Quasi Multiplication Modules

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Abstract: In this paper we study weakly prime submodules of a module over a commutative ring with identity. First, a number of results concerning weakly prime submodules are given. Second, for a commutative ring \( R \), we define the notion quasi multiplication module over \( R \). Also, we give a number of results concerning quasi multiplication modules.

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1 Introduction

Throughout this work \( R \) will denote a commutative ring with non-zero identity and all modules are unitary. Several authors have extended the notion of prime ideal to modules, see, for example [3], [6]. A proper ideal \( P \) of \( R \) to be weakly prime ideal if \( 0 \neq ab \in P \) implies \( a \in P \) or \( b \in P \). Weakly prime ideals in a commutative ring with non-zero identity have been introduced and studied by D. D. Anderson and E. Smith in [1]. A proper submodule \( N \) of a module \( M \) over a commutative ring \( R \) is said to be weakly prime submodule if whenever \( 0 \neq rm \in N \), for some \( r \in R \), \( m \in M \), then \( m \in N \) or \( rM \subseteq N \) (see for example [4]). Here we study some properties of weakly prime submodules. For example, we show that weakly prime submodules of secondary modules are secondary.

Now we define the concepts that we will use. A commutative ring \( R \) is called a quasi local ring if it has a unique maximal ideal \( P \), and denoted by \((R, P)\). If \( R \) is a ring and \( N \) is a submodule of an \( R \)-module \( M \), the ideal \( \{ r \in R : rM \subseteq N \} \) be denoted by \( (N :_R M) \). A proper submodule \( N \) of a module \( M \) over a commutative ring \( R \) is said to be prime submodule if whenever \( rm \in N \), for some \( r \in R \), \( m \in M \), then \( m \in N \) or \( rM \subseteq N \) (see [3]). An \( R \)-module \( M \) is called a secondary module provided that for every element \( r \in R \), the \( R \)-endomorphism of \( M \) produced by multiplication by \( r \) is either surjective or nilpotent. This implies

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that \( \sqrt{(0:_RM)} = P \) is a prime ideal of \( R \), and \( M \) is said to be \( P \)-secondary [7]. Recall that if \( R \) is an integral domain with the quotient field \( K \), the rank of an \( R \)-module \( M \) (\textit{rank} \( M \)) is defined to be the maximal number of elements of \( M \) linearly independent over \( R \). We have \textit{rank} \( M \) = the dimension of the vector space \( KM \) over \( K \), that is \textit{rank} \( M \) = rank\(_K\) \( KM \). An \( R \)-module \( M \) is called a multiplication module if for each submodule \( N \) of \( M \), \( N = I M \) for some ideal \( I \) of \( R \). In this case we can take \( I = (N :_RM) \). An \( R \)-module \( M \) is called weak multiplication if spec\( M = \emptyset \) or for every prime submodule \( N \) of \( M \), \( N = IM \) where \( I \) is an ideal of \( R \) ([2]). If \( R \) is a ring and \( M \) an \( R \)-module, the subset \( T(M) \) of \( M \) is defined by \( T(M) = \{m \in M : rm = 0 \text{ for some } 0 \neq r \in R \} \). Obviously, if \( R \) is an integral domain, then \( T(M) \) is a submodule of \( M \).

## 2 Weakly prime submodules

It is clear that every prime submodule is a weakly prime submodule. However, since 0 is always weakly prime (by definition), a weakly prime submodule need not be prime, but we have the following results:

**Proposition 2.1.** Let \( M \) be an \( R \)-module with \( T(M) = 0 \). Then every weakly prime submodule of \( M \) is prime.

**Proof.** Let \( N \) be weakly prime submodule of \( M \). Suppose that \( rm \in N \) where \( r \in R, m \in M \). If \( 0 \neq rm \in N \), \( N \) weakly prime gives \( m \in N \) or \( rM \subseteq N \). If \( rm = 0 \), then \( r = 0 \) or \( m = 0 \) since \( T(M) = 0 \). So \( N \) is prime. \( \square \)

**Proposition 2.2.** Let \( M \) be a module over a commutative ring \( R \). Then

1. If every proper submodule of \( M \) is a weakly prime submodule and \( T(M) \neq M \), then \((R, P)\) is a quasi local ring with \( P^2 = 0 \) or \( R = F_1 \times F_2 \) where \( F_1 \) and \( F_2 \) are fields.
2. If \( M \) is an \( R \)-module over a quasi local domain \((R, P)\) with \( P^2 = 0 \), then every proper submodule of \( M \) is weakly prime.

**Proof.** (1) Let \( a \in M \setminus T(M) \), \( Ann(a) = 0 \). It is easy to see that every proper submodule of \( M^* = Ra \) is a weakly prime submodule of \( M^* \), and \( M^* \cong R/(0 : a) \cong R \) as \( R \)-modules. Therefore every proper ideal of \( R \) is a weakly prime ideal, hence by [1, Theorem 8], \((R, P)\) is a quasi local ring with \( P^2 = 0 \) or \( R = F_1 \times F_2 \).

(2) Let \( N \) be a proper submodule of \( M \). Assume that \( 0 \neq rm \in N \) where \( r \in R \) and \( m \in M \). If \( r \) is a unite, then \( m \in N \). Let \( r \) is not a unite. Then \( r \in P \), and \( r^2 \in P^2 = 0 \), hence \( r = 0 \) since \( R \) is a domain, which is a contradiction. Therefore \( N \) is a weakly prime submodule of \( M \).

**Proposition 2.3.** Let \( M \) be a module over a quasi local ring \((R, P)\) with \( PM = 0 \). Then every proper submodule of \( M \) is weakly prime.

**Proof.** Let \( N \) be a proper submodule of \( M \), and \( 0 \neq rm \in N \) where \( r \in R \) and \( m \in M \). If \( r \) is a unite, then \( m \in N \). Let \( r \) is not a unite, so \( rm \in PM = 0 \), a contradiction. Hence \( N \) is weakly prime. \( \square \)
**Lemma 2.4.** Let $M$ be an $R$-module. Assume that $N$ and $K$ are submodules of $M$ such that $K \subseteq N$ with $N \neq M$. Then the following hold:
(i) If $N$ is a weakly prime submodule of $M$, then $N/K$ is a weakly prime submodule of $M/K$.
(ii) If $K$ and $N/K$ are weakly prime submodules, then $N$ is weakly prime.

Proof. (i) Let $0 \neq r(m + K) = rm + K \in N/K$ where $r \in R$ and $m \in M$. If $rm = 0$, then $r(m + K) = 0$, which is a contradiction. If $rm \neq 0$, $N$ weakly prime gives either $m \in N$ or $r \in (N :_RM)$; hence either $m + K = N/K$ or $r \in (N/K :_RM/K)$ (since we have $(N :_RM) = (N/K :_RM/K)$), as required.
(ii) Let $0 \neq rm \in N$ where $r \in R$ and $m \in M$, so $r(m + K) = rm + K \in N/K$. If $rm \in K$, then $K$ weakly prime gives either $m \in K \subseteq N$ or $r \in (K :_RM) \subseteq (N :_RM)$. So we may assume that $rm \notin K$. Then $0 \neq r(m + K) \in N/K$. Since $N/K$ is weakly prime, we get either $m \in N$ or $r \in (N/K :_R M/K) = (N :_M)$, as required.

**Theorem 2.5.** Let $M$ be a secondary $R$-module and $N$ a non-zero weakly prime $R$-submodule of $M$. Then $N$ is secondary.

Proof. Let $r \in R$. If $r^n M = 0$ for some $n \in N$, then $r^n N \subseteq r^n M = 0$, so $r$ is nilpotent on $N$. Suppose that $rM = M$; we show that $r$ divides $N$. Assume that $n \in N$. So $n = rm$ for some $m \in M$. We may assume that $0 \neq rm$. Hence $0 \neq rm \in N$, then $N$ weakly prime gives $m \in N$. Thus $rN = N$, as needed.

**Corollary 2.6.** Let $M$ be an $R$-module, $N$ a secondary $R$-submodule of $M$ and $K$ a weakly prime submodule of $M$. Then $N \cap K$ is secondary.

Proof. The proof is straightforward.

**Proposition 2.7.** Let $M$ be a module over a commutative ring $R$, and $S$ a multiplicatively closed subset of $R$. Let $N$ be a weakly prime submodule of $M$ such that $(N :_M S) = 0$, then $S^{-1}N$ is a weakly prime submodule of $S^{-1}M$.

Proof. Let $0/1 \neq r/s.m/t \in S^{-1}N$ where $r/s \in S^{-1}R$ and $m/t \in S^{-1}M$. So $0/1 \neq rm/st = n/t'$ for some $n \in N$ and $t' \in S$, hence there exists $s' \in S$ such that $0 \neq s't'rn = s's'tn \in N$ (because if $s't'rn = 0$, $rn/st = s't'rn/s't's'ts = 0/1$, a contradiction) and $s't' \notin (N :_MS)$, so $N$ weakly prime gives $0 \neq rm \in N$. Hence $m \in N$ or $r \in (N :_MS)$, thus $r/s \in S^{-1}(N :_R M) \subseteq (S^{-1}N :_{S^{-1}R} S^{-1}M)$ or $m/t \in S^{-1}N$, as needed.

**Lemma 2.8.** Let $M$ be a module over a quasi local ring $(R, P)$. Then $M_P = 0$ if and only if $M = 0$.

Proof. Let $M_P = 0$. Assume that $M \neq 0$, and $0 \neq m \in M$. Hence $m/1 \neq 0/1$, because if $m/1 = 0/1$, then there exists $t \in S$ such that $tm = 0$. So $t \in (0 :_m S) \cap S \subseteq P \cap S = 0$, a contradiction. Thus $M_P = 0$, a contradiction. So $M = 0$. The converse is clear.
Lemma 2.9. Let $M$ be a module over a quasi local ring $(R, P)$. Let $N$ be a weakly prime submodule of $R$-module $M$, then $(N :_R M)_P = (N :_{R_P} M_P)$.

Proof. Let $r/s \in (N_P :_{R_P} M_P)$ and $m \in M$. We show that $rm \in N$. We may assume that $rm \neq 0$. We have $r/s.m/1 \in S^{-1}N$; so $rm/s = n/t$ for some $t \in S$ and $n \in N$. There exists $t' \in S$ such that $t'trm = t'sn \in N$. If $t'trm = 0$, then $tt' \in (0 : rm) \cap P \subseteq \emptyset$, a contradiction. So $0 \neq tt'rm \in N$ and $tt' \notin (N : M)$; then $rm \in N$. Thus $(N_P :_{R_P} M_P) \subseteq (N :_R M)_P$. Clearly, $(N :_R M)_P \subseteq (N :_{R_P} M_P)$, so the proof is complete.

Theorem 2.10. Let $M$ be a module over a quasi local ring $(R, P)$. Then there exists a one to one correspondence between the weakly prime submodules of $M$ and the weakly prime submodules of $R_P$-module $M_P$.

Proof. Let $K$ be a weakly prime submodule of $M_P$. So $K = N_P$ for some submodule $N$ of $M$. We show that $N$ is weakly prime submodule of $M$. Let $0 \neq rm \in N$, so $0/1 \neq rm/1 \in N_P$ (if $rm/1 = 0/1$, then $srm = 0$ for some $s \in S$, $s \in (0 : rm) \cap P \subseteq \emptyset$, a contradiction). Hence $r/1 \in (N_P :_{R_P} M_P) \subseteq (N :_P M)_P$ by Lemma 2.9 or $m/1 \in N_P$ since $N_P$ is weakly prime. Thus $r \in (N : M)$ or $m \in N$, as required. Let $N$ be a weakly prime submodule of $M$, then by Proposition 2.7, $N_P$ is weakly prime submodule of $M_P$.

3 Quasi multiplication modules

An $R$-module $M$ is called quasi multiplication module if for every weakly prime submodule $N$ of $M$, we have $N = IM$, where $I$ is an ideal of $R$. One can easily show that if $M$ is a quasi multiplication module, then $N = (N : M)M$ for every weakly prime submodule $N$ of $M$.

Clearly, every multiplication module is quasi multiplication and every quasi multiplication is weak multiplication.

Lemma 3.1. Let $M$ be weak multiplication $R$-module with $T(M) = 0$. Then $M$ is quasi multiplication $R$-module.

Proof. It is clear by Proposition 2.1.

As seen in [2], $Q$ is a weak multiplication $Z$-module which is not multiplication. Since $T(Q) = 0$, by Lemma 3.1, $Q$ is quasi multiplication $Z$-module, so quasi multiplication modules need not be multiplication module.

Proposition 3.2. Let $M$ be quasi multiplication $R$-module and $K$ a weakly prime submodule of $M$, then $M/K$ is quasi multiplication $R$-module.

Proof. Let $L$ be weakly prime submodule of $M/K$, so $L = N/K$ for some weakly prime submodule $N$ of $M$ by Lemma 2.4. Therefore, $N = IM$ for some ideal $I$ of $R$ since $M$ is quasi multiplication. Thus $N/K = I(M/K)$, as required.
**Theorem 3.3.** Let $M$ be a module over a quasi local ring $(R, P)$. Then $M$ is quasi multiplication $R$-module if and only if $M_P$ is quasi multiplication $R_P$-module.

*Proof.* Let $M$ be a quasi multiplication and $K$ be a weakly prime submodule of $M$. Hence $K = N_P$ for some weakly prime submodule of $M$ by Theorem 2.10. So $N = IM$ for some ideal $I$ of $R$ since $M$ is quasi multiplication. Therefore, $K = N_P = (IM)_P = I_P M_P$, as required. Conversely, let $M_P$ is quasi multiplication module and $N$ a weakly prime submodule of $M$. So $N_P$ is weakly prime submodule of $M_P$ by Theorem 2.10. Hence $N_P = JM_P$ for some ideal $J$ of $R_P$. Thus $N_P = I_P M_P = (IM)_P$, then $(N/IM)_P = 0$, so $N/IM = 0$ by Lemma 2.8. Therefore, $M$ is quasi multiplication.

**Proposition 3.4.** Let $M$ be quasi multiplication module over a quasi local domain $(R, P)$ with $P^2 = 0$. Then $M$ is multiplication.

*Proof.* This follows from Proposition 2.2.

**Corollary 3.5.** Every finitely generated quasi multiplication module is a multiplication module.

*Proof.* The proof follows from [2, Theorem 2.7] and the fact that every quasi multiplication module is weak multiplication module.

**Corollary 3.6.** Let $M$ be a quasi multiplication module over an integral domain. Then:

(i) If $M$ is a non-zero torsion-free, then $\text{rank} M = 1$.

(ii) If $M$ is a torsion module, then $\text{rank} M = 0$.

(iii) $M$ is either torsion or torsion free.

*Proof.* Since every quasi multiplication module is weak multiplication, so by [2, Proposition 2.4] the proof is hold.

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**References**


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