Generalized Projection Methods for Nonlinear Mappings

Preedaporn Kanjanasamranwong, Sarapee Chairat and Siwaporn Saewan

Department of Mathematics and Statistics, Faculty of Science Thaksin University, Thailand
e-mail: ypreedaporn@hotmail.com (P. Kanjanasamranwong) sarapee@tsu.ac.th (S. Chairat) siwaporn@scholar.tsu.ac.th (S. Saewan)

Abstract: We present a new hybrid iterative process for finding an element in the solution of variational inequality problem and the fixed point set of relatively nonexpansive multi-valued mapping in Banach spaces. This theorem improve and extend some recent results.

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1 Introduction

Let C be a nonempty closed and convex subset of a Banach space E with dual $E^*$. A mapping $A : C \rightarrow E^*$ is said to be:

(1) monotone if

\[ \langle x - y, Ax - Ay \rangle \geq 0 \]

for all $x, y \in C$;

(2) $\alpha$-inverse-strongly monotone if there exists a constant $\alpha > 0$ such that

\[ \langle x - y, Ax - Ay \rangle \geq \alpha \| Ax - Ay \|^2 \]
for all \( x, y \in C \).

If \( A : C \to E^* \) is \( \alpha \)-inverse-strongly monotone, then it is Lipschitz continuous with constant \( \frac{1}{\alpha} \), that is,

\[
\|Ax - Ay\| = \frac{1}{\alpha}\|x - y\|
\]

for all \( x, y \in C \). Clearly, the class of monotone mappings include the class of \( \alpha \)-inverse-strongly monotone mappings.

The class of inverse-strongly monotone have been studied by many authors to approximate a common fixed point (see [1, 2] for more details).

The variational inequality problem for an operator \( A \) is to find \( \hat{z} \in C \) such that

\[
\langle y - \hat{z}, A\hat{z} \rangle \geq 0, \quad \forall y \in C. \tag{1.1}
\]

The set of solution of (1.1) is denoted by \( VI(A, C) \).

Let \( E \) be a Banach space with the dual space \( E^* \). The normalized duality mapping from \( E \) to \( 2E^* \) defined by

\[
J(x) = \{ f^* \in E^* : \langle x, f^* \rangle = \|x\|^2, \|f^*\| = \|x\| \}.
\]

If \( E \) is a Hilbert space, then \( J = I \), where \( I \) is the identity mapping.

Consider the functional \( \phi : E \times E \to \mathbb{R} \) defined by

\[
\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2 \tag{1.2}
\]

for all \( x, y \in E \), where \( J \) is the normalized duality mapping. It is obvious from the definition of function \( \phi \) that

\[
(\|y\| - \|x\|)^2 \leq \phi(y, x) \leq (\|y\| + \|x\|)^2 \tag{1.3}
\]

for all \( x, y \in E \) and

\[
\phi(x, y) = \phi(z, y) + \phi(x, z) + 2(z - x, Jy - Jz) \tag{1.4}
\]

for all \( x, y, z \in E \).

If \( E \) is a Hilbert space, then \( \phi(y, x) = \|y - x\|^2 \).

**Remark 1.1.** If \( E \) is a reflexive, strictly convex and smooth Banach space, then, for any \( x, y \in E \), \( \phi(x, y) = 0 \) if and only if \( x = y \). It is sufficient to show that, if \( \phi(x, y) = 0 \), then \( x = y \). From (1.2), we have \( \|x\| = \|y\| \). This implies that \( \langle x, Jy \rangle = \|x\|^2 = \|y\|^2 \). From the definition of \( J \), one has \( Jx = Jy \). Therefore, we have \( x = y \) (see [3, 4] for more details).

The generalized projection \( \Pi_C : E \to C \) is a mapping that assigns to an arbitrary point \( x \in E \) the minimum point of the functional \( \phi(x, y) \), that is, \( \Pi_C x = \bar{x} \), where \( \bar{x} \) is the solution to the minimization problem:

\[
\phi(\bar{x}, x) = \inf_{y \in C} \phi(y, x). \tag{1.5}
\]
The existence and uniqueness of the operator $\Pi_C$ follows from the properties of the functional $\phi(y, x)$ and the strict monotonicity of the mapping $J$ (see, for example, [8]). If $E$ is a Hilbert space, then $\Pi_C$ becomes the metric projection of $E$ onto $C$. If $E$ is a smooth, strictly convex and reflexive Banach space, then $\Pi_C$ is a closed relatively quasi-nonexpansive mapping from $E$ onto $C$ with $F(\Pi_C) = C$ ([8]).

Let $C$ be a nonempty closed and convex subset of a real Banach space $E$. A mapping $T : C \to C$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in C$. A point $x \in C$ is a fixed point of $T$ provided $Tx = x$. Denoted by $F(T)$ the set of fixed points of $T$, that is, $F(T) = \{x \in C : Tx = x\}$. A point $p$ in $C$ is called an asymptotic fixed point of $T$ ([8]) if $C$ contains a sequence $\{x_n\}$ which converges weakly to $p$ such that $\lim_{n \to \infty} \|x_n - Tx_n\| = 0$. The asymptotic fixed point set of $T$ is denoted by $\bar{F}(T)$.

Recall that a mapping $T : C \to C$ is closed if, for each $\{x_n\}$ in $C$, $x_n \to x$ and $Tx_n \to y$ imply that $Tx = y$.

A mapping $T : C \to C$ is called relatively nonexpansive ([8]) if

1. $F(T)$ is nonempty;
2. $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(T)$;
3. $\bar{F}(T) = F(T)$.

Let $CB(C)$ and $N(C)$ denoted the family of nonempty closed bounded subsets of $C$ and nonempty subsets, respectively. The Hausdorff metric on $CB(C)$ is defined by

$$H(A_1, A_2) = \max \{ \sup_{x \in A_1} d(x, A_2), \sup_{y \in A_2} d(y, A_1) \}$$

for all $A_1, A_2 \in CB(C)$, where $d(x, A_1) = \inf \{\|x - y\| : y \in A_1\}, x \in C$. A multi-valued mapping $T : C \to CB(C)$ is said to be nonexpansive if

$$H(T(x), T(y)) \leq \|x - y\|$$

for all $x, y \in C$. A multi-valued mapping $T : C \to CB(C)$ is said to be quasi-nonexpansive if $F(T)$ is nonempty and

$$H(T(x), T(p)) \leq \|x - p\|$$

for all $x \in C$ and all $p \in F(T)$. An element $p \in C$ is called a fixed point of $T : C \to N(C)$ if $p \in T(p)$. The set of fixed point of $T$ is denoted by $F(T)$.

Let $C$ be a nonempty closed convex subset of a smooth Banach space $E$. A point $p \in C$ is called an asymptotic fixed point of a multi-valued mapping $T : C \to N(C)$ if there exists a sequence $\{x_n\}$ in $C$ which converges weakly to $p$ and $\lim_{n \to \infty} d(x_n, T(x_n)) = 0$.

A multi-valued mapping $T : C \to N(C)$ is said to be relatively nonexpansive if
(R1) \( F(T) \) is nonempty;
(R2) \( \phi(p, z) \leq \phi(p, x) \) for all \( x \in C, z \in T(x) \) and, \( p \in F(T) \);
(R3) \( \hat{F}(T) = F(T) \).

Sastry and Babu [12] proved that the Mann and Ishikawa iteration schemes for a multi-valued mapping \( T \) with a fixed point \( p \) converge to a fixed point \( q \) of \( T \) under certain conditions. Panyanak [13] extended the result of Sastry and Babu to uniformly convex Banach spaces. In 2009, Shahzad and Zegeye [14] proved strong convergence theorems for the Ishikawa iteration scheme involving quasi-nonexpansive multi-valued mappings in uniformly convex Banach spaces.

In 2014, Homaeipour and Razani [15] introduced an iterative sequence for two relatively nonexpansive multi-valued mappings in Banach spaces. Further, they proved that \( \{x_n\} \) converges strongly to \( \Pi_{F(T) \cap EP(f)}(x_0) \) under appropriate condition.

From the recent works, in this paper, we obtain new hybrid iterative scheme to find a common element of the fixed point set of relatively nonexpansive multi-valued mapping and the solution set of variational inequality problem in Banach spaces.

2 Preliminaries

Let \( E \) be a Banach space and let \( U = \{x \in E : \|x\| = 1\} \) be the unit sphere of \( E \).

1. A Banach space \( E \) is said to be \textit{strictly convex} if \( \|\frac{x+y}{2}\| < 1 \) for all \( x, y \in E \) with \( \|x\| = \|y\| = 1 \) and \( x \neq y \).
2. A Banach space \( E \) is said to be \textit{smooth} if the limit \( \lim_{t \to 0} \frac{\|x+ty\| - \|x\|}{t} \) exists for each \( x, y \in U \).
3. The norm of \( E \) is said to be \textit{Fréchet differentiable} if, for each \( x \in U \), the limit is attained uniformly for \( y \in U \).
4. A Banach space \( E \) is said to be \textit{uniformly smooth} if the limit exists uniformly in \( x, y \in U \).
5. The \textit{modulus of convexity} of \( E \) is the function \( \delta : [0, 2] \to [0, 1] \) defined by
   \[
   \delta(\varepsilon) = \inf \left\{ 1 - \frac{\|x+y\|}{2} : x, y \in E, \|x\| = \|y\| = 1, \|x-y\| \geq \varepsilon \right\}.
   \]
6. \( E \) is said to be \textit{uniformly convex} if \( \delta(\varepsilon) > 0 \) for all \( \varepsilon \in (0, 2] \).

\begin{remark}
Let \( E \) be a Banach space. Then the following are well known (see [3] for more details):
\begin{enumerate}
\item If \( E \) is an arbitrary Banach space, then \( J \) is monotone and bounded.
\item If \( E \) is a strictly convex, then \( J \) is strictly monotone.
\item If \( E \) is a smooth, then \( J \) is single valued and semi-continuous.
\end{enumerate}
\end{remark}
(4) If $E$ is uniformly smooth, then $J$ is uniformly norm-to-norm continuous on each bounded subset of $E$.

(5) If $E$ is reflexive, smooth and strictly convex, then the normalized duality mapping $J$ is single valued, one-to-one and onto.

(6) If $E$ is reflexive, smooth and strictly convex, then $J^{-1}$ is also single valued, one-to-one, onto and it is the duality mapping from $E^*$ into $E$.

(7) If $E$ is uniformly smooth, then $E$ is smooth and reflexive.

(8) $E$ is uniformly smooth if and only if $E^*$ is uniformly convex.

We also need the following lemmas for the proof of our main results.

**Lemma 2.2** ([16]). Let $E$ be a strictly convex and smooth Banach space. Then, for all $x, y \in E$, $\phi(x, y) = 0$ if and only if $x = y$.

**Lemma 2.3** ([6]). Let $C$ be a nonempty closed convex subset of a smooth Banach space $E$ and $x \in E$. Then $x_0 = \Pi_C x$ if and only if
\[
\langle x_0 - y, Jx - Jx_0 \rangle \geq 0
\]
for all $y \in C$.

**Lemma 2.4** ([6]). Let $E$ be a reflexive, strictly convex and smooth Banach space, $C$ be a nonempty closed convex subset of $E$ and $x \in E$. Then
\[
\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x)
\]
for all $y \in C$.

**Lemma 2.5** ([15]). Let $E$ be a smooth and strictly convex Banach space and $C$ be a nonempty closed convex subset of $E$. Suppose $T : C \to 2^{C}$ is a relatively nonexpansive multi-valued mapping. Then $F(T)$ is a closed convex subset of $C$.

**Lemma 2.6** ([17]). Let $E$ be a uniformly convex and smooth Banach space and $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, \infty) \to [0, \infty]$ with $g(0) = 0$ such that
\[
g(\|y - z\|) \leq \phi(y, z)
\]
for all $y, z \in B_r(0) = \{\|x\| \leq r\}$.

**Lemma 2.7** ([18]). Let $E$ be a uniformly convex Banach space and $B_r(0)$ be a closed ball of $E$. Then there exists a strictly increasing, continuous and convex function $h : [0, \infty) \to [0, \infty)$ with $h(0) = 0$ such that
\[
\|\lambda x + \mu y + \gamma z\|^2 \leq \lambda\|x\|^2 + \mu\|y\|^2 + \gamma\|z\|^2 - \lambda \mu h(\|x - y\|)
\]
for all $x, y, z \in B_r(0)$ and $\lambda, \mu, \gamma \in [0, 1]$ with $\lambda + \mu + \gamma = 1$. 
Lemma 2.8 ([19]). Let $C$ be a nonempty closed convex subset of a uniformly smooth, strictly convex real Banach space $E$ and $A : C \to E^*$ be a continuous monotone mapping. For any $r > 0$, define a mapping $F_r : E \to C$ as follows:

$$F_r x = \{ z \in C : \langle y - z, Az \rangle + \frac{1}{r} \langle y - z, Jz - Jx \rangle \geq 0, \forall y \in C \}$$

for all $x \in C$. Then the following hold:

1. $F_r$ is a single-valued mapping;
2. $F(F_r) = VI(A,C)$;
3. $VI(A,C)$ is a closed and convex subset of $C$;
4. $\phi(q,F_r x) + \phi(F_r x, x) \leq \phi(q,x)$ for all $q \in F(F_r)$.

3 Main Results

In this section, we prove some new convergence theorems for finding a common solution of the set of common fixed points of relatively nonexpansive multi-valued mappings and the set of the variational inequality problems in a real uniformly smooth and uniformly convex Banach space.

Theorem 3.1. Let $C$ be a nonempty closed and convex subset of a uniformly convex and uniformly smooth Banach space $E$. Let $T : C \to N(C)$ be a relatively nonexpansive multi-valued mapping and $A$ be a continuous monotone mapping of $C$ into $E^*$. Define a mapping $F_{r_n} : E \to C$ by

$$F_{r_n} x = \{ z \in C : \langle y - z, Az \rangle + \frac{1}{r_n} \langle y - z, Jz - Jx \rangle \geq 0, \forall y \in C \}.$$

Assume that $\Theta := F(T) \cap VI(A,C) \neq \emptyset$. For an initial point $x_1 \in C$, define the iterative sequence $\{x_n\}$ in $C$ as follows:

$$\begin{align*}
\left\{ \begin{array}{l}
    u_n = F_{r_n} x_n, \\
    x_{n+1} = \Pi_C J^{-1}(\alpha_n Jx_n + \beta_n Ju_n + \gamma_n Jz_n)
\end{array} \right. \quad (3.1)
\end{align*}$$

for all $n \geq 1$, where $z_n \in Tx_n$. Assume that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are the sequences in $[0,1]$ such that $\alpha_n + \beta_n + \gamma_n = 1$. Then $\{\Pi_C x_n\}$ converges strongly to a point in $\Theta$, where $\Pi_C$ is the generalized projection from $E$ onto $\Theta$.

Proof. Let $T$ be a relatively nonexpansive multi-value mapping. Since $\Theta$ is closed and convex, for any $p \in \Theta$, we have

$$\begin{align*}
\phi(p,x_{n+1}) & = \phi(p,\Pi_C J^{-1}(\alpha_n Jx_n + \beta_n Ju_n + \gamma_n Jz_n)) \\
& \leq \phi(p,J^{-1}(\alpha_n Jx_n + \beta_n Ju_n + \gamma_n Jz_n)) \\
& = \|p\|^2 - 2\alpha_n \langle p, Jx_n \rangle - 2\beta_n \langle p, Ju_n \rangle - 2\gamma_n \langle p, Jz_n \rangle \\
& \quad + \|\alpha_n Jx_n + \beta_n Ju_n + \gamma_n Jz_n\|^2 \\
& \leq \|p\|^2 - 2\alpha_n \langle p, Jx_n \rangle - 2\beta_n \langle p, Ju_n \rangle - 2\gamma_n \langle p, Jz_n \rangle
\end{align*}$$
\[ + \alpha_n \| Jx_n \|^2 + \beta_n \| Ju_n \|^2 + \gamma_n \| Jz_n \|^2 = \alpha_n \phi(p, x_n) + \beta_n \phi(p, u_n) + \gamma_n \phi(p, z_n) = \alpha_n \phi(p, x_n) + \beta_n \phi(p, x_n) + \gamma_n \phi(p, x_n) \]

\[ = \phi(p, x_n). \quad (3.2) \]

Hence \( \lim_{n \to \infty} \phi(p, x_n) \) exist. Thus \( \{ \phi(p, x_n) \} \) is bounded and, further, by (1.3), \( \{ x_n \} \) is bounded and so are \( \{ z_n \} \) and \( \{ u_n \} \). Let \( y_n = \Pi \Theta x_n \) for all \( n \geq 1 \). It follows from (3.2) that

\[ \phi(y_n, x_{n+1}) \leq \phi(y_n, x_n) \quad (3.3) \]

and so, for all \( m \geq 1 \),

\[ \phi(y_n, x_{n+m}) \leq \phi(y_n, x_n). \quad (3.4) \]

Thus it follows from Lemma 2.4 that

\[ \phi(y_{n+1}, x_{n+1}) = \phi(\Pi \Theta x_n, x_{n+1}) \leq \phi(y_n, x_{n+1}) - \phi(y_n, \Pi \Theta x_{n+1}) \]

\[ = \phi(y_n, x_{n+1}) - \phi(y_n, y_{n+1}) \]

\[ \leq \phi(y_n, x_{n+1}) \]

\[ \leq \phi(y_n, x_n). \quad (3.5) \]

Therefore \( \{ \phi(y_n, x_n) \} \) is a convergence sequence. For all \( m, n \geq 1 \) with \( n > m \), it follows from Lemma 2.6 that

\[ g(\| y_m - y_n \|) \leq \phi(y_m, y_n) \]

\[ \leq \phi(y_m, x_n) - \phi(y_n, x_n). \]

Let \( r = \sup_{n \in \mathbb{N}} \| y_n \| \). It follows from Lemma 2.6 that there exist a continuous, strictly increasing, and convex function \( g : [0, \infty) \to [0, \infty) \) with \( g(0) = 0 \) such that

\[ g(\| y_m - y_n \|) \leq \phi(y_m, y_n) \]

\[ \leq \phi(y_m, x_n) - \phi(y_n, x_n). \quad (3.6) \]

(3.7)

Thus, from the property of \( g \), we can show that \( \{ y_n \} \) is a Cauchy sequence for all \( m, n \geq 1 \). Since \( E \) is complete and \( \Theta := F(T) \cap VI(A, C) \) is closed and convex, there exist \( q \in \Theta \) such that \( \{ y_n \} \) converges strongly to a point \( q \in \Theta \), where \( y_n = \Pi \Theta x_n \). This completes the proof.

In Theorem 3.1 if \( \beta_n = 0 \), then we have the following:
Corollary 3.2. Let $C$ be a nonempty closed and convex subset of a uniformly convex and uniformly smooth Banach space $E$. Let $T : C \to N(C)$ be a relatively nonexpansive multi-valued mapping. For an initial point $x_1 \in C$, define the iterative sequence $\{x_n\}$ in $C$ as follows:

$$
\begin{align*}
\{ x_{n+1} &= \Pi_C J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)Jz_n) \} \quad (3.8)
\end{align*}
$$

for all $n \geq 1$, where $z_n \in Tx_n$. Assume that $\{\alpha_n\}$ is a sequence in $[0, 1]$. Then $\{\Pi_{F(T)}x_n\}$ converges strongly to some point of $T$. Where $\Pi_{F(T)}$ is the generalized projection from $C$ onto $F(T)$.

Let $f$ be a bifunction from $C \times C$ to $\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers. The equilibrium problem (for short, EP) is to find $x^* \in C$ such that

$$
f(x^*, y) \geq 0, \quad \forall y \in C. \quad (3.9)
$$

The set of solutions of (3.9) is denoted by $EP(f)$.

For solving the equilibrium problem for a bifunction $f : C \times C \to \mathbb{R}$, let us assume that $f$ satisfies the following conditions:

(A1) $f(x, x) = 0$ for all $x \in C$;
(A2) $f$ is monotone, i.e., $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;
(A3) for each $x, y, z \in C$,

$$
\lim_{t \downarrow 0} f(tz + (1 - t)x, y) \leq f(x, y);
$$

(A4) for each $x \in C$, $y \mapsto f(x, y)$ is convex and lower semi-continuous.

Lemma 3.3 (Blum and Oettli [20]). Let $C$ be a closed convex subset of a smooth, strictly convex and reflexive Banach space $E$, let $f$ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4), and let $r > 0$ and $x \in E$. Then, there exists $z \in C$ such that

$$
f(z, y) + \frac{1}{r}(y - z, Jz - Jx) \geq 0, \quad \forall y \in C.
$$

Lemma 3.4 (Takahashi and Zembayashi [21]). Let $C$ be a closed convex subset of a uniformly smooth, strictly convex and reflexive Banach space $E$ and let $f$ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4). For $r > 0$ and $x \in E$, define a mapping $T_r : E \to C$ as follows:

$$
T_r x = \{ z \in C : f(z, y) + \frac{1}{r}(y - z, Jz - Jx) \geq 0, \quad \forall y \in C \},
$$

for all $x \in C$. Then the following hold:

(1) $T_r$ is single-valued;
(2) $T_r$ is a firmly nonexpansive-type mapping, for all $x, y \in E$,
\[ (T_r x - T_r y, JT_r x - JT_r y) \leq (T_r x - T_r y, Jx - Jy); \]

(3) $F(T_r) = EP(f)$;

(4) $EP(f)$ is closed and convex.

**Lemma 3.5** (Takahashi and Zembayashi [21]). Let $C$ be a closed convex subset of a smooth, strictly convex, and reflexive Banach space $E$, let $f$ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) and let $r > 0$. Then, for $x \in E$ and $q \in F(T_r)$,

\[ \phi(q, T_r x) + \phi(T_r x, x) \leq \phi(q, x). \]

In Theorem 3.1 if $\langle y - z, Az \rangle = f(z, y)$, then we have the following:

**Corollary 3.6.** Let $C$ be a nonempty closed and convex subset of a uniformly convex and uniformly smooth Banach space $E$. Let $T : C \to N(C)$ be a relatively nonexpansive multi-valued mapping and Let $f$ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4). Define a mapping $F_{r_n} : E \to C$ by

\[ T_{r_n} x = \{ z \in C : f(z, y) + \frac{1}{r_n} \langle y - z, Jz - Jx \rangle \geq 0, \forall y \in C \}. \]

Assume that $\Theta := F(T) \cap EP(f) \neq \emptyset$. For an initial point $x_1 \in C$, define the iterative sequence $\{x_n\}$ in $C$ as follows:

\[
\begin{cases}
  u_n = T_{r_n} x_n, \\
  x_{n+1} = \Pi_C J^{-1} (\alpha_n Jx_n + \beta_n Ju_n + \gamma_n Jz_n)
\end{cases}
\]  \\
(3.10)

for all $n \geq 1$, where $z_n \in TX_n$. Assume that $\{ \alpha_n \}$, $\{ \beta_n \}$ and $\{ \gamma_n \}$ are the sequences in $[0, 1]$ such that $\alpha_n + \beta_n + \gamma_n = 1$. Then $\{ \Pi_\Theta x_n \}$ converges strongly to a point in $\Theta$, where $\Pi_\Theta$ is the generalized projection from $E$ onto $\Theta$.

In the following theorem, we can show that the sequence $\{x_n\}$ defined in (3.1) also converges strongly to some point of $\Theta$.

**Theorem 3.7.** Let $C$ be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space $E$. Let $T : C \to N(C)$ be a relatively nonexpansive multi-valued mapping and $A$ be a continuous monotone mapping of $C$ into $E^*$. Define a mapping $F_{r_n} : E \to C$ by

\[ F_{r_n} x = \{ z \in C : \langle y - z, Az \rangle + \frac{1}{r_n} \langle y - z, Jz - Jx \rangle \geq 0, \forall y \in C \}. \]

Assume that $\Theta := F(T) \cap VI(A, C) \neq \emptyset$. For an initial point $x_1 \in C$, define the iterative sequence $\{x_n\}$ in $C$ as follows:

\[
\begin{cases}
  u_n = F_{r_n} x_n, \\
  x_{n+1} = \Pi_C J^{-1} (\alpha_n Jx_n + \beta_n Ju_n + \gamma_n Jz_n)
\end{cases}
\]  \\
(3.11)
for all \( n \geq 1 \), where \( z_n \in Tx_n \). Assume that \( \{\alpha_n\}, \{\beta_n\} \) and \( \{\gamma_n\} \) are the sequences in \([0, 1]\) satisfying the conditions:

(a) \( \alpha_n + \beta_n + \gamma_n = 1 \);
(b) \( \liminf_{n \to \infty} \alpha_n \beta_n > 0 \), \( \liminf_{n \to \infty} \alpha_n \gamma_n > 0 \);
(c) \( \{r_n\} \subset [d, \infty) \) for some \( d > 0 \).

Then \( \{x_n\} \) converges strongly to some point of \( \Theta \).

**Proof.** As in the proof of Theorem 3.1, we have \( \{x_n\} \) and \( \{z_n\} \) are bounded. So, there exists \( r_1 = \sup_{n \geq 1} \{\|x_n\|, \|z_n\|, \|u_n\|\} \) such that \( x_n, z_n \in B_r(0) \) for all \( n \geq 1 \). Since \( E \) is a uniformly convex Banach space, \( E^* \) is a uniformly smooth Banach space. Since \( \Theta \) is nonempty, there exist \( p \in \Theta \). By Lemma 2.7, there exists a continuous, strictly increasing and convex function \( h : [0, \infty) \to [0, \infty) \) with \( h(0) = 0 \) such that

\[
\phi(p, x_{n+1}) = \phi(p, \Pi_C J^{-1}(\alpha_n Jx_n + \beta_n Ju_n + \gamma_n Jz_n)) \\
\leq \phi(p, J^{-1}(\alpha_n Jx_n + \beta_n Ju_n + \gamma_n Jz_n)) \\
= \|p\|^2 - 2\alpha_n \langle p, Jx_n \rangle - 2\beta_n \langle p, Ju_n \rangle - 2\gamma_n \langle p, Jz_n \rangle \\
+ \|\alpha_n Jx_n + \beta_n Ju_n + \gamma_n Jz_n\|^2 \\
\leq \|p\|^2 - 2\alpha_n \langle p, Jx_n \rangle - 2\beta_n \langle p, Ju_n \rangle - 2\gamma_n \langle p, Jz_n \rangle \\
+ \alpha_n \|Jx_n\|^2 + \beta_n \|Ju_n\|^2 + \gamma_n \|Jz_n\|^2 - \alpha_n \gamma_n h(\|Jx_n - Jz_n\|) \\
= \alpha_n \phi(p, x_n) + \beta_n \phi(p, u_n) + \gamma_n \phi(p, z_n) - \alpha_n \gamma_n h(\|Jx_n - Jz_n\|) \\
\leq \phi(p, x_n) - \alpha_n \gamma_n h(\|Jx_n - Jz_n\|) \\
\tag{3.12}
\]

and so

\[
\alpha_n \gamma_n h(\|Jx_n - Jz_n\|) \leq \phi(p, x_n) - \phi(p, x_{n+1}).
\]

Since \( \{\phi(p, x_n)\} \) is convergent and \( \liminf_{n \to \infty} \alpha_n \gamma_n > 0 \), we have

\[
\lim_{n \to \infty} h(\|Jx_n - Jz_n\|) = 0 \\
\tag{3.13}
\]

and so

\[
\lim_{n \to \infty} \|Jx_n - Jz_n\| = 0. \\
\tag{3.14}
\]

Since \( J^{-1} \) is uniformly norm-to-norm continuous on bounded sets, we have

\[
\lim_{n \to \infty} \|x_n - z_n\| = 0. \\
\tag{3.15}
\]

Therefore,

\[
\lim_{n \to \infty} d(x_n, Tx_n) = 0. \\
\tag{3.16}
\]

Let \( p \in \Theta \) and \( r > 0 \). Then there exists \( p + rk \in \Theta \), whenever \( \|k\| \leq 1 \). Thus, by (1.23), for any \( q \in \Theta \), we have

\[
\phi(q, x_n) = \phi(x_{n+1}, x_n) + \phi(q, x_{n+1}) + 2\langle x_{n+1} - q, Jx_n - Jx_{n+1} \rangle, \\
\tag{3.17}
\]

which implies

\[
\frac{1}{2}(\phi(q, x_n) - \phi(q, x_{n+1})) = \frac{1}{2}\phi(x_{n+1}, x_n) + \langle x_{n+1} - q, Jx_n - Jx_{n+1} \rangle. \\
\tag{3.18}
\]
Follow from (3.2), we have

\[ 0 \leq \frac{1}{2} \phi(x_{n+1}, x_n) + (x_{n+1} - q, Jx_n - Jx_{n+1}) \]  

(3.19)

and

\[ -(x_{n+1} - q, Jx_n - Jx_{n+1}) \leq \frac{1}{2} \phi(x_{n+1}, x_n). \]  

(3.20)

Since

\[ (x_{n+1} - p, Jx_n - Jx_{n+1}) = (x_{n+1} - (p + rk) + rk, Jx_n - Jx_{n+1}) \]

\[ = (x_{n+1} - (p + rk), Jx_n - Jx_{n+1}) + r(k, Jx_n - Jx_{n+1}) \]  

(3.21)

and so

\[ r(k, Jx_n - Jx_{n+1}) = (x_{n+1} - p, Jx_n - Jx_{n+1}) - (x_{n+1} - (p + rk) + rk, Jx_n - Jx_{n+1}). \]

Thus it follows from (3.19), we have

\[ - (x_{n+1} - (p + rk), Jx_n - Jx_{n+1}) \leq \frac{1}{2} \phi(x_{n+1}, x_n), \]

we obtain that

\[ r(k, Jx_n - Jx_{n+1}) \leq \frac{1}{2r} (\phi(p, x_n) - \phi(p, x_{n+1})). \]  

(3.22)

On the other hand, since \( p + rk \in \Theta \), it follows from Theorem 3.1 that

\[ \phi(p + rk, x_{n+1}) \leq \phi(p + rk, x_n). \]  

(3.23)

Since \( \|k\| \leq 1 \), we obtain

\[ \|Jx_n - Jx_{n+1}\| \leq \frac{1}{2r} (\phi(p, x_n) - \phi(p, x_{n+1})) \]  

(3.24)

and so, for all \( m, n \geq 1 \) with \( n > m \), we have

\[ \|Jx_m - Jx_n\| \leq \frac{1}{2r} \sum_{i=m}^{n-1} \|Jx_i - Jx_{i+1}\| \]

\[ \leq \frac{1}{2r} \sum_{i=m}^{n-1} (\phi(p, x_i) - \phi(p, x_{i+1})) \]

(3.25)

Since \{\phi(p, x_n)\} converges, \{Jx_n\} is a Cauchy sequence. Since \( E \) is uniformly convex and uniformly smooth and \( E^* \) is complete, \{Jx_n\} converge strongly to
some point in $E^*$. Since $E^*$ has a Fréchet differentiable norm, $J^{-1}$ is norm-to-norm continuous on $E^*$. Hence $\{x_n\}$ converges strongly to some point $x$ in $C$. Thus, from (3.16) and $T$ is a relatively nonexpansive, we have $x \in F(T)$.

Also, from (3.12) and $\lim \inf_{n \to \infty} \alpha_n \beta_n > 0$, we have

$$\lim_{n \to \infty} h(\|Jx_n - Ju_n\|) = 0$$

(3.25)

and so

$$\lim_{n \to \infty} \|Jx_n - Ju_n\| = 0.$$  

(3.26)

Since $J^{-1}$ is norm-to-norm continuous on $E^*$, it follows that

$$\lim_{n \to \infty} \|x_n - u_n\| = 0.$$  

(3.27)

Thus, from (3.26), for all $r_n > 0$, we obtain

$$\lim_{n \to \infty} \frac{\|Jx_n - Ju_n\|}{r_n} = 0.$$  

(3.28)

From (3.28), we have $Ju_n - Jx_n \to 0$. Since $A$ is monotone, we have

$$\langle v - u_n, Av \rangle \geq \langle v - u_n, Av_t - Av \rangle \geq 0$$

for all $v \in C$ and so $x \in VI(A, C)$. Therefore, $x \in F(T) \cap VI(A, C)$. This completes the proof.

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References


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