Fixed Point Theorems for $s - \alpha$ Contractions in Dislocated and $b$-Dislocated Metric Spaces

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Abstract: In this paper, we prove some unique fixed point results for quasicontraction and $T$-Hardy Rogers contraction in the setting of complete dislocated and $b$-dislocated metric spaces. Our theorems involve one and two self-mappings and extend and generalize some several known results of literature in a wider class as $b$-spaces.

Keywords: dislocated metric; $b$-dislocated metric; $s - \alpha$ quasicontraction; $T$-Hardy-Rogers contraction; common fixed point.

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1 Introduction

Fixed point theory is one of the cornerstones in the development of mathematics since it plays a basic role in applications of many branches of mathematics. The famous Banach contraction principle is one of the power tools to study in this field. In 1989, Bakhtin [1] (and also Czerwik [2]) introduced the concept of $b$-metric space and presented the contraction mapping in $b$-metric spaces that is a generalization of Banach contraction principle in metric spaces. Recently there are

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a number of generalizations of metric space. Some of them are the notions of dislocated metric spaces and $b$-dislocated metric spaces where the distance of a point in the self may not be zero, introduced and studied by Hitzler and Seda [3], Nawab Hussain et.al [4]. Also in [4] are presented some topological aspects and properties of $b$-dislocated metrics. Subsequently, several authors have studied the problem of existence and uniqueness of a fixed point for single-valued and set-valued mappings and different types of contractions in these spaces. Quasicontractions and $g$-quasicontractions in metric spaces were first studied in [2, 5]. The purpose of this paper is to present some fixed point theorems for $s-\alpha$-quasicontractions and $T$-Hardy-Rogers contractions in the context of dislocated and $b$-dislocated metric spaces. The presented theorems extend and generalize some comparable results in the literature in a larger class of spaces.

2 Preliminaries

Definition 2.1. [6] Let $X$ be a nonempty set and a mapping $d_l : X \times X \to [0, \infty)$ is called a dislocated metric (or simply $d_l$-metric) if the following conditions hold for any $x, y, z \in X$:

1. If $d_l (x, y) = 0$, then $x = y$;
2. $d_l (x, y) = d_l (y, x)$;
3. $d_l (x, y) \leq d_l (x, z) + d_l (z, y)$.

The pair $(X, d_l)$ is called a dislocated metric space (or $d_l$-metric space for short). Note that when $x = y$, $d_l (x, y)$ may not be 0.

Example 2.2. If $X = R$, then $d (x, y) = |x| + |y|$ defines a dislocated metric on $X$.

Definition 2.3. [6] A sequence $(x_n)$ in $d_l$-metric space $(X, d_l)$ is called: (1) a Cauchy sequence if, for given $\varepsilon > 0$, there exists $n_0 \in N$ such that for all $m, n \geq n_0$, we have $d_l (x_m, x_n) < \varepsilon$ or $\lim_{n,m \to \infty} d_l (x_n, x_m) = 0$, (2) convergent with respect to $d_l$ if there exists $x \in X$ such that $d_l (x_n, x) \to 0$ as $n \to \infty$. In this case, $x$ is called the limit of $(x_n)$ and we write $x_n \to x$.

A $d_l$-metric space $X$ is called complete if every Cauchy sequence in $X$ converges to a point in $X$.

Definition 2.4. [7] Let $X$ be a nonempty set and a mapping $b_d : X \times X \to [0, \infty)$ is called a $b$-dislocated metric (or simply $b_d$-dislocated metric) if the following conditions hold for any $x, y, z \in X$ and $s \geq 1$:

1. If $b_d (x, y) = 0$, then $x = y$;
2. $b_d (x, y) = b_d (y, x)$;
Definition 2.9. Let \((X, b_d)\) be a \(b\)-dislocated metric space. And the class of \(b\)-dislocated metric space is larger than that of dislocated metric spaces, since a \(b\)-dislocated metric is a dislocated metric when \(s = 1\).

In [7] it was showed that each \(b_d\)-metric on \(X\) generates a topology \(\tau_{b_d}\) whose base is the family of open \(b_d\)-balls \(B_{b_d}(x, \varepsilon) = \{y \in X : b_d(x, y) < \varepsilon\}\).

Also in [7] there are presented some topological properties of \(b_d\)-metric spaces.

**Definition 2.5.** Let \((X, b_d)\) be a \(b_d\)-metric space, and \(\{x_n\}\) be a sequence of points in \(X\). A point \(x \in X\) is said to be the limit of the sequence \(\{x_n\}\) if \(\lim_{n \to \infty} b_d(x_n, x) = 0\) and we say that the sequence \(\{x_n\}\) is \(b_d\)-convergent to \(x\) and denote it by \(x_n \to x\) as \(n \to \infty\).

The limit of a \(b_d\)-convergent sequence in a \(b_d\)-metric space is unique [7, Proposition 1.27].

**Definition 2.6.** A sequence \(\{x_n\}\) in a \(b_d\)-metric space \((X, b_d)\) is called a \(b_d\)-Cauchy sequence iff, given \(\varepsilon > 0\), there exists \(n_0 \in \mathbb{N}\) such that for all \(n, m > n_0\), we have \(b_d(x_n, x_m) < \varepsilon\) or \(\lim_{n,m \to \infty} b_d(x_n, x_m) = 0\). Every \(b_d\)-convergent sequence in a \(b_d\)-metric space is a \(b_d\)-Cauchy sequence.

**Remark 2.7.** The sequence \(\{x_n\}\) in a \(b_d\)-metric space \((X, b_d)\) is called a \(b_d\)-Cauchy sequence iff \(\lim_{n,m \to \infty} b_d(x_n, x_{n+p}) = 0\) for all \(p \in \mathbb{N}^+\).

**Definition 2.8.** A \(b_d\)-metric space \((X, b_d)\) is called complete if every \(b_d\)-Cauchy sequence in \(X\) is \(b_d\)-convergent.

In general a \(b_d\)-metric is not continuous, as in Example 1.31 in [7] showed.

**Definition 2.9.** [3] Let \((X, d)\) be a metric space. A mapping \(T : X \to X\) is said to be sequentially convergent (respectively, subsequentially convergent) if, for a sequence \(\{x_n\}\) in \(X\) for which \(\{Tx_n\}\) is convergent, \(\{x_n\}\) is also convergent (respectively, \(\{x_n\}\) has a convergent subsequence).

**Lemma 2.10.** Let \((X, b_d)\) be a \(b\)-dislocated metric space with parameters \(\geq 1\). Suppose that \(\{x_n\}\) and \(\{y_n\}\) are \(b_d\)-convergent to \(x, y \in X\), respectively. Then we have

\[
\frac{1}{s^2} b_d(x, y) \leq \lim_{n \to \infty} \inf b_d(x_n, y_n) \leq \lim_{n \to \infty} \sup b_d(x_n, y_n) \leq s^2 b_d(x, y)
\]

In particular, if \(b_d(x, y) = 0\), then we have \(\lim_{n \to \infty} b_d(x_n, y_n) = 0 = b_d(x, y)\).

Moreover, for each \(z \in X\), we have

\[
\frac{1}{s} b_d(x, z) \leq \lim_{n \to \infty} \inf b_d(x_n, z) \leq \lim_{n \to \infty} \sup b_d(x_n, z) \leq sb_d(x, z)
\]

In particular, if \(b_d(x, z) = 0\), then we have \(\lim_{n \to \infty} b_d(x_n, z) = 0 = b_d(x, z)\).
Some examples in the literature shows that in general a $b$-dislocated metric is not continuous.

**Example 2.11.** Let $X = \mathbb{R}^+ \cup \{0\}$ and any constant $\alpha > 0$. Define the function $d_1 : X \times X \to [0, \infty)$ by $d_1 (x, y) = \alpha (x + y)$. Then, the pair $(X, d_1)$ is a dislocated metric space.

**Example 2.12.** If $X = \mathbb{R}^+ \cup \{0\}$, then $b_d (x, y) = (x + y)^2$ defines a $b$-dislocated metric on $X$ with parameter $s = 2$.

### 3 Main Results

Based in the definition of quasi-contraction from Ciric we introduced the following definition in the setting of $b$-dislocated metric space.

**Definition 3.1.** Let $(X, b_d)$ be complete $b$-dislocated metric space with parameter $s \geq 1$. If $T : X \to X$ is a self mapping that satisfies:

$$s^2 b_d (Tx, Ty) \leq \alpha \max \{ b_d (x, y), b_d (x, Tx), b_d (y, Ty), b_d (x, Tx), b_d (y, Tx) \}$$

for all $x, y \in X$ and $\alpha \in [0, \frac{1}{2})$. Then $T$ is called a $s - \alpha$ quasi-contraction.

In this section, we obtain the existence of some fixed point theorems for $s - \alpha$ quasi-contraction mappings in a class of space which is larger than metric and $b$-metric spaces.

**Theorem 3.2.** Let $(X, b_d)$ be complete $b$-dislocated metric space with parameter $s \geq 1$. If $T : X \to X$ is a self mapping that is a $s - \alpha$ quasi-contraction, then $T$ has a unique fixed point in $X$.

**Proof.** Let $x_0$ be an arbitrary point in $X$. Define the iterative sequence $\{ x_n \}$ as follows: $x_1 = T(x_0), x_2 = T(x_1), ...., x_{n+1} = T(x_n), ....$

If assume that $x_{n+1} = x_n$ for some $n \in N$, then we have $x_n = x_{n+1} = T(x_n)$, so $x_n$ is a fixed point of $T$ and the proof is completed. From now on we will assume that for each $n \in N, x_{n+1} \neq x_n$. By condition (3.1) we have:

$$s^2 b_d (x_n, x_{n+1}) + s^2 b_d (Tx_n, Tx_{n+1}) \leq \alpha \max \{ b_d (x_n, x_{n+1}), b_d (x_{n+1}, Tx_n), b_d (x_n, x_{n+1}), b_d (x_n, x_{n+1}) \}$$

$$= \alpha \max \{ b_d (x_n, x_{n+1}), b_d (x_n, x_{n+1}), b_d (x_n, x_{n+1}) \}$$

$$\leq \alpha \max \{ b_d (x_n, x_{n+1}), b_d (x_n, x_{n+1}), b_d (x_n, x_{n+1}) \}.$$ 

If $b_d (x_{n-1}, x_n) \leq b_d (x_n, x_{n+1})$ for some $n \in N$, then from the above inequality (3.2) we have

$$b_d (x_n, x_{n+1}) \leq \frac{2\alpha}{s} b_d (x_n, x_{n+1}) \text{ a contradiction since } \frac{2\alpha}{s} < 1.$$
Hence for all \( n \in \mathbb{N} \), \( b_d(x_n, x_{n+1}) \leq b_d(x_{n-1}, x_n) \) and also by the above inequality (3.2) we get

\[
b_d(x_n, x_{n+1}) \leq \frac{2\alpha}{s} b_d(x_{n-1}, x_n).
\] (3.3)

Similarly by the contractive condition of theorem we have:

\[
b_d(x_{n-1}, x_n) \leq \frac{2\alpha}{s} b_d(x_{n-2}, x_{n-1}).
\] (3.4)

Generally from (3.3) and (3.4) we have for all \( n \geq 2 \)

\[
b_d(x_n, x_{n+1}) \leq cb_d(x_{n-1}, x_n) \leq \ldots \leq c^n b_d(x_0, x_1)
\] (3.5)

where \( c = \frac{2\alpha}{s} \) and \( 0 \leq c < 1 \). Taking limit as \( n \to \infty \) in (3.5) we have

\[
b_d(x_n, x_{n+1}) \to 0.
\] (3.6)

Now, we prove that \( \{x_n\} \) is a \( b_d \)-Cauchy sequence, and to do this let be \( m, n > 0 \) with \( m > n \), and using definition 2.4 (3) we have

\[
b_d(x_n, x_m) \leq s [b_d(x_n, x_{n+1}) + b_d(x_{n+1}, x_{n+2}) + \ldots + b_d(x_{m-1}, x_m)]
\leq sb_d(x_n, x_{n+1}) + s^2 b_d(x_{n+1}, x_{n+2}) + \ldots
\leq s c \sum_{i=n}^{m-1} b_d(x_i, x_{i+1})
\leq \frac{sc^n}{1-sc} b_d(x_0, x_1).
\]

On taking limit for \( n, m \to \infty \) we have \( b_d(x_n, x_m) \to 0 \) as \( cs < 1 \). Therefore \( \{x_n\} \) is a \( b_d \)-Cauchy sequence in complete \( b \)-dislocated metric space \((X, b_d)\). So there is some \( u \in X \) such that \( \{x_n\} \) dislocated converges to \( u \).

If \( T \) is a continuous mapping we get:

\[
T(u) = T(\lim_{n \to \infty} x_n) = \lim_{n \to \infty} T(x_n) = \lim_{n \to \infty} (x_{n+1}) = u.
\]

Thus \( u \) is a fixed point of \( T \).

If the self-map \( T \) is not continuous then, we consider,

\[
s^2 b_d(x_{n+1}, Tu) = s^2 b_d(Tx_n, Tu)
\leq \alpha \max \{b_d(x_n, u), b_d(x_n, Tx_n), b_d(u, Tu)\}
\leq \alpha \max \{b_d(x_n, u), b_d(x_n, x_{n+1}), b_d(u, Tu), b_d(u, x_{n+1})\}.
\] (3.7)

Using Lemma 2.10, result 3.6 and taking the upper limit in (3.7) follows that

\[
s^2 \frac{1}{s} b_d(u, Tu) \leq \alpha b_d(u, Tu).
\]

From this inequality have \( b_d(u, Tu) \leq \alpha b_d(u, Tu) \) and this implies \( Tu = u \) since \( \alpha < \frac{1}{2} \). Hence \( u \) is a fixed point of \( T \).
Example 3.3. Let $b$ be a complete metric space. Therefore, $Tu = u$ and $Tv = v$. Using condition (3.1), we have:

$$s^2b_d (u, v) = s^2b_d (Tu, Tv)$$

$$\leq \alpha \max \{b_d (u, v), b_d (u, Tu), b_d (v, Tv), b_d (u, TV), b_d (Tu, Tu)\}$$

$$= \alpha \max \{b_d (u, v), b_d (u, u), b_d (v, v), b_d (u, u), b_d (v, v)\}$$

$$\leq 2\alpha s^2b_d (u, v).$$

So $b_d (u, v) \leq cb_d (u, v)$ where $c = \frac{2\alpha}{s^2}$, since $0 \leq c < 1$ we get $b_d (u, v) = 0$. Therefore, $b_d (u, v) = b_d (v, u) = 0$ implies $u = v$. Hence the fixed point is unique.

Example 3.5. Let $X = [0, 1]$ and $b_d (x, y) = (x + y)^2$ for all $x, y \in X$. It is clear that $b_d$ is a $b$-dislocated metric on $X$ with parameter $s = 2$ and $(X, b_d)$ is complete. Also $b_d$ is not a dislocated metric or a $b$-metric or a metric on $X$. Define the self-mapping $T : X \rightarrow X$ by $Tx = \frac{x}{2}$. For $x, y \in [0, 1]$, we have

$$s^2b_d (Tx, Ty) = 2^2 \left(\frac{x}{2} + \frac{y}{2}\right)^2$$

$$= 4 (x + y)^2$$

$$= 4 \frac{b_d (x, y)}{2s}\;\text{for all}\;x, y \in X$$

$$\leq \alpha \max \{b_d (x, y), b_d (Tx, Ty), b_d (y, Ty), b_d (x, Tx), b_d (x, Ty), b_d (y, Tx)\}$$

for $\frac{x}{2} \leq \alpha < \frac{1}{2}$. Clearly $x = 0$ is a unique fixed point of $T$.

If we take parameter $s = 1$ in Theorem 3.2, we obtain the following corollary in the setting of dislocated metric spaces.

Corollary 3.4. Let $(X, d_l)$ be a complete dislocated metric space. If $T : X \rightarrow X$ is a self mapping that satisfies:

$$d_l (Tx, Ty) \leq \alpha \max \{d_l (x, y), d_l (x, Tx), d_l (y, Ty), d_l (x, Ty), d_l (y, Tx)\}$$

for all $x, y \in X$ and $\alpha \in \left[0, \frac{1}{2}\right)$. Then $T$ has a unique fixed point in $X$.

The following example shows that Theorem 3.4 is a proper generalization.

Example 3.5. Let $X = [0, 1]$ and $d_l : X^2 \rightarrow R^+$ by $d_l (x, y) = (x + y)$ for all $x, y \in X$. It is clear that $d_l$ is a $b$-dislocated metric on $X$ and $(X, d_l)$ is complete. Also $d_l$ is not a metric on $X$. Define the self-mapping $T : X \rightarrow X$ by

$$Tx = \begin{cases} \frac{x}{8}, & 0 \leq x < 1 \\ \frac{1}{16}, & x = 1. \end{cases}$$

We have the following cases.

Case 1. For $x = y = 0$ have $d_l (Tx, Ty) = d_l (0, 0) = 0 \leq d_l (0, 0)$.

Case 2. If $1 > x = y > 0$, then

$$d_l (Tx, Ty) = d_l \left(\frac{x}{8}, \frac{x}{8}\right) = \frac{2x}{8} = \frac{1}{8} 2x = \frac{1}{8} d_l (x, y) < d_l (x, y).$$
Case 3. If \( x = 1, y = \frac{1}{2} \), then

\[
d_l(Tx, Ty) = d_l\left(T(1), T\left(\frac{1}{2}\right)\right) = d_l\left(\frac{1}{16}, \frac{1}{16}\right) = \frac{1}{8} < \frac{3}{2} = d_l(x, y).
\]

Case 4. If \( 0 < x = 1 < y \), then

\[
d_l(Tx, T1) = d_l\left(\frac{x}{8}, \frac{1}{16}\right) = \frac{x}{8} + \frac{1}{8} < \frac{x}{8} + \frac{1}{8} = \frac{1}{8} (x + 1) = \frac{1}{8} d_l(x, 1) < d_l(x, 1).
\]

Case 5. If \( 1 > x > y > 0 \), then

\[
d_l(Tx, Ty) = d_l\left(\frac{x}{8}, \frac{y}{8}\right) = \frac{x}{8} + \frac{y}{8} = \frac{1}{8} (x + y) < d_l(x, y).
\]

Thus all conditions of theorem are satisfied and \( T \) has \( x = 0 \) a unique fixed point in \( X \).

Therefore, we note that for \( x = 1, y = \frac{99}{100} \) in the usual metric space \((X, d)\) where \( d(x, y) = |x - y| \) in the special case of Banach contraction, we have

\[
d\left(T(1), T\left(\frac{99}{100}\right)\right) = d\left(\frac{1}{16}, \frac{99}{800}\right) = \frac{49}{800} \leq \alpha \frac{1}{100} = d\left(1, \frac{99}{100}\right).
\]

This inequality implies that \( \alpha \geq \frac{49}{8} \) and this means that the contractive condition is not true in the usual metric on \( X \). Also, this example shows that the contractive condition of theorem failed in the setting of \( b \)-metric space \((X, d)\) where \( d(x, y) = |x - y|^2 \).

In the following we are giving a result in which \( T \) is not continuous in \( X \), but \( T^p \) is continuous for some positive integer \( p \).

**Theorem 3.6.** Let \((X, b_d)\) be a complete \( b \)-dislocated metric space with parameter \( s \geq 1 \) and \( T : X \to X \) a self-mapping satisfying the condition \((3.1)\)

\[
s^2 b_d(Tx, Ty) \leq \alpha \max \{b_d(x, y), b_d(x, Tx), b_d(y, Ty), b_d(x, Ty), b_d(y, Tx)\}
\]

for all \( x, y \in X \) and \( \alpha \in \left[0, \frac{1}{2}\right) \). If for some positive integer \( p \), \( T^p \) is continuous, then \( T \) has a unique fixed point in \( X \).

**Proof.** Similarly as in above theorem we can construct a sequence \( \{x_n\} \) and conclude that the sequence \( \{x_n\} \) converges to some point \( u \in X \). Thus its subsequence \( \{x_{n_k}\} \) \( (n_k = k_p) \) converges to \( u \). Also, we have

\[
T^p(u) = T^p\left(\lim_{k \to \infty} x_{n_k}\right) = \lim_{k \to \infty} (T^p(x_{n_k})) = \lim_{k \to \infty} x_{n_{k+1}} = u.
\]

Therefore, \( u \) is a fixed point of \( T^p \). Further we have to show that \( u \) is a fixed point of \( T \).
Let \( m \) be the smallest positive integer such that \( T^m u = u \). If suppose that \( m > 1 \) we consider:

\[
s^2 b_d(u, T u) = s^2 b_d(T^m u, T u) = s^2 b_d(T^{m-1} u, T^m u) \\
\leq \alpha \max \{ b_d(T^{m-1} u, u), b_d(T^m u, T u) \} \\
= \alpha \max \{ b_d(T^{m-1} u, u), b_d(T^m u, T u) \} \\
\leq \alpha \max \{ s b_d(T^{m-1} u, u) + b_d(u, T u), 2 s b_d(T^{m-1} u, u) \} \\
\leq 2 \alpha s b_d(T^{m-1} u, u) \Rightarrow b_d(u, T u) < b_d(T^{m-1} u, u)
\]

Again from the condition of theorem, have

\[
s^2 b_d(T^{m-1} u, u) = s^2 b_d(T^{m-1} u, T^m u) = s^2 b_d(T T^{m-2} u, T^m u) \\
\leq \alpha \max \{ b_d(T^{m-2} u, u), b_d(T^{m-2} u, T T^{m-2} u), b_d(u, T^m u), b_d(T^{m-2} u, T^m u), b_d(T^{m-2} u, T u), b_d(T^{m-2} u, T^m u), b_d(u, T T^{m-2} u) \} \\
= \alpha \max \{ b_d(T^{m-2} u, u), b_d(T^{m-2} u, T u), b_d(T^{m-2} u, T^m u), b_d(u, T^{m-1} u) \} \\
\leq \alpha \max \{ s b_d(T^{m-2} u, u) + b_d(u, T^{m-1} u), 2 \alpha s b_d(T^{m-2} u, u), b_d(T^{m-2} u, T u), b_d(u, T^{m-1} u) \} \\
\leq 2 \alpha s b_d(T^{m-2} u, u) \Rightarrow b_d(T^{m-1} u, u) < b_d(T^{m-2} u, u).
\]

In general using this process inductively, we get

\[ b_d(u, T u) < b_d(T^{m-1} u, u) < b_d(T^{m-2} u, T^{m-1} u) < \cdots < b_d(u, T^2 u) \]

As a result we have, \( b_d(u, T u) < b_d(u, T u) \) that is a contradiction. Hence \( T u = u \) and \( u \) is a fixed point of \( T \).

Clearly the uniqueness of fixed point follows as in above theorem. \( \square \)

**Theorem 3.7.** Let \((X, b_d)\) be a complete \( b \)-dislocated metric space with parameter \( s \geq 1 \) and \( T : X \to X \) a self-mapping such that for some positive integer \( m \), \( T \) satisfies the following condition (3.1):

\[
s^2 b_d(T^m x, T^m y) \leq \alpha \max \{ b_d(x, y), b_d(x, T^m x), b_d(y, T^m y), b_d(x, T^m y), b_d(y, T^m x) \} \\
\]

for all \( x, y \in X \) and \( \alpha \in [0, \frac{1}{2}] \). If \( T^m \) is continuous, then \( T \) has a unique fixed point in \( X \).

**Proof.** If we set \( F = T^m \), then from Theorem 3.2 \( F \) has a unique fixed point. We call it \( u \). Then \( T^m u = u \) and this implies,

\[ T^{m+1} u = T^m (T u) = T (T^m u) = T u. \]

From this \( T u \) is a fixed point of \( T^m \). Since \( T^m \) has a unique fixed point, then \( T u = u \).

**Uniqueness.** From condition of theorem, we get the uniqueness of fixed point \( u \). \( \square \)
If we take the parameter \( s = 1 \) in Theorem 3.6 and 3.7, we reduce the following corollaries in the setting of dislocated metric spaces.

**Corollary 3.8.** Let \((X, d_l)\) be a complete dislocated metric space and \( T : X \to X \) a self- mapping satisfying the condition:

\[
d_l(Tx, Ty) \leq \alpha \max \{ d_l(x, y), d_l(x, Tx), d_l(y, Ty), d_l(x, Ty), d_l(y, Tx) \}
\]

for all \( x, y \in X \) and \( \alpha \in [0, \frac{1}{2}) \). If for some positive integer \( p \) \( T^p \) is continuous, then \( T \) has a unique fixed point in \( X \).

**Corollary 3.9.** Let \((X, d_l)\) be a complete dislocated metric space and \( T : X \to X \) a self- mapping such that for some positive integer \( m \), \( T \) satisfies the following condition:

\[
d_l(T^m x, T^m y) \leq \alpha \max \{ d_l(x, y), d_l(x, T^m x), d_l(y, T^m y), d_l(x, T^m y), d_l(y, T^m x) \}
\]

for all \( x, y \in X \) and \( \alpha \in [0, \frac{1}{2}) \). If \( T^m \) is continuous, then \( T \) has a unique fixed point in \( X \).

Further we prove existence of unique fixed point for a mapping that is said to be a \( T \)-Hardy-Rogers contraction in the setup of a \( b \)-dislocated metric space.

**Theorem 3.10.** Let \((X, b_d)\) be a complete \( b \)-dislocated metric space with parameter \( s \geq 1 \) and \( T, f : X \to X \) are such that \( T \) is one-to-one, continuous and the contractive condition,

\[
s^2 b_d(Tfx, Tfy) \leq Ab_d(Tx, Ty) + Bb_d(Tx, Tf x) + Cb_d(Ty, Tfy) + Db_d(Tx, Tfy) + Eb_d(Ty, Tfx)
\]

holds for all \( x, y \in X \), where the constants \( A, B, C, D, E \) are non negative and \( 0 \leq A + B + C + 2D + 2E < 1 \). Then we have the following:

1. For every \( x_0 \in X \) the sequence \( \{T^nx_0\} \) is convergent;
2. If \( T \) is subsequentially convergent, then \( f \) has a unique fixed point in \( X \);
3. If \( T \) is sequentially convergent, then for each \( x_0 \in X \) the sequence \( \{f^n x_0\} \) converges to the fixed point of \( f \).

**Proof.** Let \( x_0 \in X \) be an arbitrary point. We define the sequence \( \{x_n\} \) by \( x_{n+1} = f x_n = f^{n+1}x_0, n = 0, 1, 2, \ldots \). If there exists \( n \in N \) such that \( x_n = x_{n+1} \) then we get \( x_n = x_{n+1} = f x_n \) thus \( f \) has a fixed point and the proof is completed. Thus we can suppose \( x_n \neq x_{n+1} \) for all \( n \in N \).

From contractive condition of theorem, we have:

\[
s^2 b_d(Tx_{n+1}, Tx_n) = s^2 b_d(Tfx_n, Tfx_{n-1}) \leq Ab_d(Tx_n, Tx_{n-1}) + Bb_d(Tx_n, Tfx_n) + Cb_d(Tx_{n-1}, Tfx_{n-1}) + Db_d(Tx_n, Tfx_n) + Eb_d(Tx_{n-1}, Tfx_{n})
\]

Further we prove existence of unique fixed point for a mapping that is said to be a \( T \)-Hardy-Rogers contraction in the setup of a \( b \)-dislocated metric space.

Let \((X, d_l)\) be a complete dislocated metric space and \( T : X \to X \) are such that \( T \) is one-to-one, continuous and the contractive condition,

\[
s^2 b_d(Tfx, Tfy) \leq Ab_d(Tx, Ty) + Bb_d(Tx, Tf x) + Cb_d(Ty, Tfy) + Db_d(Tx, Tfy) + Eb_d(Ty, Tfx)
\]

holds for all \( x, y \in X \), where the constants \( A, B, C, D, E \) are non negative and \( 0 \leq A + B + C + 2D + 2E < 1 \). Then we have the following:

1. For every \( x_0 \in X \) the sequence \( \{T^nx_0\} \) is convergent;
2. If \( T \) is subsequentially convergent, then \( f \) has a unique fixed point in \( X \);
3. If \( T \) is sequentially convergent, then for each \( x_0 \in X \) the sequence \( \{f^n x_0\} \) converges to the fixed point of \( f \).

**Proof.** Let \( x_0 \in X \) be an arbitrary point. We define the sequence \( \{x_n\} \) by \( x_{n+1} = f x_n = f^{n+1}x_0, n = 0, 1, 2, \ldots \). If there exists \( n \in N \) such that \( x_n = x_{n+1} \) then we get \( x_n = x_{n+1} = f x_n \) thus \( f \) has a fixed point and the proof is completed. Thus we can suppose \( x_n \neq x_{n+1} \) for all \( n \in N \).

From contractive condition of theorem, we have:

\[
s^2 b_d(Tx_{n+1}, Tx_n) = s^2 b_d(Tfx_n, Tfx_{n-1}) \leq Ab_d(Tx_n, Tx_{n-1}) + Bb_d(Tx_n, Tfx_n) + Cb_d(Tx_{n-1}, Tfx_{n-1}) + Db_d(Tx_n, Tfx_n) + Eb_d(Tx_{n-1}, Tfx_{n})
\]

Further we prove existence of unique fixed point for a mapping that is said to be a \( T \)-Hardy-Rogers contraction in the setup of a \( b \)-dislocated metric space.
Hence
\[ b_d(T_{x_{n+1}}, T_{x_n}) \]
\[ \leq \frac{1}{s} \left[ sAb_d(T_{x_n}, T_{x_{n-1}}) + sBb_d(T_{x_n}, T_{x_{n+1}}) + sCb_d(T_{x_{n-1}}, T_{x_n}) + 2sDb_d(T_{x_n}, T_{x_{n-1}}) + sE[b_d(T_{x_{n-1}}, T_{x_n}) + b_d(T_{x_n}, T_{x_{n+1}})] \right] \]
\[ = \frac{1}{s} \left[ Ab_d(T_{x_n}, T_{x_{n-1}}) + Bb_d(T_{x_n}, T_{x_{n+1}}) + Cb_d(T_{x_{n-1}}, T_{x_n}) + 2Db_d(T_{x_n}, T_{x_{n-1}}) + E[b_d(T_{x_{n-1}}, T_{x_n}) + b_d(T_{x_n}, T_{x_{n+1}})] \right]. \]

If \( b_d(T_{x_n}, T_{x_{n-1}}) \leq b_d(T_{x_{n+1}}, T_{x_n}) \) for some \( n \in N \), then from the above inequality (3.8) we have
\[ b_d(T_{x_{n+1}}, T_{x_n}) \leq \frac{1}{s} [A + B + C + 2D + 2E] b_d(T_{x_n}, T_{x_{n-1}}). \]  
(3.9)

Also in a same way we have
\[ b_d(T_{x_n}, T_{x_{n-1}}) \leq \frac{1}{s} [A + B + C + 2D + 2E] b_d(T_{x_{n-1}}, T_{x_{n-2}}). \]  
(3.10)

Now from (3.9) and (3.10) we have
\[ b_d(T_{x_{n+1}}, T_{x_n}) \leq k b_d(T_{x_n}, T_{x_{n-1}}) \leq \ldots \leq k^n b_d(T_{x_1}, T_{x_0}) \]  
(3.11)

where \( k = \frac{A + B + C + 2D + 2E}{s} \), so \( 0 < k < 1 \) as \( s \geq 1 \). Taking in limit in inequality (3.11) we get
\[ b_d(T_{x_{n+1}}, T_{x_n}) \to 0. \]  
(3.12)

Let we prove that \( \{T_{x_n}\} \) is a \( b_d \)-Cauchy sequence.

By the triangle inequality, for \( m \geq n \) we have:
\[ b_d(T_{x_n}, T_{x_m}) \leq s [b_d(T_{x_n}, T_{x_{n+1}}) + b_d(T_{x_{n+1}}, T_{x_m})] \]
\[ \leq s b_d(T_{x_n}, T_{x_{n+1}}) + s^2 b_d(T_{x_{n+1}}, T_{x_{n+2}}) + s^3 b_d(T_{x_{n+2}}, T_{x_{n+3}}) + \ldots \]
\[ \leq s^k b_d(T_{x_0}, T_{x_1}) + s^k b_d(T_{x_0}, T_{x_1}) + s^k b_d(T_{x_0}, T_{x_1}) + \ldots \]
\[ = s^k b_d(T_{x_0}, T_{x_1}) \left[ 1 + s + (sk)^2 + (sk)^3 + \ldots \right] \]
\[ \leq \frac{s^k}{1-sk} b_d(T_{x_0}, T_{x_1}). \]

As \( 0 \leq sk < 1 \) letting \( n, m \to \infty \), we have \( \lim_{n,m \to \infty} b_d(T_{x_n}, T_{x_m}) = 0 \). So \( \{T_{x_n}\} \) is a \( b_d \)-Cauchy sequence in \( X \).
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Since $(X, b_d)$ is a complete $b$-dislocated metric space then $\{T^nx_0\} = \{Tf^nx_0\}$ is a $b_d$-Cauchy convergent sequence, so there exists a point $z \in X$ such that

$$\lim_{n \to \infty} T^nx_0 = z. \quad (3.13)$$

Assuming that $T$ is subsequentially convergent then $\{f^nx_0\}$ has a $b_d$-convergent subsequence. Hence there exists $u \in X$ and a subsequence $\{n_i\}$ such that $\lim_{i \to \infty} f^{n_i}x_0 = u$. Since the mapping $T$ is continuous, we obtain

$$\lim_{i \to \infty} Tf^{n_i}x_0 = Tu \quad (3.14)$$

and by $(3.13)$, $(3.14)$ we conclude that $Tu = z$.

In the contractive condition of theorem, we have

$$s^2b_d(Tfu, Tfux_n) \leq Ab_d(Tu, Tx_n) + Bb_d(Tu, Tfux_n) + Cb_d(Tx_n, Tfux_n) + Db_d(Tu, Tfux_n) + Eb_d(Tfux_n, Tfux_n).$$

Taking upper limit as $n \to \infty$ and using Lemma 2.10, and results in $(3.12)$ and $(3.13)$, we have

$$s^2 \frac{1}{2}b_d(Tfu, Tu) \leq s(b + E)b_d(Tu, Tu) \leq s(A + B + C + 2D + 2E)b_d(Tu, Tu)$$

which implies that

$$b_d(Tfu, Tu) \leq (A + B + C + 2D + 2E)b_d(Tfu, Tu).$$

Since $0 \leq A + B + C + 2D + 2E < 1$ we obtain $b_d(Tfu, Tu) = 0$ that means $Tfu = Tu$.

As $T$ is one-to-one, we get $fu = u$. Thus $f$ has a fixed point.

Also if $T$ is sequentially convergent, similarly we get that $\lim_{n \to \infty} f^nx_0 = u$ replacing $\{n\}$ with $\{n_i\}$.

**Uniqueness.** Firstly we will prove that if $u$ is a fixed point of $f$ then $b_d(Tu, Tu) = 0$. Using the contractive condition of Theorem 3.10 replacing $x = y = u$, we have

$$s^2b_d(Tu, Tu) \leq Ab_d(Tu, Tu) + Bb_d(Tu, Tu) + Cb_d(Tu, Tu) + Db_d(Tu, Tu) + Eb_d(Tu, Tu)$$

$$(A + B + C + D + E)b_d(Tu, Tu) \leq (A + B + C + 2D + 2E)b_d(Tu, Tu)$$

From this inequality we get,
\[ b_d(Tu, Tu) \leq \frac{A + B + C + 2D + 2E}{s^2} b_d(Tu, Tu) \]

This implies \( b_d(Tu, Tu) = 0 \) since \( s^2 \leq \frac{A + B + C + 2D + 2E}{s^2} < 1 \).

If we assume that \( w \) is another fixed point of \( f \), then we have,

\[
\begin{align*}
\quad s^2 b_d(Tu, Tw) & = s^2 b_d(Tfu, Tf w) \\
& \leq Ab_d(Tu, Tw) + Bb_d(Tu, Tf u) + Cb_d(Tw, Tf w) + Db_d(Tu, Tf w) + Eb_d(Tw, Tf u) \\
& = Ab_d(Tu, Tw) + Bb_d(Tu, Tu) + Cb_d(Tw, Tw) + Db_d(Tu, Tw) + Eb_d(Tw, Tu) \\
& \leq (A + B + C + 2D + 2E) b_d(Tu, Tw) .
\end{align*}
\]

The above inequality implies that \( b_d(Tu, Tw) \leq \frac{A + B + C + 2D + 2E}{s^2} b_d(Tu, Tw) \) and this implies \( b_d(Tu, Tw) = 0 \) and by property 2, we have \( Tu = Tw \). Since \( T \) is continuous and one-to-one, we get \( u = w \).

Thus the fixed point is unique.

**Example 3.11.** Let \( X = [0, \infty) \) be equipped with the \( b \)-dislocated metric \( b_d(x, y) = (x + y)^2 \) for all \( x, y \in X \), where \( s = 2 \). It is clear that \((X, b_d)\) is a complete \( b \)-dislocated metric space. Also let be the self-mappings \( T, f : X \to X \) defined by \( T(x) = \frac{x}{4}, f(x) = \frac{x}{2} \). We note, that \( f \) is a \( T \)-Hardy-Rogers contraction, also \( T \) is continuous and subsequentially convergent.

For each \( x, y \in X \), we have

\[
\begin{align*}
\quad s^2 b_d(Tfx, Tf y) & = 2^2 b_d\left(\frac{x}{18}, \frac{y}{18}\right) \\
& = \frac{4(x+y)^2}{144} \\
& \leq \frac{1}{4} b_d(Tx, Ty) \\
& \leq \frac{1}{4} \left(\frac{x}{4} + \frac{y}{2}\right)^2 \\
& = \frac{1}{4} b_d(Tx, Ty) \\
& \leq Ab_d(Tx, Ty) + Bb_d(Tx, Tfx) + Cb_d(Ty, Tf y) + Db_d(Tx, Tf y) + Eb_d(Ty, Tfx).
\end{align*}
\]

Thus \( T, f \) satisfy all the conditions of Theorem 3.10. Moreover \( 0 \) is the unique fixed point of \( f \).

As a consequence of Theorem 3.10 for taking the parameter \( s = 1 \) or the identity mapping \( Tx = x \) we can establish the following corollaries.

**Corollary 3.12.** Let \((X, d_i)\) be a complete dislocated metric space and \( T, f : X \to X \) are such that \( T \) is one-to-one, continuous and the contractive condition

\[
\begin{align*}
d_i(Tfx, Tf y) & \leq Ad_i(Tx, Ty) + Bd_i(Tx, Tfx) + Cb_i(Ty, Tf y) + Db_i(Tx, Tf y) \\
& + Ed_i(Ty, Tfx)
\end{align*}
\]


holds for all $x, y \in X$, where the constants $A, B, C, D, E$ are non negative and $0 \leq A + B + C + 2D + 2E < 1$. Then we have the following

1. For every $x_0 \in X$ the sequence $\{Tf^n x_0\}$ is convergent;
2. If $T$ is subsequentially convergent, then $f$ has a unique fixed point in $X$;
3. If $T$ is sequentially convergent, then for each $x_0 \in X$ the sequence $\{f^n x_0\}$ converges to the fixed point of $f$.

**Corollary 3.13.** Let $(X, b_d)$ be a complete $b$-dislocated metric space with parameter $s \geq 1$ and $f : X \to X$ is a self-mapping such that the contractive condition

$$s^2 b_d (f x, f y) \leq A b_d (x, y) + B b_d (x, f x) + C b_d (y, f y) + D b_d (x, f y) + E b_d (y, f x)$$

holds for all $x, y \in X$, where the constants $A, B, C, D, E$ are non negative and $0 \leq A + B + C + 2D + 2E < 1$. Then $f$ has a unique fixed point in $X$.

**Corollary 3.14.** Let $(X, b_d)$ be a complete dislocated metric space and $f : X \to X$ is a self-mapping such that the contractive condition

$$b_d (f x, f y) \leq A b_d (x, y) + B b_d (x, f x) + C b_d (y, f y) + D b_d (x, f y) + E b_d (y, f x)$$

holds for all $x, y \in X$, where the constants $A, B, C, D, E$ are non negative and $0 \leq A + B + C + 2D + 2E < 1$. Then $f$ has a unique fixed point in $X$.

**Remark 3.15.** From Theorem 3.10 and its corollaries by specifying condition on the given constants we derive as corollaries (special cases) fixed point results for $T$-Kannan contraction, $T$-Chatterjea contractions and $T$-Reich contraction in the framework of $b$-dislocated metric spaces.

**References**


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