \to Y \text{ is intuitionistic fuzzy continuous; in this case, } F(rx, ry) = r^2 F(x, y) \text{ for all } r \in \mathbb{R}. \]

\[
\text{Proof. Define } \varphi : X \times X \times X \times X \to Z \text{ by } \varphi(x, y, z, w) = (\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)z_0 \text{ for all } x, y, z, w \in X. \text{ Existence and uniqueness of the mapping } F \text{ satisfying } (1.1) \text{ and } (4.1) \text{ are deduced from Theorem 3.1. Note that, for all } x, y \in X,\]

\[
\text{4 Intuitionistic Fuzzy Continuity}
\]

In this section we apply the intuitionistic fuzzy continuity, which is discussed in [13], to study continuous mapping satisfying (1.1) approximately.
all $n \in \mathbb{N}$ and all $t > 0$, we get

\[
\begin{align*}
\mu \left( F(x, y) - \frac{f(2^n x, 2^n y)}{4^n}, t \right) &= \mu \left( f(2^n x, 2^n y), t \right) \\
&= \mu \left( f(2^n x, 2^n y) - f(2^n x, 2^n y), 4^n t \right) \\
&\geq \ast \infty \mu' \left( \|x\|^p + \|y\|^p \right)_{2^{np+1}} \left( \frac{4^n(4 - 2^p)}{8} \right) t \\
&\ast \infty \mu' \left( \|x\|^p + \|y\|^p \right)_{2^{np+1}} \left( \frac{4^n(4 - 2^p)}{8} \right) t \\
&\ast \infty \mu' \left( \|x\|^p + \|y\|^p \right)_{2^{np+1}} \left( \frac{4^n(4 - 2^p)}{8} \right) t,
\end{align*}
\]

(4.3)

\[
\nu \left( F(x, y) - \frac{f(2^n x, 2^n y)}{4^n}, t \right) = \nu \left( f(2^n x, 2^n y), t \right) \\
= \nu \left( f(2^n x, 2^n y) - f(2^n x, 2^n y), 4^n t \right) \\
\leq \circ \infty \nu' \left( \|x\|^p + \|y\|^p \right)_{2^{np+1}} \left( \frac{4^n(4 - 2^p)}{8} \right) t \\
\circ \infty \nu' \left( \|x\|^p + \|y\|^p \right)_{2^{np+1}} \left( \frac{4^n(4 - 2^p)}{8} \right) t \\
\circ \infty \nu' \left( \|x\|^p + \|y\|^p \right)_{2^{np+1}} \left( \frac{4^n(4 - 2^p)}{8} \right) t.
\]

By putting $x = y = 0$ in (4.3), we have

\[
\begin{align*}
\mu \left( f(0, 0) - \frac{1}{t} f(0, 0), t \right) &\geq 1, \\
\nu \left( f(0, 0) - \frac{1}{t} f(0, 0), t \right) &\leq 0
\end{align*}
\]

for all $n \in \mathbb{N}$ and $t > 0$.

Consider fix $x, y \in X$. From (4.3), we obtain

\[
\begin{align*}
\mu \left( F(rx, ry) - \frac{f(2^n rx, 2^n ry)}{4^n}, t \right) \geq \ast \infty \mu' \left( \|x\|^p + \|y\|^p \right)_{2^{np+1}} \left( \frac{4^n(4 - 2^p)}{8} \right) t \\
\ast \infty \mu' \left( \|x\|^p + \|y\|^p \right)_{2^{np+1}} \left( \frac{4^n(4 - 2^p)}{8} \right) t + \ast \infty \mu' \left( \|x\|^p + \|y\|^p \right)_{2^{np+1}} \left( \frac{4^n(4 - 2^p)}{8} \right) t \\
\ast \infty \mu' \left( \|x\|^p + \|y\|^p \right)_{2^{np+1}} \left( \frac{4^n(4 - 2^p)}{8} \right) t \\
\ast \infty \mu' \left( \|x\|^p + \|y\|^p \right)_{2^{np+1}} \left( \frac{4^n(4 - 2^p)}{8} \right) t \\
\ast \infty \mu' \left( \|x\|^p + \|y\|^p \right)_{2^{np+1}} \left( \frac{4^n(4 - 2^p)}{8} \right) t.
\end{align*}
\]

for all $r \in \mathbb{R}\backslash\{0\}$. Since $\lim_{n \to \infty} 4^n(4 - 2^p) t_2^{np+1} = \infty$ for all $t > 0$, then we get

\[
\begin{align*}
\lim_{n \to \infty} \mu \left( F(rx, ry) - \frac{f(2^n rx, 2^n ry)}{4^n}, t \right) &= 1, \\
\lim_{n \to \infty} \nu \left( F(rx, ry) - \frac{f(2^n rx, 2^n ry)}{4^n}, t \right) &= 0
\end{align*}
\]

for all $r \in \mathbb{R}\backslash\{0\}$. Consider fix $r_0 \in \mathbb{R}$, from the intuitionistic fuzzy continuity of the mapping $t \to \frac{f(2^n x, 2^n y)}{4^n}$, we have

\[
\begin{align*}
\lim_{n \to \infty} \mu \left( \frac{f(2^n rx, 2^n ry)}{4^n} - \frac{f(2^n r_0 x, 2^n r_0 y)}{4^n}, t \right) &= 1, \\
\lim_{n \to \infty} \nu \left( \frac{f(2^n rx, 2^n ry)}{4^n} - \frac{f(2^n r_0 x, 2^n r_0 y)}{4^n}, t \right) &= 0.
\end{align*}
\]
It is concluded that
\[ \mu(F(rx, ry) - F(r_0x, r_0y), t) \geq \mu(F(rx, ry) - \frac{f(2^nrx, 2^ny)}{4^n}, \frac{t}{3}) \] 
\* \mu\left(\frac{f(2^nrx, 2^ny)}{4^n} - F(r_0x, r_0y), \frac{t}{3}\right) \geq 1 \]

and
\[ \nu(F(rx, ry) - F(r_0x, r_0y), t) \leq 0 \]
as \( r \to r_0 \) for all \( t > 0 \). Therefore it is concluded that mapping \( r \to F(rx, ry) \) is intuitionistic fuzzy continuous.

By using the intuitionistic fuzzy continuity of the mapping \( r \to F(rx, ry) \) we show that \( f(sx, sy) = s^2F(x, y) \) for all \( s \in \mathbb{R} \).

Consider rational number \( r \) such that \( 0 < |r - s| < \delta \) and \( |r^2 - s^2| < 1 - \alpha \), then we will have
\[ \mu(F(sx, sy) - s^2(x, y), t) \geq \mu\left(F(sx, sy) - F(rx, ry), \frac{t}{3}\right) \* \mu\left(F(rx, ry) - r^2F(x, y), \frac{t}{3}\right) \* \mu\left(\frac{\nu^2F(x, y) - s^2F(x, y)}{3(1 - \alpha)}, \frac{t}{3}\right) \geq 1 \* \mu\left(F(x, y), \frac{t}{3(1 - \alpha)}\right) \]

and
\[ \nu(F(sx, sy) - s^2(x, y), t) \leq (1 - \alpha) \circ 0 \circ \nu\left(F(x, y), \frac{t}{3(1 - \alpha)}\right) \]

When \( \alpha \to 1 \) and using the definition of IFNS, we get
\[ \mu(F(sx, sy) - s^2F(x, y), t) = 1 \quad \text{and} \quad \nu(F(sx, sy) - s^2F(x, y), t) = 0. \]

So we conclude that
\[ F(sx, sy) = s^2F(x, y). \]

In the following we prove a result similar to Theorem 4.2 for case \( p > 2 \).

**Theorem 4.3.** Let \( X \) be a normed space and \((Y, \mu, \nu)\) be an intuitionistic fuzzy Banach space. Let \((Z, \mu', \nu')\) be an IFNS and let \( p > 2 \) and \( z_0 \in Z \). Let \( f : \)
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$X \times X \to Y$ be a mapping such that satisfies in (1.1). Then there exists a unique mapping $F : X \times X \to Y$ satisfies (1.1) such that

\begin{align*}
\mu(F(x, y) - f(x, y), t) &\geq *^\infty \mu'(2\|x\|^p + \|y\|^p) z_0, \left(\frac{(2^p-4)}{8} t\right) \\
\nu(F(x, y) - f(x, y), t) &\leq *^\infty \nu'(2\|x\|^p + \|y\|^p) z_0, \left(\frac{(2^p-4)}{8} t\right)
\end{align*}

(4.4)

for all $x, y \in X$ and all $t > 0$. Furthermore, if for some $x, y \in X$ and all $n \in \mathbb{N}$, the mapping $g : \mathbb{R} \to Y$ defined by $g(r) := 4^n f(x, y)^2 + 4^n f(x, y)^2$ is intuitionistic fuzzy continuous for some $x, y \in X$ and all $n \in \mathbb{N}$, then the mapping $r \to F(rx, ry)$ from $\mathbb{R}$ to $Y$ is intuitionistic fuzzy continuous, in this case, $F(rx, ry) = r^2 F(x, y)$ for all $r \in \mathbb{R}$.

**Proof.** Define a mapping $\varphi : X \times X \times X \times X \to Z$ by $\varphi(x, y, z, w) = (\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p) z_0$ for all $x, y, z, w \in X$. Then

\[
\mu'(\varphi(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}, \frac{x}{2}), t) = \mu'(\frac{1}{2^{p-1}}(\|x\|^p + \|y\|^p) z_0, t)
\]

for all $x, y \in X$ and all $t > 0$. From $p > 2$, then $2^p > 4$. By Theorem 4.3 there exists a unique mapping $F$ which satisfies (1.1) and (4.4). The rest of the proof is similar as in Theorem 4.2.

**References**


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