A Related Fixed Point Theorem for Three Pairs of Mappings on Three Metric Spaces

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Abstract: A new related fixed point theorem for three pairs of mappings on three complete metric spaces is obtained.

Keywords: Complete metric space; Common fixed point; Related fixed point mappings.

2000 Mathematics Subject Classification: 54H25.

The following related fixed point theorem was for two pairs of mappings on two complete metric spaces was proved in [3]. See also [1] and [2].

Theorem 1 Let \((X,d)\) and \((Y,\rho)\) be complete metric spaces. Let \(A,B\) be mappings of \(X\) into \(Y\) and let \(S,T\) be mappings of \(Y\) into \(X\) satisfying the inequalities

\[
\rho(BSy, ATy') \leq c \frac{f(x, x', y, y')}{h(x, x', y, y')},
\]

\[
d(SAx, TBx') \leq c \frac{g(x, x', y, y')}{h(x, x', y, y')}
\]

for all \(x, x'\) in \(X\) and \(y, y'\) in \(Y\) for which \(h(x, x', y, y') \neq 0\), where

\[
f(x, x', y, y') = \max \{d(x, x')\rho(y, y'), d(x, Sy)d(x', Ty'), \}
\]

\[
d(x, Ty')d(x', Sy), \rho(y, Bx')\rho(y', Ax')\}
\]

\[
g(x, x', y, y') = \max \{\rho(Ax, Bx')d(Sy, Ty'), \rho(Ax, BSy)\rho(Bx', ATy'), \}
\]

\[
\rho(Ax, ATy')\rho(Bx', BSy), d(Sy, TBx')d(Ty', SAx)\}
\]

\[
h(x, x', y, y') = \max \{\rho(Ax, Bx'), d(SAx, TBx'), d(Sx, Ty'), \rho(BSy, ATy')\}
\]

and \(0 \leq c < 1\). If one of the mappings \(A, B, S\) and \(T\) is continuous, then \(SA\) and \(TB\) have a unique common fixed point \(z\) in \(X\) and \(BS\) and \(AT\) have a unique common fixed point \(w\) in \(Y\). Further, \(Az = Bz = w\) and \(Sw = Tw = z\).

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We now prove a related fixed point theorem for three pairs of mappings on three complete metric spaces.

**Theorem 2** Let \((X,d), (Y, \rho), \text{ and } (Z, \sigma)\) be complete metric spaces. Let \(A, B\) be mappings of \(X\) into \(Y\), let \(C, D\) be mappings of \(Y\) into \(Z\) and let \(E, F\) be mappings of \(Z\) into \(X\) satisfying the inequalities

\[
d(ECAx, FDBx') \leq c \frac{f_1(y, y', z, z')}{g_1(x, x')},
\]

\[
\rho(BECy, AFDy') \leq c \frac{f_2(z, z', x, x')}{g_2(y, y')},
\]

\[
\sigma(DBEz, CAFz') \leq c \frac{f_3(x, x', y, y')}{g_3(z, z')}
\]

for all \(x, x'\) in \(X\); \(y, y'\) in \(Y\) and \(z, z'\) in \(Z\) for which \(g_1(x, x') \neq 0\); \(g_2(y, y') \neq 0\), \(g_3(z, z') \neq 0\), where

\[
f_1(y, y', z, z') = \max\{\rho(y, y')d(Ez, Fz'), \sigma(Cy, Dy')\rho(BEz, AFz'),
\]

\[
d(ECy, FDy')\sigma(DBEz, CAFz')\}
\]

\[
f_2(z, z', x, x') = \max\{\sigma(z, z')\rho(Ax, Bx'), \rho(BEz, AFz')d(ECAx, DBx'),
\]

\[
\sigma(DBEz, CAFz')\}
\]

\[
f_3(x, x', y, y') = \max\{d(x, x'), \sigma(Cy, Dy'), \rho(Ax, Bx'), d(ECy, FDy'),
\]

\[
\sigma(CAx, DBx')\rho(BECy, AFDy')\}
\]

\[
g_1(x, x') = \max\{d(x, x'), \rho(Ax, Bx'), \sigma(CAx, DBx')d(ECAx, FDBx')\}
\]

\[
g_2(y, y') = \max\{\rho(y, y'), \sigma(Cy, Dy'), d(ECy, FDy'), \rho(BECy, AFDy')\}
\]

\[
g_3(z, z') = \max\{\sigma(z, z'), d(Ez, Fz'), \rho(BEz, AFz'), \sigma(DBEz, CAFz')\}
\]

and \(0 \leq c < 1\). If \(A\) and \(C\) or \(B\) and \(D\) are continuous, then \(ECA\) and \(FDB\) have a unique common fixed point \(u\) in \(X\), \(BEC\) and \(AFD\) have a unique common fixed point \(v\) in \(Y\), and \(DBE\) and \(CAF\) have a unique common fixed point \(w\) in \(Z\). Further, \(Au = Bu = v, Cv = Dw = w\) and \(Eu = Fw = u\).

**Proof.** Let \(x = x_0\) be an arbitrary point in \(X\). We define the sequences \(\{x_n\}\) in \(X\), \(\{y_n\}\) in \(Y\) and \(\{z_n\}\) in \(Z\) inductively by

\[
Ax_{2n-2} = y_{2n-1}, Cy_{2n-1} = z_{2n-1}, Ez_{2n-1} = x_{2n-1},
\]

\[
Bx_{2n-1} = y_{2n}, Dy_{2n} = z_{2n}, Fz_{2n} = x_{2n}
\]

for \(n = 1, 2, \ldots\).

We will first of all suppose that for some \(n\),

\[
g_1(x_{2n}, x_{2n-1}) = \max\{d(x_{2n}, x_{2n-1}), \rho(Ax_{2n}, Bx_{2n-1}), \sigma(CAx_{2n}, DBx_{2n-1}),
\]

\[
d(ECAx_{2n}, FDBx_{2n-1})\}
\]

\[
= \max\{d(x_{2n}, x_{2n-1}), \rho(y_{2n+1}, y_{2n}), \sigma(z_{2n+1}, z_{2n}), d(x_{2n+1}, x_{2n})\}
\]

\[
= 0.
\]
Then putting
\[ x_{2n-1} = x_{2n} = x_{2n+1} = u, \quad y_{2n} = y_{2n+1} = v, \quad z_{2n} = z_{2n+1} = w, \]
we see that
\[ ECAu = FDBu = u = Ew = Fw, \quad AFDv = v = Au = Bu, \]
\[ CAFw = w = Cv = Dv, \]
from which it follows that
\[ BECv = v, \quad DBEw = w. \]
Similarly, \( g_1(x_{2n}, x_{2n+1}) = 0 \) for some \( n \) implies that there exist points \( u \) in \( X, v \) in \( V \) and \( w \) in \( Z \) such that
\[ ECAu = FDBu = u = Ew = Fw, \quad BECv = AFDv = v = Au = Bu \]
\[ DBEw = CAFw = w = CvDv. \]
Similarly, if one \( g_2(y_{2n-1}, y_{2n}), g_2(y_{2n+1}, y_{2n}), g_3(z_{2n-1}, z_{2n}), g_3(z_{2n+1}, z_{2n}) \)
is equal to zero for some \( n \), then equations (4) follow.
We will therefore suppose that \( g_1(x_{2n-1}, x_{2n}), g_1(x_{2n}, x_{2n+1}), g_2(y_{2n-1}, y_{2n}), g_2(y_{2n+1}, y_{2n}), g_3(z_{2n-1}, z_{2n}) \)
and \( g_3(z_{2n+1}, z_{2n}) \) are all non-zero for all \( n \). We have
\[
\begin{align*}
\ell_1(y_{2n-1}, y_{2n}, z_{2n-1}, z_{2n}) &= \max\{\rho(y_{2n-1}, y_{2n})d(x_{2n-1}, x_{2n}), \sigma(z_{2n-1}, z_{2n})\rho(y_{2n}, y_{2n+1}), d(x_{2n-1}, x_{2n})\sigma(z_{2n}, z_{2n+1})\}, \\
\ell_2(z_{2n-1}, z_{2n}, x_{2n}, x_{2n-1}) &= \max\{\sigma(z_{2n-1}, z_{2n})\rho(y_{2n}, y_{2n+1}), d(x_{2n-1}, x_{2n})\sigma(z_{2n}, z_{2n+1}), \rho(y_{2n}, y_{2n+1})d(x_{2n}, x_{2n+1})\}, \\
\ell_3(x_{2n}, x_{2n-1}, y_{2n-1}, y_{2n}) &= \max\{d(x_{2n-1}, x_{2n}), \sigma(z_{2n-1}, z_{2n})\rho(y_{2n}, y_{2n+1}), d(x_{2n-1}, x_{2n})\sigma(z_{2n}, z_{2n+1})\rho(y_{2n}, y_{2n+1})\},
\end{align*}
\]
\[
\begin{align*}
g_1(x_{2n}, x_{2n-1}) &= \max\{d(x_{2n-1}, x_{2n}), \rho(y_{2n}, y_{2n+1}), \sigma(z_{2n}, z_{2n+1}), d(x_{2n}, x_{2n+1})\}, \\
g_2(y_{2n-1}, y_{2n}) &= \max\{\rho(y_{2n-1}, y_{2n}), \sigma(z_{2n-1}, z_{2n}), d(x_{2n-1}, x_{2n}), \rho(y_{2n}, y_{2n+1})\}, \\
g_3(z_{2n-1}, z_{2n}) &= \max\{\sigma(z_{2n-1}, z_{2n}), d(x_{2n-1}, x_{2n}), \rho(y_{2n}, y_{2n+1}), \sigma(z_{2n}, z_{2n+1})\}.
\end{align*}
\]
Applying inequality (1), we get
\[
d(x_{2n+1}, x_{2n}) = d(ECAx_{2n}, FDBx_{2n-1}) \\
\leq c \frac{f_1(y_{2n-1}, y_{2n}, z_{2n-1}, z_{2n})}{g_1(x_{2n}, x_{2n-1})} \]
(11)
and it now follows from (5), (8) and (11) that
\[
d(x_{2n}, x_{2n+1}) \leq c \max\{d(x_{2n-1}, x_{2n}), \rho(y_{2n-1}, y_{2n}), \sigma(z_{2n-1}, z_{2n})\}.
\] (12)

Applying inequality (2), we get
\[
\rho(y_{2n}, y_{2n+1}) = \rho(BECy_{2n-1}, AFDy_{2n})
\leq c \frac{f_2(z_{2n-1}, z_{2n}, x_{2n}, x_{2n-1})}{g_2(y_{2n-1}, y_{2n})}
\] (13)

and it now follows from (6), (9) and (13) that
\[
\rho(y_{2n}, y_{2n+1}) \leq c \max\{d(x_{2n}, x_{2n+1}), \sigma(z_{2n}, z_{2n+1})\}.
\] (14)

Applying inequality (3), we get
\[
\sigma(z_{2n}, z_{2n+1}) = \sigma(DBEz_{2n-1}, CAFz_{2n})
\leq c \frac{f_3(x_{2n}, x_{2n-1}, y_{2n-1}, y_{2n})}{g_3(z_{2n-1}, z_{2n})}
\] (15)

and it now follows from (7), (10) and (15) that
\[
\sigma(z_{2n}, z_{2n+1}) \leq c \max\{d(x_{2n-1}, x_{2n}), \sigma(z_{2n-1}, z_{2n})\}.
\] (16)

Using inequalities (12), (14) and (16) we now get
\[
\rho(y_{2n}, y_{2n+1}) \leq c \max\{cd(x_{2n-1}, x_{2n}), c\rho(y_{2n-1}, y_{2n}), c\sigma(z_{2n-1}, z_{2n})\}
\leq c \max\{d(x_{2n-1}, x_{2n}), \rho(y_{2n-1}, y_{2n}), \sigma(z_{2n-1}, z_{2n})\}.
\] (17)

On applying inequality (1) again, we get
\[
d(x_{2n-1}, x_{2n}) = d(ECAx_{2n-2}, FDBx_{2n-1})
\leq c \frac{f_1(y_{2n-1}, y_{2n-2}, z_{2n-2}, z_{2n-1})}{g_1(x_{2n-2}, x_{2n-1})}
\]

from which it follows that
\[
d(x_{2n-1}, x_{2n}) \leq c \max\{d(x_{2n-2}, x_{2n-1}), \rho(y_{2n-2}, y_{2n-1}), \sigma(z_{2n-2}, z_{2n-1})\}
\] (18)

and similarly on using inequalities (2) and (3), we get
\[
\rho(y_{2n-1}, y_{2n}) \leq c \max\{d(x_{2n-1}, x_{2n}), \sigma(z_{2n-1}, z_{2n})\},
\] (19)

\[
\sigma(z_{2n-1}, z_{2n}) \leq c \max\{d(x_{2n-2}, x_{2n-1}), \sigma(z_{2n-2}, z_{2n-1})\}.
\] (20)

On using inequalities (18), (19) and (20), we get
\[
\rho(y_{2n-1}, y_{2n}) \leq c \max\{d(x_{2n-2}, x_{2n-1}), \rho(y_{2n-2}, y_{2n-1}), \sigma(z_{2n-2}, z_{2n-1})\}.
\] (21)
It now follows from inequalities (12) and (18) that
\[ d(x_n, x_{n+1}) \leq \max \{ d(x_{n-1}, x_n), \rho(y_{n-1}, y_n), \sigma(z_{n-1}, z_n) \} \]
\[ \leq k^{n-1} \max \{ d(x_1, x_2), \rho(y_1, y_2), \sigma(z_1, z_2) \}. \] (22)

Similarly, on using inequalities (17), (21), (16) and (20), we get
\[ \rho(y_n, y_{n+1}) \leq k^{n-1} \max \{ d(x_1, x_2), \rho(y_1, y_2), \sigma(z_1, z_2) \}, \] (23)
\[ \sigma(z_n, z_{n+1}) \leq k^{n-1} \max \{ d(x_1, x_2), \rho(y_1, y_2), \sigma(z_1, z_2) \}. \] (24)

Since \( c < 1 \), it follows from inequalities (22), (23) and (24) that \( \{x_n\} \) is a Cauchy sequence in \( X \) with a limit \( u \), \( \{y_n\} \) is a Cauchy sequence in \( Y \) with a limit \( v \) and \( \{z_n\} \) is a Cauchy sequence in \( Z \) with a limit \( w \).

Now suppose that \( A \) and \( C \) are continuous. Then
\[ v = \lim_{n \to \infty} y_{2n+1} = \lim_{n \to \infty} Ax_{2n} = Au, \quad w = \lim_{n \to \infty} z_{2n-1} = \lim_{n \to \infty} Cy_{2n-1} = Cv \] (25)
and hence
\[ \lim_{n \to \infty} f_1(v, y_{2n}, w, z_{2n}) = d(Ew, u)\sigma(DBEw, u), \] (26)
\[ \lim_{n \to \infty} f_2(w, z_{2n}, u, x_{2n-1}) = \rho(BEw, v)d(Ew, u), \] (27)
\[ \lim_{n \to \infty} f_3(v, y_{2n}, w, z_{2n}) = 0, \] (28)
\[ \lim_{n \to \infty} g_1(u, x_{2n-1}) = d(Ew, u), \] (29)
\[ \lim_{n \to \infty} g_2(v, v_n) = \max \{ d(Ew, u)\rho(BEw, v) \}, \] (30)
\[ \lim_{n \to \infty} g_3(w, z_{2n}) = \max \{ d(Ew, u)\rho(BEw, w), \sigma(DBEw, w) \}. \] (31)

If \( \lim_{n \to \infty} g_1(u, x_{2n-1}) = 0 \), then \( Ew = u \) and \( ECAu = u \).

If it were possible that
\[ \lim_{n \to \infty} g_1(u, x_{2n-1}) = d(Ew, u) \neq 0, \]
then on applying inequality (1) and equations (25), (26) and (29), we get
\[ d(Ew, u) = \lim_{n \to \infty} d(ECAu, FDBx_{2n-1}) \leq \sigma(DBEw, w). \] (32)

On using inequality (3) and equations (28) and (31), we get
\[ \sigma(DBEw, w) = \lim_{n \to \infty} \sigma(DBEw, CAFz_{2n}) = 0 \]
which implies that \( DBEw = w \) and hence from (32) we must have \( Ew = u \).

On using inequality (2) and equations (25), (27) and (30), we have
\[ \rho(BEw, v) = \lim_{n \to \infty} \rho(BEw, v) \leq cd(Ew, u) = 0 \]
which implies that

\[ Bu = v, \quad Dv = w, \quad ECAu = u, \quad BECv = v. \]

Now suppose that \( Fw \neq u \). On applying inequality (1), we have

\[
d(u, Fw) = \lim_{n \to \infty} d(ECAx_{2n}, FDBu) \\
\leq c \left\{ \frac{\lim_{n \to \infty} f_1(y_{2n-1}, v, z_{2n-1}, w)}{\lim_{n \to \infty} \lambda_1(x_{2n}, u)} \right\} \\
= c \sigma(w, CAFw).
\]

Applying inequality (3), we now have

\[
\sigma(w, CAFw) = \lim_{n \to \infty} \sigma(DBEz_{2n-1}, CAFw) \\
\leq c \left\{ \frac{\lim_{n \to \infty} f_3(x_{2n}, u, y_{2n-1}, v)}{\lim_{n \to \infty} \lambda_3(z_{2n-1}, w)} \right\} \\
= 0.
\]

This implies that \( w = CAFw \) and hence from (33), we must have \( Fw = u \). Equations (4) follows.

Equations (4) follow similarly if \( B \) and \( D \) are continuous.

To prove the uniqueness, let \( ECA \) and \( FDB \) have a second distinct fixed point \( u' \). Then, using inequalities (1), (2) and (3) respectively, we have

\[
d(u, u') = d(ECAu, FDBu') \leq c \frac{f_1(Au, Bu', CAu, DBu)}{\lambda_1(u, u')} \\
\]

which implies that

\[
d(u, u') \leq c \max\{\rho(v, Au'), \rho(v, Bu'), \sigma(w, CAu')\}, \tag{34}
\]

\[
\rho(v, Au') = \rho(BECAu, AFDBu') \leq c \frac{f_2(CAu, DBu', u, u')}{\lambda_2(Au, Bu')} \\
\]

which implies that

\[
\rho(v, Au') \leq c \max\{d(u, u'), \rho(v, Bu')\} \tag{35}
\]

and

\[
\sigma(w, CAu') = \sigma(DBECAu, CAFDBu') \leq c \frac{f_3(u, u', Au, Bu')}{\lambda_3(CAu, DBu')} \\
\]

which implies that

\[
\sigma(w, CAu') \leq c \max\{d(u, u'), \rho(v, Au'), \rho(v, Bu')\}. \tag{36}
\]
On applying inequality (2) again, we have
\[
\rho(Bu', v) = \rho(\text{BEC}Au', \text{AFD}Bu) \leq c \frac{f_2(CAu', DBu, u', u)}{g_2(Au, Bu')}
\]
which implies that
\[
\rho(v, Bu') \leq c \max\{d(u, u'), \rho(v, Au')\}. \tag{37}
\]
It now follows from (34) to (37) that
\[
d(u, u') \leq c \max\{\rho(v, Au'), \rho(v, Bu')\} \tag{38}
\]
and then (35), (37) and (38) imply that \( u = u' \), proving the uniqueness of \( u \).

We can prove similarly that \( v \) is the unique common fixed point of \text{BEC} and \text{AFD} and \( w \) is the unique common fixed point of \text{DBE} and \text{CAF}.

\section*{References}


\( \text{(Received 25 May 2005)} \)

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