A Congruence Relation in Partially Ordered Sets

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Abstract : A concept of congruence relations in partially ordered sets is introduced in this article. A fundamental theorem for homomorphism is obtained. Further extensions to “doubly directed” sets are obtained.

Keywords : directed set; congruence relation; partition.

2010 Mathematics Subject Classification : 06B10; 18B35.

1 Introduction

Congruence relations are studied in lattices and semilattices (see, for example, [1] and [2]). A definition for congruence relations in partially ordered sets is proposed in this article. A partial order on a set is an order which is reflexive, anti-symmetric and transitive. A set with a partial order is called a poset (partially ordered set). A poset in which any two elements have a least upper bound and a greatest lower bound is called a lattice. An equivalence relation in a set is a relation which is reflexive, symmetric and transitive. The collection of all equivalence classes on a set corresponding to an equivalence relation form a partition. On the other hand each partition on a set corresponds to an equivalence relation. The books ([3],[4]) are referred to fundamental concepts and results. In a lattice...
(L, ≤) or (L, ∨, ∧), an equivalence relation \( \theta \), denoted by \( x \equiv y \pmod{\theta} \) when \( x \) and \( y \) are related in \( L \), is called a congruence relation, if it has the substitution properties: \( x \land y \equiv y \land z \pmod{\theta} \) and \( x \lor y \equiv y \lor z \pmod{\theta} \) whenever \( x \equiv y \pmod{\theta} \) in \( L \) and \( z \in L \). A definition for congruence relations in posets will be given in the next section. A corresponding fundamental theorem of homomorphism is obtained. The definitions and the theorem are also extended to “doubly directed sets”. Let us say that a poset \( (P, \leq) \) is a doubly directed set, if for given \( a, b \in P \), there are \( c \) and \( d \) in \( P \) such that \( c \leq a \leq d \) and \( c \leq b \leq d \).

2 Some definitions and examples

**Definition 2.1.** Let \( (P, \leq) \) be a poset. Let \( \theta \) be an equivalence relation on \( P \). It is called a congruence relation, if the following hold in \( P \).

(i) If \( x_1 \leq x_2 \) and \( x_1 \equiv y_1 \pmod{\theta} \), then there is an element \( y_2 \) in \( P \) such that \( x_2 \equiv y_2 \pmod{\theta} \) and \( y_1 \leq y_2 \).

(ii) If \( x_1 \leq x_2 \) and \( x_2 \equiv y_2 \pmod{\theta} \), then there is an element \( y_1 \) in \( P \) such that \( x_1 \equiv y_1 \pmod{\theta} \) and \( y_1 \leq y_2 \).

(iii) If \( x \equiv z \pmod{\theta} \) and \( x \leq y \leq z \), then \( x \equiv y \pmod{\theta} \).

**Example 2.1.** Let \( \theta \) be an usual (lattice) congruence relation on a lattice \( (L, \leq) \) or \( (L, \lor, \land) \) mentioned in the previous section. If \( x_1 \leq x_2 \) and \( x_1 \equiv y_1 \pmod{\theta} \) in \( L \), then \( x_2 = x_1 \lor x_2 \equiv y_1 \lor x_2 \pmod{\theta} \) and \( y_1 \leq y_2 \lor x_2 \). If \( x_1 \leq x_2 \) and \( x_2 \equiv y_2 \pmod{\theta} \) in \( L \), then \( x_1 = x_1 \land x_2 \equiv x_1 \land y_2 \pmod{\theta} \) and \( x_1 \land y_2 \leq y_2 \).

Thus the conditions (i) and (ii) of the previous definition are satisfied. In view of the lemma in page 22 in [3], the condition (iii) of the previous definition is also satisfied.

**Example 2.2.** Consider the lattice given by the Hasse diagram in figure 1. Consider the partition \( \{ \{1\}, \{a, b\}, \{c, d\}, \{e, 0\} \} \). This defines a congruence relation \( \theta \) mentioned in definition 2.1. However, \( e \lor d = b \), \( 0 \lor d = d \), \( e \equiv 0 \pmod{\theta} \) and \( b \neq d \pmod{\theta} \).
Example 2.3. Consider the poset given by the Hasse diagram in figure 2. Then the partition \{\{a, d\}, \{b, c\}\} defines an equivalence relation. It is not a congruence relation that satisfies the conditions (i), (ii) and (iii) of definition 2.1.

Example 2.4. Let \((P, \leq)\) be a poset. Define an equivalence relation \(\theta\) on \(P\) by \(a \equiv b \ (\text{mod} \ \theta)\) if and only if \(\{x \in P : x < a\} = \{x \in P : x < b\}\) and \(\{x \in P : x > a\} = \{x \in P : x > b\}\), (see definition 4.3 in chapter 2 in [3]). If \(x_1 < x_2\) and \(x_1 \equiv y_1 \ (\text{mod} \ \theta)\), then \(x_2 \in \{x \in P : x > x_1\} = \{x \in P : x > y_1\}\), so that \(y_1 < x_2\). If \(x_1 < x_2\) and \(x_2 \equiv y_2 \ (\text{mod} \ \theta)\), then \(x_1 \in \{x \in P : x < y_2\}\) so that \(x_1 < y_2\). If \(x_1 \equiv z_1 \ (\text{mod} \ \theta)\) and \(x_1 \leq y_1 \leq z_1\), then \(\{x \in P : x < z_1\} = \{x \in P : x < x_1\} = \{x \in P : x < y_1\}\) and \(\{x \in P : x > x_1\} = \{x \in P : x > y_1\}\) so that \(x_1 \equiv y_1 \ (\text{mod} \ \theta)\). These three statements prove that \(\theta\) is a congruence relation of definition 2.1, because the other cases of verification are trivial.

Let us recall that a mapping \(T : P_1 \to P_2\) from a poset \(P_1\) to a poset \(P_2\) is said to be order preserving, if \(T(a) \leq T(b)\) in \(P_2\), whenever \(a \leq b\) in \(P_1\).

Definition 2.2. A mapping \(T : P_1 \to P_2\) from a poset \(P_1\) to a poset \(P_2\) is inversely order preserving if the following are satisfied whenever \(a \leq b\) for some \(a, b \in T(P_1)\).

(I) For given \(a_1 \in T^{-1}(a)\), there is a \(b_1 \in T^{-1}(b)\) such that \(a_1 \leq b_1\)

(II) For given \(b_2 \in T^{-1}(b)\), there is an \(a_2 \in T^{-1}(a)\) such that \(a_2 \leq b_2\)

(III) If \(x, z \in T^{-1}(a)\) and \(x \leq y \leq z\) in \(P_1\), then \(y \in T^{-1}(a)\).

3 A fundamental theorem of homomorphism

Theorem 3.1. Let \(\theta\) be an equivalence relation on a poset \((P, \leq)\). If \(\theta\) is a congruence relation mentioned in definition 2.1, then \(P/\theta\) becomes a poset and the natural quotient mapping \(\pi : P \to P/\theta\) is a surjective, order preserving and inversely order preserving mapping. On the other hand, for a given mapping \(T : P \to P_1\) from \(P\) onto a poset \(P_1\) which is order preserving and inversely order preserving, the partition \(\{T^{-1}(a) : a \in P_1\}\) leads to a congruence relation of definition 2.1.

Proof First part: When \(\theta\) is a congruence relation, let \([x]\) denote the equivalence class containing \(x\). Define an order relation \(\leq\) on \(P/\theta\) by the rule: \([x] \leq [y]\) if and only if for given \(x_1 \in [x]\) and \(y_1 \in [y]\), there are \(x_2 \in [x]\) and \(y_2 \in [y]\) such that \(x_1 \leq y_2\) and \(x_2 \leq y_1\). Note that \([x] \leq [x]\), \(\forall x \in P\).

To prove anti-symmetry, suppose \([x] \leq [y]\) and \([y] \leq [x]\) in \(P/\theta\). Then there are \(y_1 \in [y]\) and \(x_1 \in [x]\) such that \(x \leq y_1 \leq x_1\). Then, by (iii) of definition 2.1, \(y \equiv x \ (\text{mod} \ \theta)\). This proves the anti-symmetry of the relation in \(P/\theta\). To prove transitivity, consider the relations \([x] \leq [y]\) and \([y] \leq [z]\) in \(P/\theta\). Then, to given \(x_1 \in [x]\), there are \(y_1 \in [y]\) and \(z_1 \in [z]\) such that \(x_1 \leq y_1\) and \(y_1 \leq z_1\) so that \(x_1 \leq z_1\). Similarly, to given \(z_1 \in [z]\), there are \(y_1 \in [y]\) and \(x_1 \in [x]\) such that \(x_1 \leq y_1\) and \(y_1 \leq z_1\) so that \(x_1 \leq z_1\). This proves transitivity and hence \(P/\theta\) is also a poset.
Suppose \( x \leq y \) in \( P \). Then, by (i) of definition 2.1, for given \( x_1 \in [x] \), there is a \( y_1 \in [y] \), such that \( x_1 \leq y_1 \). Similarly, for given \( y_2 \in [y] \), there is a \( x_2 \in [x] \), such that \( x_2 \leq y_2 \). So \( [x] \leq [y] \). Thus the mapping \( \pi : P \to P/\theta \) defined by \( \pi(x) = [x] \) is order preserving. The definition 2.1 and the definition 2.6 imply that \( \pi \) is inversely order preserving.

**Second part:** Now, let \( \theta \) denote the equivalence relation defined by \( \{ T^{-1}(a) : a \in P \} \) for the given mapping \( T : P \to P_1 \). The definition 2.1 and the definition 2.6 imply that \( \theta \) is a congruence relation. This completes the proof of the theorem.

## 4 Doubly directed sets

A poset \((P, \leq)\) is a doubly directed set if any two elements in \( P \) have an upper bound and a lower bound.

**Definition 4.1.** Let \((P, \leq)\) be a doubly directed set. Let \( \theta \) be an equivalence relation on \( P \). It is called a congruence relation if it satisfies the following:

(i) If \( x \) and \( y \) are given, if \( z \geq x, z \geq y, \) and if \( x \equiv x_1 \mod \theta, y \equiv y_1 \mod \theta \) in \( P \), then there is a \( z_1 \) in \( P \) such that \( z_1 \geq x_1, z_1 \geq y_1 \) and \( z \equiv z_1 \mod \theta \).

(ii) If \( x \) and \( y \) are given, if \( z \leq x, z \leq y \), and if \( x \equiv x_2 \mod \theta, y \equiv y_2 \mod \theta \) in \( P \), then there is a \( z_2 \) in \( P \) such that \( z_2 \leq x_2, z_2 \leq y_2 \), and \( z \equiv z_2 \mod \theta \).

(iii) If \( x \equiv z \mod \theta \) and \( x \leq y \leq z \), then \( x \equiv y \mod \theta \)

Let us observe that the conditions (i),(ii) and (iii) of definition 4.1 imply the corresponding conditions (i),(ii) and (iii) of definition 2.1, where \( z_1 = y_2, z = x_2 \) and \( x = y = x_1 \) (for(i)). Let us now rephrase the definition 2.6.

**Definition 4.2.** A mapping \( T : P_1 \to P_2 \) from a doubly directed set \( P_1 \) to a doubly directed set \( P_2 \) is inversely direction preserving, if the following are satisfied:

(I) If \( a, b, c \) are in \( T(P_1) \) satisfying \( a \leq c \) and \( b \leq c \), then for given \( a_1 \in T^{-1}(a), b_1 \in T^{-1}(b) \), there is a \( c_1 \in T^{-1}(c) \) such that \( a_1 \leq c_1 \) and \( b_1 \leq c_1 \).

(II) If \( a, b, c \) are in \( T(P_1) \) satisfying \( a \geq c \) and \( b \geq c \), then for given \( a_2 \in T^{-1}(a), b_2 \in T^{-1}(b) \), there is a \( c_2 \in T^{-1}(c) \) such that \( a_2 \geq c_2 \) and \( b_2 \geq c_2 \).
(III) If \( a \in T(P_1) \) and \( x, z \in T^{-1}(a) \), and \( x \leq y \leq z \) in \( P_1 \), then \( y \in T^{-1}(a) \).

Let us again observe that the conditions (I),(II) and (III) of definition 4.2 imply that corresponding conditions (I),(II) and (III) of definition 2.6, where \( c_1 = b, c = b, b_1 = a_1 \) and \( b = a \) (for(I)). Let us now rephrase the theorem 3.1.

**Theorem 4.1.** Let \( \theta \) be an equivalence relation on a doubly directed set \((P, \leq)\). If \( \theta \) is a congruence relation mentioned in definition 4.1, then \( P/\theta \) becomes a doubly directed set and the natural quotient mapping \( \pi : P \to P/\theta \) is a surjective, order preserving and inversely direction preserving mapping. On the other hand, for a given mapping \( T \) from \( P \) onto a doubly directed set \( P_1 \) which is order preserving and inversely direction preserving, the partition \( \{ T^{-1}(a) : a \in P_1 \} \) leads to a congruence relation of definition 4.1.

**Proof**  
**First part:** Let us follow the notations and definitions used in the proof for the first part of the theorem 3.1. Then \( P/\theta \) is a poset. Let us fix \( x \) and \( y \) in \( P \) and hence \([x]\) and \([y]\) in \( P/\theta \) to verify that \( P/\theta \) is a doubly directed set. Since \( P \) is a doubly directed set, there are \( a \) and \( b \) in \( P \) such that \( a \leq x \leq b \) and \( a \leq y \leq b \). Then it follows from the definition 4.1 that \([a] \leq [x] \leq [b]\) and \([a] \leq [y] \leq [b]\) in \( P/\theta \). Thus \( P/\theta \) is a doubly directed set; and \( \pi \) is inversely direction preserving. **Second part:** The definitions 4.1 and 4.2 imply the second part to complete the proof of the theorem.

**References**


(Received 5 December 2013)  
(accepted 2 January 2015)