Common Fixed Point Theorems

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Abstract: In this note we establish a common fixed point theorem for a quadruple of self-mappings satisfying a generalized contractive condition in a normed space which extends the result of Rashwan [2]. We also prove some fixed point theorems with asymptotic regularity condition for a quadruple of mappings. These theorems generalize and extend results of Sastry et al. [3] and Zeqing Liu et al. [4].

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1 Introduction

The following definitions were used in [1] and [2] respectively.

Definition 1.1 Let \((N, ||.||)\) be a normed space. Then \(T_1\) and \(T_2\) be two self-mappings of \(N\) called a \textit{generalized contractive pair of mappings} if

\[
||T_1x - T_2y|| \leq \max \left\{ ||x - y||, \frac{||x - T_1x||[1 - ||x - T_2y||]}{1 + ||x - T_1x||}, \frac{||x - T_2y||[1 - ||x - T_1x||]}{1 + ||x - T_2y||}, \frac{||T_1x - y||[1 - ||T_1x - y||]}{1 + ||T_1x - y||} \right\},
\]

for all \(x, y\) in \(X\), where \(0 < q < 1\).

Definition 1.2 Let \(T_1\) and \(T_2\) be two self-mappings of a Banach space \(B\). The \textit{Mann iterative process} associated with \(T_1\) and \(T_2\) is defined in the following
Theorem 1.4

Let $x_0$ be in $N$ and let

\[
x_{2n+1} = (1 - c_{2n})x_{2n} + c_{2n}T_1x_{2n},
\]
\[
x_{2n+2} = (1 - c_{2n+1})x_{2n+1} + c_{2n+1}T_2x_{2n+1},
\]

for $n = 0, 1, 2, \ldots$, where $c_n$ satisfies (i) $c_0 = 1$, (ii) $0 < c_n < 1$, $n = 1, 2, \ldots$ and (iii) $\lim_{n \to \infty} c_n = h > 0$.

In [1], Pathak proved the following common fixed point theorem:

**Theorem 1.3** Let $X$ be a closed convex subset of a normed space $N$ and let $T_1$ and $T_2$ be two continuous self mappings satisfying Definition 1.1 on $X$. Let $x_0$ be an arbitrary point in $X$. Then sequence of Mann iterates $\{x_n\}$ associated with $T_1$ and $T_2$ is defined by

\[
x_{2n+1} = (1 - c_{2n})x_{2n} + c_{2n}T_1x_{2n},
\]
\[
x_{2n+2} = (1 - c_{2n+1})x_{2n+1} + c_{2n+1}T_2x_{2n+1},
\]

for $n = 0, 1, 2, \ldots$, where $\{c_n\}$ satisfies conditions (i), (ii) and (iii) of Definition 1.2. If $\{x_n\}$ converges to $u$ in $X$ and if $u$ is fixed point of either $T_1$ or $T_2$, then $u$ is the common fixed point of $T_1$ and $T_2$.

In [2], Rashwan extended Theorem 1.3 for three mappings as follows:

**Theorem 1.4** Let $X$ be a closed convex subset of a normed space $N$. Let $T_1$ and $T_2$ be mappings of $X$ into $X$ and $f$ a continuous mapping of $X$ into $X$ such that

\[
\|T_1x - T_2y\| \leq q \max \left\{ \frac{\|fx - fy\|}{1 + \|fx - T_1x\|}, \frac{\|fx - T_2y\|}{1 + \|fx - T_2y\|}, \frac{\|fx - T_1x\|}{1 + \|fx - T_1x\|}, \frac{\|T_1x - f_2y\|}{1 + \|f_2y - T_2y\|}, \frac{\|T_1x - f_2y\|}{1 + \|T_1x - f_2y\|} \right\},
\]

\[
\|fx - fy\| \leq \|T_1x - f_2x\| + \|T_1x - T_2y\| + \|T_2y - f_2y\|
\]

for all $x, y$ in $X$, where $0 < q < 1$, and the sequence $\{fx_n\}$ associated with $T_1$ and $T_2$ is given by

\[
fx_{2n+1} = (1 - c_{2n})fx_{2n} + c_{2n}T_1x_{2n},
\]
\[
fx_{2n+2} = (1 - c_{2n+1})fx_{2n+1} + c_{2n+1}T_2x_{2n+1},
\]

for $n = 0, 1, 2, \ldots$, where $\{c_n\}$ satisfies conditions (i), (ii) and (iii) as given above and $x_0$ is an arbitrary point in $X$. If $\{fx_n\}$ converges to a point $u$ in $X$, then $u$ is a common fixed point of $T_1, T_2$ and $f$. 
2 Main Results

We extend Theorem 1.4 for a quadruple of self-mappings as follows:

**Theorem 2.1** Let \( X \) be a closed convex subset of a normed space \( N \). Let \( T_1, T_2 \) be mappings of \( X \) into \( X \) and let \( f \) and \( g \) be injective and continuous mappings of \( X \) into \( X \) satisfying

\[
\|T_1x - T_2y\| \leq q \max \left\{ \frac{\|fx - gy\|}{1 + \|fx - T_1x\|}, \frac{\|fx - T_2y\|}{1 + \|fx - T_1x\|}, \frac{\|T_1x - gy\|}{1 + \|T_1x - gy\|}, \frac{\|T_1x - gy\|}{1 + \|T_1x - gy\|} \right\}
\]

(2.1)

for all \( x, y \in X \), where \( 0 < q < 1 \),

\[
(1 - \lambda)f(X) + \lambda T_1(X) \subseteq g(X),
\]

(2.4)

\[
(1 - \mu)g(X) + \mu T_2(X) \subseteq f(X)
\]

(2.5)

for all \( \lambda, \mu \in (0, 1] \), the sequence \( \{x_n\} \) associated with the mappings \( T_1, T_2, f \) and \( g \) is defined by

\[
x_{2n+1} = g^{-1}[(1 - c_{2n})fx_{2n} + c_{2n}T_1x_{2n}],
\]

(2.6)

\[
x_{2n+2} = f^{-1}[(1 - c_{2n+1})gx_{2n+1} + c_{2n+1}T_2x_{2n+1}]
\]

(2.7)

for \( n = 0, 1, 2, \ldots \), where \( x_0 \) is an arbitrary point in \( X \) and \( \{y_n\} \) is the sequence defined by \( y_{2n-1} = fx_{2n-1} \) and \( y_{2n} = gx_{2n} \) for \( n = 1, 2, \ldots \) and \( \{c_n\} \) satisfies conditions (i), (ii) and (iii) given above. If \( \{y_n\} \) converges to a point \( u \) in \( X \), then \( u \) is the unique common fixed point of \( T_1, T_2, f \) and \( g \).

**Proof.** Since \( f \) and \( g \) are injective and satisfy conditions (2.4) and (2.5), the sequence \( \{x_n\} \) defined by equations (2.6) and (2.7) is unique. Also from equation (2.6), we have

\[
T_1x_{2n} = \frac{gx_{2n+1} - (1 - c_{2n})fx_{2n}}{c_{2n}}
\]

and so

\[
\lim_{n \to \infty} T_1x_{2n} = \lim_{n \to \infty} \frac{gx_{2n+1} - (1 - c_{2n})fx_{2n}}{c_{2n}} = \frac{u - (1 - h)u}{h} = u.
\]
Similarly
\[
\lim_{n \to \infty} T_2x_{2n+1} = u.
\]

From equation (2.2), we have
\[
\|f_{x_{2n}} - f_{g_{x_{2n}}+1}\| \leq \|T_1x_{2n} - f_{x_{2n}}\| + \|T_1x_{2n} - T_2x_{2n+1}\| + \|T_2x_{2n+1} - g_{x_{2n}} - f_{x_{2n}}\|
\]
and so
\[
\lim_{n \to \infty} \|f_{x_{2n}} - f_{g_{x_{2n}}+1}\| = \lim_{n \to \infty} \|y_{2n} - f_{y_{2n+1}}\| = \|u - f_{u}\| \leq 0.
\]

It follows that \(u = f_{u}\).

Also from (2.3), we have
\[
\|g_{x_{2n+1}} - g_{f_{x_{2n}}}\| \leq \|T_1x_{2n} - g_{x_{2n+1}}\| + \|T_1x_{2n} - T_2x_{2n+1}\| + \|T_2x_{2n+1} - f_{x_{2n}}\| + \|g_{x_{2n+1}} - f_{x_{2n}}\|
\]
and so
\[
\lim_{n \to \infty} \|g_{x_{2n+1}} - g_{f_{x_{2n}}}\| = \lim_{n \to \infty} \|y_{2n+1} - g_{y_{2n}}\| = \|u - g_{u}\| \leq 0.
\]

It follows that \(u = g_{u}\).

Further, using inequality (2.1), we have
\[
\|u - T_2u\| \leq \|u - g_{x_{2n+1}}\| + \|g_{x_{2n+1}} - T_2u\|
\]
\[
\leq \|u - g_{x_{2n+1}}\| + \|(1 - c_{2n})f_{x_{2n}} + c_{2n}T_1x_{2n} - T_2u\|
\]
\[
\leq \|u - g_{x_{2n+1}}\| + (1 - c_{2n})\|f_{x_{2n}} - T_2u\| + c_{2n}\|T_1x_{2n} - T_2u\|
\]
\[
\leq \|u - g_{x_{2n+1}}\| + (1 - c_{2n})\|f_{x_{2n}} - T_2u\|
\]
\[
+ c_{2n}q \max \left\{ \|f_{x_{2n}} - g_{u}\|, \frac{\|f_{x_{2n}} - T_1x_{2n}\|}{1 + \|f_{x_{2n}} - T_1x_{2n}\|}, \frac{\|T_1x_{2n} - g_{u}\|}{1 + \|T_1x_{2n} - g_{u}\|}, \frac{\|g_{u} - T_2u\|}{1 + \|g_{u} - T_2u\|} \right\}.
\]
Assuming that $T_2 u \neq u$, we have on letting $n$ tends to infinity
\[
||u - T_2 u|| \leq 0 + (1 - h)||u - T_2 u|| + hq \max \left\{ 0, 0, \frac{||u - T_2 u||}{1 + ||u - T_2 u||}, 0, \frac{||u - T_2 u||}{1 + ||u - T_2 u||} \right\}
\]
\[
\leq (1 - h)||u - T_2 u|| + hq \frac{||u - T_2 u||}{1 + ||u - T_2 u||}
\]
\[
< (1 - h + hq)||u - T_2 u||
\]
\[
< ||u - T_2 u||,
\]
a contradiction, and so $u = T_2 u$.

Similarly
\[
||u - T_1 u|| \leq ||u - f x_{2n+2}|| + ||f x_{2n+2} - T_1 u||
\]
\[
\leq ||u - f x_{2n+2}|| + |(1 - c_{2n+1})gx_{2n+1} + c_{2n+1}T_2 x_{2n+1} - T_1 u||
\]
\[
\leq ||u - f x_{2n+2}|| + (1 - c_{2n+1})||gx_{2n+1} - T_1 u||
\]
\[
+ c_{2n+1}||T_1 u - T_2 x_{2n+1}||
\]
\[
\leq ||u - f x_{2n+2}|| + (1 - c_{2n+1})||gx_{2n+1} - T_1 u||
\]
\[
+ c_{2n+1}q \max \left\{ ||fu - gx_{2n+1}|, \frac{||fu - T_1 u||}{1 + ||fu - T_1 u||}, \frac{||fu - T_2 x_{2n+1}||}{1 + ||fu - T_2 x_{2n+1}||} \right\}
\]
\[
1 + ||T_1 u - gx_{2n+1}||
\]
\[
1 + ||T_1 u - gx_{2n+1}||
\]
\[
1 + ||T_1 u - gx_{2n+1}||
\]
\[
1 + ||T_1 u - gx_{2n+1}||
\]
Assuming that $T_1 u \neq u$, we have on letting $n$ tend to infinity
\[
||u - T_1 u|| \leq 0 + (1 - h)||u - T_1 u|| +
\]
\[
\leq (1 - h)||u - T_1 u|| + \frac{hq||u - T_1 u||}{1 + ||u - T_1 u||}
\]
\[
< (1 - h + hq)||u - T_1 u||
\]
\[
< ||u - T_1 u||,
\]
a contradiction, so that $u = T_1 u$. We have therefore proved that $u$ is a common fixed point of $T_1, T_2, f$ and $g$. 
To prove the uniqueness of $u$, suppose that $v$ is a second common fixed point of $T_1, T_2, f$ and $g$. Then
\[
||u - v|| = ||T_1u - T_2v||
\]
\[
\leq q \max \left\{ ||f - g||, \frac{||f - T_1u||[1 - ||f - T_2v||]}{1 + ||f - T_1u||}, \frac{||f - T_2v||[1 - ||f - T_1u||]}{1 + ||f - T_2v||}, \frac{||g - T_1u||[1 - ||g - T_2v||]}{1 + ||g - T_2v||}, \frac{||g - T_2v||[1 - ||g - T_1u||]}{1 + ||g - T_2v||} \right\}
\]
\[
= q \max \left\{ ||u - v||, \frac{||u - u||[1 - ||u - v||]}{1 + ||u - u||}, \frac{||u - v||[1 - ||u - u||]}{1 + ||u - v||}, \frac{||v - v||[1 - ||v - u||]}{1 + ||v - v||} \right\}
\]
\[
= q \max \left\{ ||u - v||, 0, \frac{||u - v||}{1 + ||u - v||} \right\}
\]
\[
= q ||u - v||,
\]
a contradiction and so $u = v$. This proves the uniqueness of $u$. \hfill \Box

When $f = g = I_X$ the identity mapping on $X$, conditions (2.2) and (2.3) are trivial and we have the following corollary:

**Corollary 2.2** Let $X$ be a closed convex subset of a normed vector space $N$. Let $T_1$ and $T_2$ be mappings of $X$ into $X$ satisfying
\[
||T_1x - T_2y|| \leq q \max \left\{ ||x - y||, \frac{||x - T_1x||[1 - ||x - T_2y||]}{1 + ||x - T_1x||}, \frac{||x - T_2y||[1 - ||x - T_1x||]}{1 + ||x - T_2y||}, \frac{||y - y||[1 - ||y - T_2y||]}{1 + ||y - T_2y||} \right\},
\]
for all $x, y$ in $X$, where $0 < q < 1$,
\[
(1 - \lambda)X + \lambda T_1(X) \subseteq X,
\]
\[
(1 - \mu)X + \mu T_2(X) \subseteq X,
\]
for all $\lambda, \mu \in (0,1]$, the sequence $\{x_n\}$ is defined as in Theorem 1.3 and $\{c_n\}$ satisfies conditions (i), (ii) and (iii), given above. If $\{x_n\}$ converges to a point $u$ in $X$, then $u$ is the unique common fixed point of $T_1$ and $T_2$. 
**Example 2.3** Let $X = [0, 1] \subset \mathbb{R}$, where $\mathbb{R}$ is the set of real numbers with the usual norm and $T_1, T_2, f, g : X \to X$

$$T_1x = \frac{x}{4}, \quad T_2x = \frac{x^{2/3}}{4},$$

$$fx = x^{1/2}, \quad gx = x^{1/3}.$$  

Clearly the mappings $g^{-1}$ and $f^{-1}$ defined by

$$g^{-1}x = x^3 \quad \text{and} \quad f^{-1}x = x^2$$  

exist.

Suppose that $\{y_n\}$ is a sequence of elements of $X$ such that

$$y_{2n+1} = gx_{2n+1} = (1 - c_{2n})fx_{2n} + c_{2n}T_1x_{2n},$$

$$y_{2n+2} = fx_{2n+2} = (1 - c_{2n+1})gx_{2n+1} + c_{2n+1}T_2x_{2n+1},$$

and

$$c_n = \frac{n + 1}{2n + 1}.$$  

If $x_0 = \frac{1}{2}$, then with the help of equations (2.6) and (2.7), we obtain the sequence $\{x_n\}$, where

$$x_1 = g^{-1}[(1 - c_0)fx_0 + c_0T_1x_0]$$

$$= g^{-1}[(1 - 1)f\left(\frac{1}{2}\right) + \frac{1}{2}] = \left(\frac{1}{8}\right)^3,$$

$$x_2 = f^{-1}[(1 - c_1)gx_1 + c_1T_2x_1]$$

$$= f^{-1}\left[(1 - \frac{2}{3})\frac{1}{8} + \frac{2}{3}\left(\frac{1}{8}\right)^{2/3}\right]$$

$$= \left(\frac{17}{3.128}\right)^2,$$

$$x_3 = g^{-1}[(1 - c_2)fx_2 + c_2T_1x_2]$$

$$= \left[\frac{17}{3.5.128}\left(2 + \frac{17}{512}\right)\right]^3,$$

$$x_4 = f^{-1}[(1 - c_3)gx_3 + c_3T_2x_3]$$

$$= \left[\frac{17}{3.5.7.128}\left(2 + \frac{17}{512}\right)\left(3 + \frac{17}{3.5.128}\left(2 + \frac{17}{128}\right)\right)\right]^2,$$

and so on. Then

$$y_1 = gx_1 = \frac{1}{3},$$

$$y_2 = fx_2 = \frac{17}{3.144},$$

$$y_3 = gx_3 = \left(\frac{17}{3.5.128}\right)\left(2 + \frac{17}{512}\right),$$

$$y_4 = fx_4 = \frac{17}{3.5.7.128}\left(2 + \frac{17}{512}\right)\left(3 + \frac{17}{3.5.128}\left(2 + \frac{17}{128}\right)\right).$$
and so on. It is evident that \( y_n \to 0 \) as \( n \to \infty \).

We note that \( T_1, T_2, f \) and \( g \) are continuous and satisfy all the conditions of Theorem 2.1 with \( 0 < q = \frac{1}{2} < 1 \). Indeed we have

\[
||T_1x - T_2y|| = \frac{1}{4}||x - y\frac{2}{3}|| \\
\leq (||x^{1/2}|| + ||y^{1/3}||)(||x^{1/2} - y^{1/3}||) \\
\leq \frac{||x^{1/2} - y^{1/3}||}{2} \\
\leq \frac{||fx - gy||}{2}.
\]

Further, 0 is the common fixed point of \( T_1, T_2, f \) and \( g \).

### 3 Fixed Point Theorems with Asymptotic Regularity Condition

Let \( \mathbb{R}^+ \) denote the set of nonnegative real numbers, \( W : \mathbb{R}^+ \to \mathbb{R}^+ \) be a continuous function such that \( 0 < W(t) < t \) for all \( t \in \mathbb{R}^+ \) and let \( T_1, T_2, f \) and \( g \) be selfmaps on a metric space \((X,d)\). For a point \( x_0 \in X \), if there exists a sequence \( \{y_n\} \) in \( X \) such that

\[
y_{2n} = T_1x_{2n} = gx_{2n+1}, \\
y_{2n+1} = T_2x_{2n+1} = fx_{2n+2},
\]

for \( n = 0, 1, 2, \ldots \), then \( O(T_1, T_2, f, g, x_0) = \{y_n : n = 1, 2, \ldots\} \) is called the orbit of \((T_1, T_2, f, g)\) at \( x_0 \). \( T_1 \) and \( T_2 \) are said to be orbitally continuous at \( x_0 \) if and only if they are continuous on \( O(T_1, T_2, f, g, x_0) \). \( X \) is said to be orbitally complete at \( x_0 \) if and only if every Cauchy sequence in \( O(T_1, T_2, f, g, x_0) \) converges in \( X \). The pair \((T_1, T_2)\) is said to be asymptotically regular (a.r.) with respect to \((g, f)\) at \( x_0 \) if there exists a sequence \( \{y_n\} \) in \( X \) such that

\[
y_{2n} = T_1x_{2n} = gx_{2n+1}, \\
y_{2n+1} = T_2x_{2n+1} = fx_{2n+2},
\]

for \( n = 0, 1, 2, \ldots \) and \( d(y_n, y_{n+1}) \to 0 \) as \( n \to \infty \).

Zeqing Liu et al. [4] proved the following theorem:

**Theorem 3.1** Let \( f, g \) and \( h \) be selfmaps on a metric space \((X,d)\) and let \( fh = hf \) or \( gh = hg \). Suppose that there exists a point \( x_0 \in X \) such that \((f,g)\) is a.r. with
respect to $h$ at $x_0$, $X$ is orbitally complete at $x_0$, and $h$ is orbitally continuous at $x_0$. If
\[ d(fx, gy) \leq M(x, y) - W(M(x, y)) \] (3.1)
holds for all $x, y \in X$, then $f, g$ and $h$ have a unique common fixed point in $X$, where
\[ M(x, y) = \max\{d(hx, hy), d(hx, fx), d(hy, gy), d(hx, gy), d(hy, fx)\}. \]

### 4 Main Result

Now we present our second main theorem:

**Theorem 4.1** Let $T_1, T_2, f$ and $g$ be selfmaps on a metric space $(X, d)$ and let $T_1f = fT_1$ and $T_2g = gT_2$. Suppose that there exists a point $x_0 \in X$ such that $(T_1, T_2)$ is a.r. with respect to $(g, f)$ at $x_0$, $X$ is orbitally complete at $x_0$, and $g, f$ are orbitally continuous at $x_0$. If
\[ d(T_1x, T_2y) \leq M'(x, y) - W(M'(x, y)) \] (4.1)
holds for all $x, y \in X$, then $T_1, T_2, f$ and $g$ have a unique common fixed point in $X$, where
\[ M'(x, y) = \max\{d(fx, gy), d(fx, T_1x), d(fx, T_2y), d(T_1x, gy), d(gy, T_2y)\}. \]

**Proof.** Since $(T_1, T_2)$ is a.r. with respect to $(g, f)$ at $x_0$, there exists a sequence $\{y_n\}$ in $X$ such that
\[
\begin{align*}
y_{2n} &= T_1x_{2n} = gx_{2n+1}, \\
y_{2n+1} &= T_2x_{2n+1} = fx_{2n+2},
\end{align*}
\]
for $n = 0, 1, 2, \ldots$ and
\[ \lim_{n \to \infty} d(y_n, y_{n+1}) = 0. \] (4.2)

In order to show that $\{y_n\}$ is a Cauchy sequence, it is sufficient to show that $\{y_{2n}\}$ is a Cauchy sequence. Suppose that the result is not true. Then there will be a positive number $\epsilon$ such that for each even integer $2k$, there are even integers $2m(k)$ and $2n(k)$ such that $2m(k) > 2n(k) > 2k$ and
\[ d(y_{2m(k)}, y_{2n(k)}) > \epsilon. \] (4.3)

For each integer $2k$, let $2m(k)$ be the least even integer exceeding $2n(k)$ and satisfying (4.3) so that
\[ d(y_{2m(k)-2}, y_{2n(k)}) \leq \epsilon. \] (4.4)
Then for each even integer $2k$,

$$d(y_{2m(k)}, y_{2n(k)}) \leq d(y_{2m(k)-2}, y_{2n(k)}) + d(y_{2m(k)-2}, y_{2m(k)-1}) + d(y_{2m(k)-1}, y_{2m(k)}).$$

From (4.2), (4.3), (4.4) and the above inequality we have,

$$\lim_{n \to \infty} d(y_{2m(k)}, y_{2n(k)}) \leq \epsilon.$$  

(4.5)

Using the triangular inequality and putting $d(y_n, y_{n+1}) = d_n$, we obtain

$$|d(y_{2m(k)+1}, y_{2n(k)}) - d(y_{2m(k)}, y_{2n(k)})| \leq d_{2m(k)},$$
$$|d(y_{2m(k)+1}, y_{2n(k)+1}) - d(y_{2m(k)+1}, y_{2n(k)})| \leq d_{2n(k)},$$
$$|d(y_{2m(k)+2}, y_{2n(k)+1}) - d(y_{2m(k)+1}, y_{2n(k)+1})| \leq d_{2m(k)+1},$$
$$|d(y_{2m(k)+2}, y_{2n(k)}) - d(y_{2m(k)+1}, y_{2n(k)})| \leq d_{2m(k)+1},$$

and from (4.2), (4.5) and the above inequalities, we have

$$\lim_{k \to \infty} d(y_{2m(k)+1}, y_{2n(k)}) \leq \epsilon,$$
$$\lim_{k \to \infty} d(y_{2m(k)+1}, y_{2n(k)+1}) \leq \epsilon,$$
$$\lim_{k \to \infty} d(y_{2m(k)+2}, y_{2n(k)+1}) \leq \epsilon,$$
$$\lim_{k \to \infty} d(y_{2m(k)+2}, y_{2n(k)}) \leq \epsilon.$$
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It follows from (4.1) that

\[ d(y_{2n(k)+1}, y_{2m(k)+2}) = d(T_2x_{2n(k)+1}, T_1x_{2m(k)+2}) \]
\[ = d(T_2x_{2n(k)+1}, T_1x_{2m(k)+2}) \]
\[ \leq \max \left\{ d(fx_{2m(k)+2}, gev_{x_{2n(k)+1}}), d(fx_{2m(k)+2}, T_1x_{2m(k)+2}), \right. \]
\[ \left. d(fx_{2m(k)+2}, T_2x_{2n(k)+1}), d(T_1x_{2m(k)+2}, gev_{x_{2n(k)+1}}), \right. \]
\[ \left. d(T_1x_{2m(k)+2}, T_2x_{2n(k)+1}) \right\} \]
\[ - W \left( d(fx_{2m(k)+2}, gev_{x_{2n(k)+1}}), d(fx_{2m(k)+2}, T_1x_{2m(k)+2}), \right. \]
\[ \left. d(fx_{2m(k)+2}, T_2x_{2n(k)+1}), d(T_1x_{2m(k)+2}, gev_{x_{2n(k)+1}}), \right. \]
\[ \left. d(T_1x_{2m(k)+2}, T_2x_{2n(k)+1}) \right\} \]
\[ \leq \max \left\{ d(y_{2m(k)+1}, y_{2n(k)}), d(y_{2m(k)+1}, y_{2m(k)+2}), \right. \]
\[ \left. d(y_{2m(k)+1}, y_{2m(k)+1}), d(y_{2m(k)+2}, y_{2n(k)}), \right. \]
\[ \left. d(y_{2n(k)}, y_{2n(k)+1}) \right\} \]
\[ - W \left( d(y_{2m(k)+1}, y_{2n(k)}), d(y_{2m(k)+1}, y_{2m(k)+2}), \right. \]
\[ \left. d(y_{2m(k)+1}, y_{2n(k)+1}), d(y_{2m(k)+2}, y_{2n(k)}), \right. \]
\[ \left. d(y_{2n(k)}, y_{2n(k)+1}) \right\}. \]

As \( k \) tends to infinity, we have

\[ \epsilon \leq \max \{ \epsilon, 0, \epsilon, 0 \} - W(\max \{ \epsilon, 0, \epsilon, 0 \}), \]

or

\[ \epsilon \leq \epsilon - W(\epsilon). \]

That is, \( W(\epsilon) \leq 0 \), which implies \( \epsilon = 0 \), a contradiction. Hence \( \{ y_{2n} \} \) is a Cauchy sequence.

Since \( X \) is \( (T_1, T_2, f, g) \) orbitally complete at \( x_0 \), there exists a point \( z \) such that \( y_n \to z \) as \( n \) tends to \( \infty \).

Now applying (4.1) to \( d(gx_{2n+1}, T_2z) \) and \( d(T_1z, fx_{2n+2}) \) and letting \( n \) tend to infinity, we have

\[ d(gx_{2n+1}, T_2z) = d(T_1x_{2n}, T_2z) \]
\[ \leq \max \left\{ d(fx_{2n}, gz), d(fx_{2n}, T_1x_{2n}), d(fx_{2n}, T_2z), \right. \]
\[ \left. d(T_1x_{2n}, gz), d(gz, T_2z) \right\} \]
\[ - W(\max \{ d(fx_{2n}, gz), d(fx_{2n}, T_1x_{2n}), d(fx_{2n}, T_2z), \right. \]
\[ \left. d(T_1x_{2n}, gz), d(gz, T_2z) \right\}). \]
Letting $n \to \infty$, we have

$$d(fz, z) \leq d(fz, z) - W(d(fz, z),$$

which implies $fz = z$. 

and

$$d(T_1 z, f(x_{2n+2}) = d(T_1 z, T_2 x_{2n+1})$$

$$\leq \max\{d(fz, z), d(fz, T_1 z), d(fz, z), d(T_1 z, z), d(z, z)\}$$

or

$$d(T_1 z, z) \leq \max\{d(fz, z), d(fz, T_1 z), d(fz, z), d(T_1 z, z), d(z, z)\}$$

$$- W(\max\{d(fz, z), d(fz, T_1 z), d(fz, z), d(T_1 z, z), d(z, z)\})$$

Since $T_1 f = fT_1$, we have $T_1 f(x_{2n+2} = fT_1 x_{2n+2} \to fz$.

Again, since $T_1$ is orbitally continuous at $x_0$ we have by (4.1)

$$d(T_1 f(x_{2n+2}, f(x_{2n+2}) = d(T_1 f(x_{2n+2}, T_2 x_{2n+1})$$

$$\leq \max\{d(f x_{2n+2}, g x_{2n+1}), d(f x_{2n+2}, T_1 f(x_{2n+2}), d(f x_{2n+2}, T_2 x_{2n+1})\}$$

$$- W(\max\{d(f x_{2n+2}, g x_{2n+1}), d(f x_{2n+2}, T_1 f(x_{2n+2}), d(f x_{2n+2}, T_2 x_{2n+1})\})$$

$$\leq \max\{d(f, z), d(f, z), d(f, z), d(f, z), d(z, z)\}$$

$$- W(\max\{d(f, z), d(f, z), d(f, z), d(f, z), d(z, z)\}).$$
Similarly if \( T_2g = gT_2 \), then \( T_2gx_{2n+1} = gT_2x_{2n+1} \rightarrow gz \).

Since \( T_2 \) is orbitally continuous on \( x_0 \), then we see on applying (4.1) to \( d(gx_{2n+1}, T_2gx_{2n+1}) \) and letting \( n \rightarrow \infty \), we obtain \( gz = z \).

In equation (4.6), if we put \( z = gz \), then we get \( T_2z = z \). Again in (4.7), if we put \( z = fz \), then we get \( T_1z = z \). Thus \( z \) is a common fixed point of \( T_1, T_2, f \) and \( g \). Uniqueness of \( z \) is obvious. This completes the proof of the theorem. \( \square \)

**Remark 4.2** When \( f = g \), Theorem 3.1 strictly extends Theorem 2.1 of Liu Zeqing et al. [4]. Furthermore, taking \( W(t) = (1 - r)t : r \in (0,1) \), we obtain Theorem 1 of Sastry et al. [3].

**References**


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