\( (\alpha, \beta) \)-Normal Composition Operators

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Abstract : The composition operators of \((\alpha, \beta)\)-normal operators and their adjoints have been characterized on \(L^2(m)\).

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1 Introduction and Statement of Results

Let \( H \) be an infinite dimensional complex separable Hilbert space and \( \mathcal{B}(H) \) be the algebra of all bounded linear operators defined on \( H \). An operator \( T \in \mathcal{B}(H) \) is called normal if \( TT^* = T^*T \), hyponormal if \( T^*T \geq TT^* \) which is equivalent to the condition \( \|T^*x\| \leq \|Tx\| \), for all \( x \in H \). For real numbers \( \alpha \) and \( \beta \) with \( 0 \leq \alpha \leq 1 \leq \beta \), an operator \( T \) acting on a Hilbert space \( H \) is called \((\alpha, \beta)\)-normal if \( \alpha^2T^*T \leq TT^* \leq \beta^2T^*T \), which is equivalent to the condition \( \alpha\|Tx\| \leq \|T^*x\| \leq \beta\|Tx\| \), for all \( x \in H \) \([1,2]\). For \( \alpha = 1 = \beta \), \( T \) is a normal operator. For \( \alpha = 1 \), we observe from the left inequality that \( T^* \) is hyponormal and for \( \beta = 1 \), from the right inequality we obtain that \( T \) is hyponormal. Takagi and K. Yokouchi \([3]\) initiated the study of multiplication and composition operators between \(L^p\)-spaces. The study of non-normal classes of composition operators initiated by R.K. Singh \([4]\)
in 1974 and later this was studied by many authors [5]-[13]. In this paper, we obtain a necessary and sufficient condition for an operator and its adjonits to be \((\alpha, \beta)\)-normal composition operator.

Let \((X, \Sigma, m)\) be a sigma-finite measure space. The space \(L^2(m)\) is defined as:

\[
L^2(m) = \left\{ f : X \to \mathbb{C} : f \text{ is a measurable function and } \int_X |f|^2 dm < \infty \right\}
\]

with \(\|f\|_2 = \left( \int_X |f|^2 dm \right)^{\frac{1}{2}}\).

**Radon Nikodym Theorem.** If \((X, \Sigma, m)\) is a \(\sigma\)-finite measure space and \(m'\) is a \(\sigma\)-finite measure on \(\Sigma\) such that \(m'\) is absolutely continuous with respect to \(m\), then there exists a finite-valued non-negative measurable function \(h\) on \(X\) such that for each \(A \in \Sigma\), \(m'(A) = \int_A h dm\). Also, \(h\) is unique in the sense that if \(m'(A) = \int_A g dm\) for each \(A \in \Sigma\), then \(h = g\) a.e.(\(m\)).

A mapping \(T : X \to X\) is said to be measurable if \(T^{-1}(A) \in \Sigma\) whenever \(A \in \Sigma\). A measurable transformation \(T : X \to X\) is called non-singular if the pre-image of every null set under \(T\) is a null set. Such a transformation induces a well defined composition operator

\[
C_T : L^2(m) \to L^2(m) \text{ as } C_T f = f \circ T \text{ for each } f \in L^2(m), \text{ if}
\]

(i) the measure \(m \circ T^{-1}\) is absolutely continuous with respect to \(m\), and

(ii) the Radon-Nikodym derivative \(h = \frac{d(m\circ T^{-1})}{dm}\) is essentially bounded.

Every essentially bounded complex-valued measurable function \(\theta\) induces a bounded operator \(M_{\theta}\) on \(L^2(m)\) which is defined by \(M_{\theta} f = \theta f\) for every \(f \in L^2(m)\).

Let \(E \subseteq X\), then the characteristic function of \(E\), written as \(\chi_E\), is the function on \(X\) defined by

\[
\chi_E(x) = 1 \text{ for } x \in E \text{ and } \chi_E(x) = 0 \text{ for } x \in (X - E)
\]

2 \((\alpha, \beta)\)-Normal Composition Operators

In this section we obtain a necessary and sufficient condition for an operator to be \((\alpha, \beta)\)-normal composition operator.

The following lemma due to Harrington and Whitley [7 Lemma 1] is instrumental in the subsequent results.

**Lemma 2.1.** Let \(P\) denote the projection of \(L^2(m)\) on \(R(C_T)\).
(\alpha, \beta)\text{-Normal composition operators}

(a) \( C_T^* C_T f = h f \) and \( C_T C_T^* f = (h \circ T)Pf \) for all \( f \) in \( L^2(m) \).

(b) \( \mathcal{R}(C_T^*) = \{ f \in L^2(m) : f \text{ is } T^{-1}(\Sigma)\text{-measurable} \} \).

**Theorem 2.2.** A composition operator \( C_T \) on \( L^2(m) \) is \((\alpha, \beta)\text{-normal}\) (0 \( \leq \) \( \alpha \leq 1 \leq \beta \)) iff \( \alpha^2 h \leq (h \circ T)P \leq \beta^2 h \) a.e.

*Proof.* By definition of \((\alpha, \beta)\text{-normal operators}, \ C_T \) is \((\alpha, \beta)\text{-normal}\) (0 \( \leq \alpha \leq 1 \leq \beta \))

iff \( \alpha^2 C_T^* C_T \leq C_T C_T^* \leq \beta^2 C_T^* C_T \)

i.e. \( \alpha^2 (C_T^* C_T f, f) \leq (C_T C_T^* f, f) \leq \beta^2 (C_T^* C_T f, f) \) \( \forall f \in L^2(m) \)

iff \( \alpha^2 (M_h f, f) \leq (M_{(h \circ T)}P f, f) \leq \beta^2 (M_h f, f) \) \( \forall f \in L^2(m) \)

iff \( \alpha^2 (M_h \chi_E, \chi_E) \leq (M_{(h \circ T)}P \chi_E, \chi_E) \leq \beta^2 (M_h \chi_E, \chi_E), \)

for every \( \chi_E \) of \( E \) in \( \Sigma \) such that \( m(E) < \infty \)

iff \( \int_E \alpha^2 h \, dm \leq \int_E (h \circ T)P \, dm \leq \int_E \beta^2 h \, dm, \)

for every \( E \) in \( \Sigma \) such that \( m(E) < \infty \)

iff \( \alpha^2 h \leq (h \circ T)P \leq \beta^2 h \) a.e., for 0 \( \leq \alpha \leq 1 \leq \beta \) . \( \square \)

**Theorem 2.3.** An operator \( T \in \mathcal{B}(H) \) is \((\alpha, \beta)\text{-normal}\) (0 \( \leq \alpha \leq 1 \leq \beta \)) iff \( k^2(TT^*) + 2k\alpha^2(T^*T) + TT^* \geq 0 \) a.e. and \( k^2(T^*T) + 2k(TT^*) + \beta^4(T^*T) \geq 0 \) a.e., for all \( k \in \mathbb{R} \).

*Proof.* For all \( x \in H \) and 0 \( \leq \alpha \leq 1 \leq \beta \).

\( k^2(TT^*) + 2k\alpha^2T^*T + TT^* \geq 0 \) a.e. and

\( k^2(T^*T) + 2k(TT^*) + \beta^4(T^*T) \geq 0 \) a.e. for all \( k \in \mathbb{R} \)

iff \( \langle (k^2TT^* + 2k\alpha^2T^*T + TT^*)x, x \rangle \geq 0 \) a.e. and

\( \langle (k^2T^*T + 2kTT^* + \beta^4T^*T)x, x \rangle \geq 0 \) a.e. for all \( k \in \mathbb{R} \)

iff \( k^2(T^*x, x) + 2k\alpha^2(T^*Tx, x) + \langle TT^*x, x \rangle \geq 0 \) a.e. and

\( k^2(T^*Tx, x) + 2k(TT^*x, x) + \beta^4(T^*Tx, x) \geq 0 \) a.e. for all \( k \in \mathbb{R} \)

iff \( k^2(T^*x, T^*x) + 2k\alpha^2(Tx, Tx) + \langle T^*x, T^*x \rangle \geq 0 \) a.e. and

\( k^2(Tx, Tx) + 2k(T^*x, T^*x) + \beta^4(Tx, Tx) \geq 0 \) a.e. for all \( k \in \mathbb{R} \)

iff \( k^2\|T^*x\|^2 + 2k\alpha^2\|Tx\|^2 + \|T^*x\|^2 \geq 0 \) a.e. and

\( k^2\|Tx\|^2 + 2k\|T^*x\|^2 + \beta^4\|Tx\|^2 \geq 0 \) a.e. for all \( k \in \mathbb{R} \).
Using elementary properties of real quadratic forms

\[ k^2TT^* + 2k\alpha^2T^*T + TT^* \geq 0 \text{ a.e. and} \]
\[ k^2T^*T + 2kTT^* + \beta^4TT^* \geq 0 \text{ a.e. for all } k \in \mathbb{R} \]
iff \[ 4\alpha^4\|Tx\|^4 \leq 4\|T^*x\|^4 \text{ and } 4\|T^*x\|^4 \leq 4\beta^4\|Tx\|^4 \]
iff \[ \alpha\|Tx\| \leq \|T^*x\| \text{ and } \|T^*x\| \leq \beta\|Tx\| \]
iff \[ T \in B(H) \text{ is } (\alpha, \beta)\text{-normal operator} \]
iff \[ \alpha\|Tx\| \leq \|T^*x\| \leq \beta\|Tx\|, \quad 0 \leq \alpha \leq 1 \leq \beta \]

**Theorem 2.4.** A composition operator \( C_T \) on \( L^2(m) \) is \((\alpha, \beta)\text{-normal operator} \) \((0 \leq \alpha \leq 1 \leq \beta)\) iff \( k^2(h \circ T)P + 2k\alpha^2h + (h \circ T)P \geq 0 \text{ a.e. and } k^2h + 2k(h \circ T)P + \beta^4h \geq 0 \text{ a.e. for all } k \in \mathbb{R} \).

**Proof.** By Theorem 2.3, \( C_T \) is \((\alpha, \beta)\text{-normal operator} \) \((0 \leq \alpha \leq 1 \leq \beta)\)
iff \[ \langle (k^2C_T^* + 2k\alpha^2C_T^* + C_T^*C_T^*)T(f), T(f) \rangle \geq 0 \text{ and} \]
\[ \langle (k^2C_T^*T + 2kC_T^*T + \beta^4C_T^*T)T(f), T(f) \rangle \geq 0 \]
for all \( f \in L^2(m) \) and for all \( k \in \mathbb{R} \)
iff \[ \langle (k^2C_T^* + 2k\alpha^2C_T^* + C_T^*C_T^*)\chi_E, \chi_E \rangle \geq 0 \text{ and} \]
\[ \langle (k^2C_T^*T + 2kC_T^*T + \beta^4C_T^*T)\chi_E, \chi_E \rangle \geq 0 \]
for every \( \chi_E \) of \( E \) in \( \Sigma \) such that \( m(E) < \infty \) and \( k \in \mathbb{R} \)
iff \[ \langle (k^2M_{(h \circ T)}P + 2k\alpha^2M_h + M_{(h \circ T)}P)\chi_E, \chi_E \rangle \geq 0 \text{ and} \]
\[ \langle (k^2M_h + 2kM_{(h \circ T)}P + \beta^4M_h)\chi_E, \chi_E \rangle \geq 0 \]
for every \( \chi_E \) of \( E \) in \( \Sigma \) such that \( m(E) < \infty \) and \( k \in \mathbb{R} \)
iff \[ \int (k^2M_{(h \circ T)}P + 2k\alpha^2M_h + M_{(h \circ T)}P)\chi_E d\mu \geq 0 \text{ and} \]
\[ \int (k^2M_h + 2kM_{(h \circ T)}P + \beta^4M_h)\chi_E d\mu \geq 0 \]
for every \( \chi_E \) of \( E \) in \( \Sigma \) such that \( m(E) < \infty \) and \( k \in \mathbb{R} \)
iff \[ \int (k^2(h \circ T)P + 2k\alpha^2h + (h \circ T)P)dm \geq 0 \text{ and} \]
\[ \int (k^2h + 2k(h \circ T)P + \beta^4h)dm \geq 0 \]
for every \( E \) in \( \Sigma \) such that \( m(E) < \infty \) and \( k \in \mathbb{R} \)
iff \[ k^2(h \circ T)P + 2k\alpha^2h + (h \circ T)P \geq 0 \text{ a.e. and} \]
\[ k^2h + 2k(h \circ T)P + \beta^4h \geq 0 \text{ a.e. for all } k \in \mathbb{R} \].

**Corollary 2.5.** A composition operator \( C_T \) on \( L^2(m) \) with dense range is \((\alpha, \beta)\text{-normal} \) \((0 \leq \alpha \leq 1 \leq \beta)\) iff \( k^2(h \circ T) + 2k\alpha^2h + (h \circ T) \geq 0 \text{ a.e. and } k^2h + 2k(h \circ T) + \beta^4h \geq 0 \text{ a.e. for all } k \in \mathbb{R} \).
Corollary 2.6. A composition operator $C_T$ on $L^2(m)$ with dense range is $(\alpha, \beta)$-normal $(0 \leq \alpha \leq 1 \leq \beta)$ iff $\alpha^2 h \leq (h \circ T) \leq \beta^2 h$ a.e.

Corollary 2.7. A composition operator $C_T$ on $L^2(m)$ is $(\alpha, \beta)$-normal $(0 \leq \alpha \leq 1 \leq \beta)$ iff for all $f \in L^2(m)$

(a) $\|\alpha h^\frac{1}{2}f\| \leq \|(h \circ T)h^\frac{1}{2}f\| \leq \|\beta h^\frac{1}{2}f\|$.

(b) $\|\alpha h^\frac{1}{2}Pf\| \leq \|(h \circ T)h^\frac{1}{2}Pf\| \leq \|\beta h^\frac{1}{2}Pf\|$

Theorem 2.8. A composition operator $C_T$ on $L^2(m)$ is $(\alpha, \beta)$-normal $(0 \leq \alpha \leq 1 \leq \beta)$ iff for all $f \in L^2(m)$

(a) $\|\alpha h^\frac{1}{2}f\| \leq \|(h \circ T)h^\frac{1}{2}f\| \leq \|\beta h^\frac{1}{2}f\|$.

(b) $\|\alpha h^\frac{1}{2}Pf\| \leq \|(h \circ T)h^\frac{1}{2}Pf\| \leq \|\beta h^\frac{1}{2}Pf\|$

Proof. Let a composition operator $C_T$ on $L^2(m)$ be a $(\alpha, \beta)$-normal operator $(0 \leq \alpha \leq 1 \leq \beta)$.

Then by Corollary 2.7(b)

$\|\alpha h^\frac{1}{2}Pf\| \leq \|(h \circ T)h^\frac{1}{2}Pf\| \leq \|\beta h^\frac{1}{2}Pf\|

Let $E$ be a set of finite measure in $\Sigma$. Let $A = T^{-1}(E)$. As $A$ is $T^{-1}(\Sigma)$ measurable, therefore $P \chi_A = \chi_A$ and

$0 \leq \|(h \circ T)h^\frac{1}{2}P \chi_A\|^2 - \|\alpha h^\frac{1}{2}P \chi_A\|^2$

$= \int_A (h \circ T - \alpha h) dm$

$= \int_A (h \circ T) dm - \alpha d(mT^{-1})(A)$

$= \int_A (h \circ T) C_T \chi_E dm - \alpha d(mT^{-1})(A)$

$= \int_A (h \circ T)(\chi_E \circ T) dm - \alpha d(mT^{-1})(A)$

$= \int_E \left( h^2 - \alpha \frac{d(mT^{-2})}{dm} \right) dm.$

Therefore,

$h^2 - \alpha \frac{d(mT^{-2})}{dm} \geq 0 \text{ a.e.}$

or $h^2 \geq \alpha \frac{d(mT^{-2})}{dm} \text{ a.e.}$ (2.1)

Also,

$0 \leq \|\beta h^\frac{1}{2}P \chi_A\|^2 - \|(h \circ T)h^\frac{1}{2}P \chi_A\|^2$

$= \int_E \left( \beta \frac{d(mT^{-2})}{dm} - h^2 \right) dm.$
Therefore
\[
\beta \frac{d m T^{-2}}{dm} - h^2 \geq 0 \text{ a.e.}
\]
or
\[
\beta \frac{d m T^{-2}}{dm} \geq h^2 \text{ a.e.}
\] (2.2)

Combining (2.1) and (2.2)
\[
\alpha \frac{d m T^{-2}}{dm} \leq h^2 \leq \beta \frac{d m T^{-2}}{dm} \text{ a.e.}
\]

Conversely, suppose that
\[
\alpha \frac{d(mT^{-2})}{dm} \leq h^2 \leq \beta \frac{d(mT^{-2})}{dm} \text{ a.e.}
\]

Then, for any \(E\) in \(\Sigma\) such that \(m(E) < \infty\), the argument above shows that the inequality of Corollary 2.7(b) holds for \(f = \chi_{T^{-1}(E)}\). Suppose that \(f\) is \(T^{-1}(\Sigma)\)-measurable and simple. Then, we can write
\[
f = \sum_j a_j A_j
\]
where \(A_j\)'s are disjoint sets in \(T^{-1}(\Sigma)\).

Then,
\[
\|h^{1/2} Pf\|^2 = \sum \|a_j h^{1/2} \chi_{A_j}\|^2 \\
\geq \sum |a_j| (h \circ T)^{1/2} \chi_{A_j}\|^2 \\
= \| (h \circ T)^{1/2} Pf\|^2
\]

Similarly,
\[
\|\alpha h^{1/2} Pf\|^2 \leq \| (h \circ T)^{1/2} Pf\|^2
\]

As \(T^{-1}(\Sigma)\)-measurable simple functions are dense in \(R(C_T)\), the inequality
\[
\|\alpha h^{1/2} Pf\| \leq \| (h \circ T)^{1/2} Pf\| \leq \| \beta h^{1/2} Pf\| \text{ holds for all } f \in L^2(m)
\]
and hence, \(C_T\) is \((\alpha, \beta)\)-normal \((0 \leq \alpha \leq 1 \leq \beta)\). \(\square\)

**Example 2.9.** Let \(X = \mathbb{N}\) and let \(m\) be the counting measure.
Define \(T : \mathbb{N} \rightarrow \mathbb{N}\) as
\[
T(n) = 2n \ \forall \ n \in \mathbb{N}
\]

Then, \(h(2n) = 1 \ \forall \ n \in \mathbb{N}\).
By Corollary 2.6, $C_T$ is $(\alpha, \beta)$-normal $(0 \leq \alpha \leq 1 \leq \beta)$ if

$$\alpha^2 h \leq h \circ T \leq \beta^2 h, \quad \text{a.e.}$$

if

$$\alpha^2 h(2n) \leq (h \circ T)(2n) \leq \beta^2 h(2n) \quad \forall \ n \in \mathbb{N}$$

if

$$\alpha^2 \cdot 1 \leq h(4n) \leq \beta^2 \cdot 1 \quad \forall \ n \in \mathbb{N}$$

if

$$\alpha^2 \leq 1 \leq \beta^2, \quad \text{which is true since } 0 \leq \alpha \leq 1 \leq \beta.$$

Hence, the composition operator induced by above $T$ is $(\alpha, \beta)$-normal operator $(0 \leq \alpha \leq 1 \leq \beta)$.

3 Adjoint of $(\alpha, \beta)$-Normal Composition Operators

In this section we explore the conditions under which the adjoint of a composition operator is $(\alpha, \beta)$-normal operator.

**Theorem 3.1.** An operator $C_T^* \in \mathcal{B}(L^2(m))$ is $(\alpha, \beta)$-normal $(0 \leq \alpha \leq 1 \leq \beta)$ if $\alpha^2(h \circ T)P \leq h \leq \beta^2(h \circ T)P$.

**Proof.** By definition of $(\alpha, \beta)$-normal operator, $C_T^*$ is $(\alpha, \beta)$-normal $(0 \leq \alpha \leq 1 \leq \beta)$

iff $\alpha^2 C_T C_T^* \leq C_T^* C_T \leq \beta^2 C_T C_T^*$

iff $\alpha^2 \langle C_T C_T^* f, f \rangle \leq \langle C_T^* C_T f, f \rangle \leq \beta^2 \langle C_T C_T^* f, f \rangle \quad \forall \ f \in L^2(m)$

iff $\alpha^2 \langle M_{(h \circ T)} P f, f \rangle \leq \langle M_h f, f \rangle \leq \beta^2 \langle M_{(h \circ T)} P f, f \rangle \quad \forall \ f \in L^2(m)$

iff $\alpha^2 \langle M_{(h \circ T)} P \chi_E, \chi_E \rangle \leq \langle M_h \chi_E, \chi_E \rangle \leq \beta^2 \langle M_{(h \circ T)} P \chi_E, \chi_E \rangle \quad \forall \ f \in L^2(m)$

and for every $\chi_E$ of $E$ in $\Sigma$ such that $m(E) < \infty$

iff $\int_E \alpha^2(h \circ T)Pdm \leq \int_E hdm \leq \int_E \beta^2(h \circ T)Pdm$

for every $E$ in $\Sigma$ such that $m(E) < \infty$

iff $\alpha^2(h \circ T)P \leq h \leq \beta^2(h \circ T)P$ a.e. for $0 \leq \alpha \leq 1 \leq \beta$.

**Theorem 3.2.** An operator $C_T^* \in \mathcal{B}(L^2(m))$ is $(\alpha, \beta)$-normal $(0 \leq \alpha \leq 1 \leq \beta)$ if

$$k^2 h + 2k \alpha^2 (h \circ T)P + h \geq 0 \quad \text{a.e. and}$$

$$k^2(h \circ T)P + 2kh + \beta^4(h \circ T)P \geq 0 \quad \text{a.e. for all } k \in \mathbb{R}.$$

**Proof.** By Theorem 2.3 $C_T^* \in \mathcal{B}(L^2(m))$ is $(\alpha, \beta)$-normal $(0 \leq \alpha \leq 1 \leq \beta)$

iff

$$\langle (k^2 M_h + 2k \alpha^2 M_{(h \circ T)} P + M_h) \chi_E, \chi_E \rangle \geq 0$$

and

$$\langle (k^2 M_{(h \circ T)} P + 2k M_h + \beta^4 M_{(h \circ T)} P) \chi_E, \chi_E \rangle \geq 0$$

for every $\chi_E$ of $E$ in $\Sigma$ such that $m(E) < \infty$ and for all $k \in \mathbb{R}$.
Corollary 3.3. A composition operator $C_T^*$ on $L^2(m)$ with dense range is $(\alpha, \beta)$-normal $\langle 0 \leq \alpha \leq 1 \leq \beta \rangle$ iff $k^2 h + 2k\alpha^2 (h \circ T) + h \geq 0$ a.e. and $k^2 (h \circ T) + 2kh + \beta^4 (h \circ T) \geq 0$ a.e. for all $k \in \mathbb{R}$.

Corollary 3.4. Let $C_T^*$ on $L^2(m)$ be a composition operator with dense range. Then, $C_T^*$ is $(\alpha, \beta)$-normal $\langle 0 \leq \alpha \leq 1 \leq \beta \rangle$ iff $\alpha^2 (h \circ T) \leq h \leq \beta^2 (h \circ T) \geq 0$ a.e.

Corollary 3.5. For an operator, the adjoint $C_T^*$ of composition operator is $(\alpha, \beta)$-Normal $\langle 0 \leq \alpha \leq 1 \leq \beta \rangle$ iff

(a) $\sum_{\sigma(h)} \subseteq T^{-1}(\Sigma)$, and

(b) $\alpha^2 (h \circ T) \leq h \leq \beta^2 (h \circ T)$ a.e., where $\sum_{\sigma(h)}$ denote the relative completion of the sigma-algebra generated by $\{ A \cap \text{support of } h : A \in \Sigma \}$.

Proof. Suppose $C_T^*$ is $(\alpha, \beta)$-Normal $\langle 0 \leq \alpha \leq 1 \leq \beta \rangle$.

Since $\sum_{\sigma(h)} \subseteq T^{-1}(\Sigma)$, therefore $\ker C_T^* \subseteq \ker C_T$.

Therefore, (a) holds and so $h$ is $T^{-1}(\Sigma)$-measurable.

Hence, the set $A = \{ s : \alpha^2 h(T(s)) > h(s) > \beta^2 h(T(s)) \}$ belongs to $T^{-1}(\Sigma)$ and so $A$ can be written as disjoint union of sets $A_n$ of finite measure which also belong to $T^{-1}(\Sigma)$.

Since, $C_T^*$ is $(\alpha, \beta)$-Normal operator

$$0 \leq (\langle C_T^* C_T - \alpha^2 C_T C_T^* \rangle \chi_{A_n}, \chi_{A_n})$$

$$= \langle h \chi_{A_n}, \chi_{A_n} \rangle - \langle \alpha^2 (h \circ T) P \chi_{A_n}, \chi_{A_n} \rangle$$

$$= \int_{A_n} (h - \alpha^2 (h \circ T)) dm \leq 0.$$ 

Hence, $m(A_n) = 0, \forall \ n \in \mathbb{N}$ and therefore (b) holds.

Conversely, let (a) and (b) hold.

Write $f = f_1 + f_2$, where $f_1 \in \overline{R(C_T)}$ and $f_2 \in \overline{R(C_T)^\perp}$.
We have,
\[\langle (C^*_TC - \alpha^2CTC^*_T), f \rangle = \langle hf - \alpha^2(h \circ T)Pf, f \rangle = \langle h(f_1 + f_2) - \alpha^2(h \circ T)P(f_1 + f_2), (f_1 + f_2) \rangle\]
since, \(\alpha^2(h \circ T)f_1\) is \(T^{-1}(\Sigma)\)-measurable, therefore it belongs to \(\text{R}(C_T)\) and so \(\langle \alpha^2(h \circ T)Pf_1, f_2 \rangle = 0\).
Since, \(f_2 \in \ker C_T\). Therefore, \(hf_2 = C^*_Tf_2 = 0\) and \(\langle hf_1, f_2 \rangle = \langle hf_2, f_1 \rangle = 0\).
So,
\[\langle (C^*_TC - \alpha^2CTC^*_T) \rangle = \langle hf_1, f_1 \rangle - \alpha^2\langle (h \circ T)f_1, f_1 \rangle = \int(h - \alpha^2(h \circ T))|f_1|^2 \, dm \geq 0\]
Similarly, \(\beta^2CTC^*_T \geq C^*_TC_T\).
Therefore, \(C^*_T\) is \((\alpha, \beta)\)-normal operator.

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