Γ-Group Congruences on E-Inversive Γ-Semigroups

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Abstract: In this paper, we give a characterization and some properties of E-inversive Γ-semigroup. Moreover, we also introduce a Γ-group congruence on any E-inversive Γ-semigroup and give its characterizations. Our main results improve and extend many results obtained by Seth [1].

Keywords: E-inversive Γ-semigroup, Γ-group congruence.

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1 Introduction

The characterization of Γ-semigroup has been studied by Sen and Saha [3], they gave some characterizations of orthodox Γ-semigroups and extended different results of orthodox semigroups to orthodox Γ-semigroups. They also studied some properties of orthodox Γ-semigroups in terms of (α, β)-inverses and regular Γ-semigroups. In 1992, Seth [1] gave the sufficient condition of being Γ-group congruences and the least Γ-group congruence on regular Γ-semigroups. In 2005, Chattopadhyay [4] introduced the concept of right (left) orthodox Γ-semigroup and gave some interesting results of this kind of Γ-semigroup. In this paper, we extend some results of E-inversive semigroup as in [5] to E-inversive Γ-semigroup.

Sen and Saha [3] defined the concepts of Γ-semigroup and regular Γ-semigroup as follows: For two non-empty sets S and Γ, S is said to be a Γ-semigroup if for all \(a, b, c \in S\) and \(\alpha, \beta \in \Gamma\), (i) \(a\alpha b \in S\) and (ii) \((a\alpha b)\beta c = a\alpha (b\beta c)\). A Γ-semigroup S is called a regular Γ-semigroup if for any \(a \in S\) there exist \(a' \in S, \alpha, \beta \in \Gamma\) such that \(a = a\alpha a'\beta a\). An element \(a' \in S\) is called an \((\alpha, \beta)\)-inverse of an element \(a \in S\) if \(a = a\alpha a'\beta a\) and...
In this case, \( a' = a'\beta a\alpha a' \). If \( S \) is regular \( \Gamma \)-semigroup then \( V_\alpha^\beta(a) \neq \emptyset \) for some \( \alpha, \beta \in \Gamma \). An element \( e \in S \) is called an \( \alpha \)-idempotent, where \( \alpha \in \Gamma \), if \( eae = e \). We denote the set of all \( \alpha \)-idempotents of \( S \) by \( E_\alpha \). Now, for any \( a \in S, \alpha, \beta \in \Gamma \) if \( a' \in V_\alpha^\beta(a) \) then \( aoa' \) is \( \beta \)-idempotent and \( a'\beta a \) is \( \alpha \)-idempotent. A non-empty set \( H \) of \( \Gamma \)-semigroup \( S \) is said to be a \( \Gamma \)-subsemigroup of \( S \) if \( HGH \subseteq H \). In 2005, Siripitukdet and Sattayaporn [5] showed that every regular, orthodox and inverse semigroups are \( E \)-inversive semigroup.

A \( \Gamma \)-semigroup \( S \) is said to be an \( E \)-inversive \( \Gamma \)-semigroup if for all \( a \in S \) there exist \( x \in S, \alpha, \beta \in \Gamma \) such that \( aax \) is \( \beta \)-idempotent. In this research, for \( \alpha, \beta \in \Gamma \) we define weak \((\alpha, \beta)\)-inverse of an element \( a \in S \) as follows:

\[
W_\alpha^\beta(a) := \left\{ x \in S \mid x = x\beta a\alpha x \right\}
\]

the set of all weak \((\alpha, \beta)\)-inverses of an element \( a \). In this paper, we replace regular \( \Gamma \)-semigroup in [1] by \( E \)-inversive \( \Gamma \)-semigroup and replace the set of all \((\alpha, \beta)\)-inverses \( V_\alpha^\beta(a) \) by the set of all weak \((\alpha, \beta)\)-inverses \( W_\alpha^\beta(a) \) of an \( E \)-inversive \( \Gamma \)-semigroup.

**Example 1.1.** Let \( Q^* \) be the set of all non-zero rational numbers and \( \Gamma \) be the set of all positive integers \((\mathbb{Z}^+)\). For \( a, b \in Q^* \) and \( \alpha \in \Gamma \), we define, \( a\alpha b = |a|\alpha b \). We will show that \( Q^* \) is \( \Gamma \)-semigroup.

Let \( \frac{p}{q} \in Q^*, p \neq 0, q \neq 0 \) and \( |p|, |q| \in \Gamma \). Then \( E_{|p|} = \left\{ -\frac{1}{p}, \frac{1}{p} \right\} \) and \( E_{|q|} = \left\{ -\frac{1}{q}, \frac{1}{q} \right\} \). Hence \( \frac{1}{|p|} \in Q^* \) and \( \frac{p}{q\left|\frac{1}{|q|}\right|} = \frac{|p|}{|q|} \frac{1}{|p|} = 1 \in E_{|p|} \).

Therefore \( Q^* \) is \( E \)-inversive \( \Gamma \)-semigroup and \( \frac{1}{|p|} \in W_{|q|}^1(\frac{p}{q}) \) where \( |p|, |q|, 1 \in \Gamma \).

2 Some Auxiliary Results

In this section, we give some conditions and some results of \( E \)-inversive \( \Gamma \)-semigroups.

**Proposition 2.1.** \( S \) is an \( E \)-inversive \( \Gamma \)-semigroup if and only if \( W_\alpha^\beta(a) \neq \emptyset \) for all \( a \in S \) and for some \( \alpha, \beta \in \Gamma \).

**Proof.** Suppose that \( S \) is an \( E \)-inversive \( \Gamma \)-semigroup and \( a \in S \). Then there exist \( x \in S, \alpha, \beta \in \Gamma \) such that \( aax \in E_\beta \). Thus

\[
(axx)\beta(aax) = aax
\]

\[
(x\beta aax)\beta a(x\beta aax) = x\beta(aax\beta aax\beta aax) = x\beta aax.
\]
Therefore $x \beta a \alpha x \in W_\alpha^\beta(a)$.

Now, let $a \in S$ and $x \in W_\alpha^\beta(a)$ for some $\alpha, \beta \in \Gamma$. Then $x = x \beta a \alpha x$ and $a \alpha x = (a \alpha x) \beta(a \alpha x)$, hence $a \alpha x \in E_\beta$ and so $S$ is an $E$-inversive $\Gamma$-semigroup.

\begin{proof}
 Let $S$ be regular $\Gamma$-semigroup and $a \in S$. Then there exist $x \in S$ and $\alpha, \beta \in \Gamma$ such that $V_\alpha^\beta(a) \neq \emptyset$. Note that $V_\alpha^\beta(a) \subseteq W_\alpha^\beta(a)$, we have $W_\alpha^\beta(a) \neq \emptyset$. The result is obtained by Theorem 2.1.
\end{proof}

\begin{Theorem}
 If $S$ is a $\Gamma$-group if and only if for all $\alpha, \beta \in \Gamma$, $e \alpha f = f a e = f$ and $e \beta f = f \beta e = e$ for any $e \in E_\alpha$ and $f \in E_\beta$.

The following definition is needed for our consideration.

\begin{Definition}
 Let $S$ be a $\Gamma$-semigroup and $\alpha \in \Gamma$, and let $H := \{H_\alpha \mid \alpha \in \Gamma \}$ where $H_\alpha$ are subsets of $S$, for all $\alpha \in \Gamma$. $H$ is called full and weakly-conjugate family of $S$ if

1. $E_\alpha \subseteq H_\alpha$ for all $\alpha \in \Gamma$,
2. for each $a \in H_\alpha$ and $b \in H_\beta$, $\alpha, \beta \in \Gamma$, $a \alpha b \in H_\beta$ and $a \beta b \in H_\alpha$,
3. for each $a' \in W_\alpha^\beta(a)$ and $c \in H_\gamma$, $\alpha, \beta, \gamma \in \Gamma$, $a \alpha c \gamma a', a' \gamma c a \in H_\beta$ and $a' \beta c \gamma a, a' \gamma c \beta a \in H_\alpha$.

\end{Definition}

\begin{Example}
 By Example 1.1, let $\frac{p}{q} \in Q^*$ where $p, q \neq 0$. We have $|q|, |p|, 1 \in \Gamma$ and

1. $E_{|q|} = \{-\frac{1}{q}, \frac{1}{q}\}$, $E_{|p|} = \{-\frac{1}{p}, \frac{1}{p}\}$, and $E_{\{1\}} = \{1\}$

   $H_{|q|} = \{-\frac{1}{q}, \frac{1}{q}, 1 \}$ and $H_{|p|} = \{-\frac{1}{p}, \frac{1}{p}, 1 \}$.

   Therefore $E_{|q|} \subseteq H_{|q|}, E_{|p|} \subseteq H_{|p|}$ and $H_{\{1\}} = \{1\} = E_{\{1\}}$.

2. Let $\frac{1}{|q|} \in H_{|q|}$ and $\frac{1}{|p|} \in H_{|p|}$.

   Then $\frac{1}{|q|} \frac{1}{|q|} \cdot \frac{1}{p} = \frac{1}{p} \in H_{|p|}$ and $\frac{1}{|q|} \frac{1}{|p|} \cdot \frac{1}{p} \in H_{|q|}$.

3. Let $a = \frac{p}{q} \in Q^*$ where $p, q \neq 0$. Note that $\frac{1}{|p|} \in W_{|q|}^{\{1\}}(\frac{p}{q})$ and $\frac{1}{|p|} \in H_{|p|}$, we have $\alpha = |q|, \beta = 1, \gamma = |p|$. Choose $a' = \frac{1}{|p|}$ and $c = \frac{1}{p}$.

   Hence $a \alpha c \gamma a', a' \gamma c a \in H_\beta$ and $a' \alpha c \gamma a, a' \gamma c a \in H_\alpha$. Let $H = \{H_{|q|}, H_{|p|}, H_{\{1\}}\}$. Then $H$ is full and weakly-conjugate family of $Q^*$.
\end{Example}
Proposition 2.6. Let $S$ be an E-inversive $Γ$-semigroup and $a, b ∈ S, θ ∈ Γ$. If $x ∈ W^δ_γ(aθb)$ for some $γ, δ ∈ Γ$ then $bγxδa$ is $θ$-idempotent of $S$.

Proof. Let $x ∈ W^δ_γ(aθb)$ for some $γ, δ ∈ Γ$. Then $(bγxδa)θ(bγxδa) = bγ(xδaθbγx)δa = bγxδa$, hence $bγxδa ∈ E_θ$.

4 Main Results

The purpose of this section is to give some characterizations of $Γ$-group congruences on E-inversive $Γ$-semigroup and those of the least $Γ$-group congruence.

Theorem 4.1. Let $S$ be an E-inversive $Γ$-semigroup and $H := \{H_α, α ∈ Γ\}$ be full and weakly-conjugate family of $S$. Then $ρ_H := \{(a, b) ∈ S × S | aαx = yβb \text{ for some } x ∈ H_α, y ∈ H_β \text{ and } α, β ∈ Γ\}$ is a $Γ$-group congruence on $S$.

Proof. Let $a ∈ S$ and $a' ∈ W^δ_γ(a)$ for some $α, β ∈ Γ$. Now, $aa(α'βa) = (aaα')βa$. Since $α'βa ∈ E_α ⊆ H_α$ and $aaα' ∈ E_β ⊆ H_β$, we have $(a, a) ∈ ρ_H$. Let $a, b ∈ S$ and $(a, b) ∈ ρ_H$. Then there exist $x ∈ H_α$ and $y ∈ H_β$ where $α, β ∈ Γ$ such that $aαx = yβb$. Let $a' ∈ W^δ_γ(a)$ and $b' ∈ W^δ_γ(b)$ for some $γ, δ, θ, φ ∈ Γ$. Now, $bθ[(bφyβb)γ(α'βa)] = [(bθb')φ(aαxγα')βa]$. Since $α'βa ∈ E_γ ⊆ H_γ$, by Definition 2.4(3), we have $b'φyβb ∈ H_θ$. Again $bθb' ∈ E_φ ⊆ H_φ$ and by Definition 2.4(3), $aαxγα' ∈ H_δ$, and by Definition 2.4(2), we have $(bθb')φ(aαxγα') ∈ H_δ$. Therefore $(b, a) ∈ ρ_H$.

Let $a, b, c ∈ S$ be such that $(a, b) ∈ ρ_H$ and $(b, c) ∈ ρ_H$. Then there exist $x ∈ H_α, y ∈ H_β, z ∈ H_γ$ and $w ∈ H_δ$ for some $α, β, γ, δ ∈ Γ$ such that $aαx = yβb$ and $bγz = wδc$. Now, $aαxγz = (aαx)γz = (yβb)γz = yβ(wδc) = (yβw)δc$. Since $xγz ∈ H_α$ and $yβw ∈ H_δ$, so $(a, c) ∈ ρ_H$, hence $ρ_H$ is an equivalence relation on $S$.

To show that $ρ_H$ is compatible, let $(a, b) ∈ ρ_H$ and $θ ∈ Γ, c ∈ S$. Then there exist $x ∈ H_α$ and $y ∈ H_β$ for some $α, β ∈ Γ$ such that $aαx = yβb$. Let $c' ∈ W^δ_γ(c)$ and $g ∈ W^δ_γ(bθc), h ∈ W^δ_γ(aθc)$. By Proposition 2.6, $(cγ2hδa) ∈ E_θ ⊆ H_θ$, so $(cγ2hδα)x ∈ H_θ$ and by Definition 2.4(3), $c'δ[c(γ2hδa)xθc] ∈ H_γ$. Again $gδ1(bθc) ∈ E_γ ⊆ H_γ$ and by Definition 2.4(3), $c'δ[cγ2hδaθcγ1]g ∈ H_δ$. Similarly, since $c'c ∈ E_γ ⊆ H_γ$ and by Definition 2.4(2), $yβ[bθcγ1g] ∈ H_δ$ and so $((aθc)γc'(δcγ2h))δ2(yβθcγ1g) ∈ H_δ$. 


Now, \((a\theta c)\gamma[c'h_2\delta_2\alpha\theta c\gamma_1\delta_1\beta\theta c] = [a\theta c\gamma'c'h_2\delta_2\alpha'\beta\theta c\gamma_1\delta_1\beta\theta c].\) Therefore \((a\theta c, b\theta c) \in \rho_H.\)

Next, we show that \((c\theta a, c\theta b) \in \rho_H.\) Let \(c' \in W^c_\gamma(c), \theta \in \Gamma\) and \(w \in W^c_\gamma(c\theta b), z \in W^c_\gamma(c\theta a)\) for some \(\gamma, \gamma_2, \delta_1, \delta_2 \in \Gamma.\) Since \(z\delta_2(c\theta a) \in E_{\gamma_2} \subseteq H_{\gamma_2}\) and by Definition 2.4(2), \(z\delta_2(c\theta a)x \in H_{\gamma_2}, c\gamma' \in E_\delta \subseteq H_\delta,\) by Definition 2.4(3), \(w\delta_1(c\gamma')\delta(c\theta b) \in H_{\gamma_1}.\)

Then \((z\delta_2(c\theta a)x)\gamma_1(w\delta_1(c\gamma')\delta(c\theta b)) \in H_{\gamma_2}.\) Similarly, by Proposition 2.6, \(a\gamma_2\zeta_2c \in E_\theta \subseteq H_\theta\) and by Definition 2.4(2), \((a\gamma_2\zeta_2c)\theta y \in H_\beta\) because \(y \in H_\beta.\) Again by Proposition 2.6, \(b\gamma_1\zeta_1c \in E_\theta \subseteq H_\theta,\) then \((a\gamma_2\zeta_2c\theta y)\zeta(b\gamma_1\zeta_1c) \in H_\theta\) and so \(c\theta((a\gamma_2\zeta_2c\theta y)\zeta(b\gamma_1\zeta_1c)) \gamma' \in H_\delta.\) Now, \((c\theta a)\gamma_2(z\delta_2(c\theta a)x)\gamma_1(w\delta_1(c\gamma')\delta(c\theta b)) = [c\theta a\gamma_2\zeta_2c\theta y\zeta(b\gamma_1\zeta_1c)c\gamma']\delta(c\theta b).\) Hence \((c\theta a, c\theta b) \in \rho_H\) and so \(\rho_H\) is a congruence on \(S.\)

To show that \(S/\rho_H\) is a \(\Gamma\)-group, we will show that \(S/\rho_H\) is a regular \(\Gamma\)-semigroup. Let \(a' \in W^a_\delta(a)\) where \(\alpha, \beta \in \Gamma.\) Then \(aa(a'\beta a) = a\alpha(a'\beta a\alpha a')\beta a = (aa'a')\beta(aaa'\beta a).\) Since \(a'\beta a \in E_\alpha \subseteq H_\alpha\) and \(aa'a' \in E_\beta \subseteq H_\beta,\) we get that \((a, a'a'a') \in \rho_H.\) Hence \(S/\rho_H\) is a regular \(\Gamma\)-semigroup.

Let \(\alpha, \beta \in \Gamma\) and \(e \in E_\alpha, f \in E_\beta.\) Since \(E_\alpha \subseteq H_\alpha\) and \(E_\beta \subseteq H_\beta\) by Definition 2.4(2), we get \(e\alpha f, f\alpha e \in H_\beta.\) Now, \((e\alpha f)\beta f = (e\alpha f)\beta f,\) hence \((e\alpha f, f) \in \rho_H\) and \((f\alpha e)\beta f = (f\alpha e)\beta f,\) hence \((f\alpha e, f) \in \rho_H.\)

Thus \((e\rho_H)\alpha(f \rho_H) = f \rho_H = (f \rho_H)\alpha(e \rho_H).\) Similarly, we can show that \((e\beta f)\alpha e = (e\beta f)\alpha e,\) hence \((e\beta f, e) \in \rho_H.\) and \((f\beta e)\alpha e = (f\beta e)\alpha e,\) hence \((f\beta e, e) \in \rho_H.\) Thus \((e\rho_H)\beta(f \rho_H) = e \rho_H = (f \rho_H)\beta(e \rho_H).\) Therefore \(S/\rho_H\) is a \(\Gamma\)-group, and \(\rho_H\) is a \(\Gamma\)-group congruence on \(S.\)

The following theorem, we give some characterizations of \(\Gamma\)-group congruence on \(E\)-inversive \(\Gamma\)-semigroup \(S\) by using a full and weakly-conjugate family of \(S\) and the following concept.

**Definition 4.2.** Let \(S\) be \(\Gamma\)-semigroup. If \(H := \{H_\alpha, \alpha \in \Gamma\}\) is a full and weakly-conjugate family of subset of \(S,\) the closure \(H_\omega\) of \(H\) is the family defined by

\[
(H_\omega)_\gamma := \{(H_\omega)_\gamma \mid \gamma \in \Gamma\}
\]

where

\[
(H_\omega)_\gamma = \{a \in S \mid h\alpha a \in H_\gamma \text{ for some } h \in H_\alpha, \alpha \in \Gamma\}.
\]

Then \(H\) is closed if \(H = H_\omega.\)

**Remark.** Let \(S\) be a \(\Gamma\)-semigroup with \(H := \{H_\alpha, \alpha \in \Gamma\}\) is a full and weakly-conjugate family of \(S.\) Then for all \(e \in E_\alpha, \alpha \in \Gamma, e\alpha e = e \in E_\alpha \subseteq H_\alpha,\) hence \(e \in (H_\omega)_\alpha\) and for all \(h \in H_\alpha,\) if \(hah \in H_\alpha,\) we get \(H_\alpha \subseteq (H_\omega)_\alpha\) for all \(\alpha \in \Gamma.\)
Theorem 4.3. Let $S$ be an $E$-inversive $\Gamma$-semigroup such that $H := \{H_\alpha, \alpha \in \Gamma\}$ is a full and weakly-conjugate family of $S$. Then

$$\rho_H^* := \{(a,b) \in S \times S \mid a \gamma b' \in (H_\omega)_\delta \text{ for some } b' \in W^\delta_\gamma(b)\},$$

hence $\rho_H^* = \rho_H$.

Proof. By Theorem 4.1, $\rho_H := \{(a,b) \in S \times S \mid a \omega x = y \gamma b \text{ for some } x \in H_\alpha, y \in H_\beta \}$ and $\alpha, \beta \in \Gamma$.

Let $(a,b) \in \rho_H$. Then $(b,a) \not\in \rho_H$ and there exist $x \in H_\alpha, y \in H_\beta, \alpha, \beta \in \Gamma$ such that $a \omega x = y \gamma b$. Let $b' \in W^\delta_\gamma(b), \gamma, \delta \in \Gamma$. Then $(y \beta a) \gamma b' = (b \alpha x) \gamma b'$. By Definition 2.4(3), $b \alpha x \gamma b' \in H_\delta$. Since $y \in H_\beta$ and $y \beta (a \gamma b') \in H_\delta$, we get that $a \gamma b' \in (H_\omega)_\delta$ and so $(a,b) \not\in \rho_H^*$, hence $\rho_H \subseteq \rho_H^*$. Let $(a,b) \in \rho_H^*$. Then there exist $b' \in W^\delta_\gamma(b), \gamma, \delta \in \Gamma$ such that $a \gamma b' \in (H_\omega)_\delta$. Then there exist $h \in H, \alpha \in \Gamma$ such that $h \alpha (a \gamma b') \in H_\delta$. Put $f = h \alpha a \gamma b' \in H_\delta$. Note that, $b \alpha (a \omega \alpha a \gamma b') \delta \alpha = b \alpha a \omega \phi \delta \alpha$ for some $\alpha' \in W^\delta(b)$. Since $a' \omega \alpha a \in H_\alpha, h \in H_\alpha, b \alpha (a' \omega \alpha a) \gamma b' \in H_\delta$ and $a' \omega \alpha \delta \alpha \in H_\delta$, it follows that $b \alpha (a' \omega \alpha a \gamma b') \delta \alpha$. Hence $(b,a) \in \rho_H$ and $(a,b) \in \rho_H$. Therefore $\rho_H^* \subseteq \rho_H$ and consequently $\rho_H^* = \rho_H$. \qed

Now, we introduce the concept of the set $\ker \rho$.

Definition 4.4. [1] Let $\rho$ be a congruence on $\Gamma$-semigroup $S$, and let $\ker \rho := \{(\ker \rho)_\alpha, \alpha \in \Gamma\}$ where $(\ker \rho)_\alpha := \{a \in S \mid e \alpha a \text{ for some } e \in E_\alpha\}$.

Example 4.5. Let $\rho$ be a congruence on $\Gamma$-semigroup $S$ with $E_\alpha \neq \emptyset$ for some $\alpha \in \Gamma$. Let $e \in E_\alpha$. Then $e \alpha e$ for all $e \in E_\alpha$, and so $e \in (\ker \rho)_\alpha$. Therefore $(\ker \rho)_\alpha \neq \emptyset$.

Theorem 4.6. Let $S$ be an $E$-inversive $\Gamma$-semigroup such that $H := \{H_\alpha, \alpha \in \Gamma\}$ is a full and weakly-conjugate family of $S$. Then $\ker \rho_H = H_\omega$, where $\rho_H$ defined as in Theorem 4.1.

Proof. To show that $(\ker \rho_H)_\alpha = (H_\omega)_\alpha$ for all $\alpha \in \Gamma$, let $x \in (\ker \rho_H)_\alpha$ for some $\alpha \in \Gamma$. Then $e \rho_H x$ for some $e \in E_\alpha$ and by Theorem 4.1, then exist $y \in H_\beta, z \in H_\gamma, \beta, \gamma \in \Gamma$ such that $e \beta y = z \gamma x$. Since $e \beta y \in H_\alpha$, we get that $z \gamma x \in H_\alpha$ and so $x \in (H_\omega)_\alpha$. Since $y \in (H_\omega)_\alpha, \alpha \in \Gamma$. then there exist $g \in H_\gamma, \gamma \in \Gamma$ such that $g \gamma y \in H_\alpha$. Now, for some $e \in E_\alpha, e \alpha (g \gamma y) = (e \alpha g) \gamma y$ where $g \gamma y \in H_\alpha$ and $e \alpha g \in H_\gamma$, it follows that $(e,y) \in \rho_H$ and by Definition 4.4, $y \in (\ker \rho_H)_\alpha$. Therefore $(\ker \rho_H)_\alpha = (H_\omega)_\alpha$ for all $\alpha \in \Gamma$. Hence $\ker \rho_H = H_\omega$. \qed
Theorem 4.7. Let $S$ be an E-inversive $\Gamma$-semigroup such that $H := \{ H_\alpha, \alpha \in \Gamma \}$ is a full and weakly-conjugate family of $S$. Then $a_\rho H b$ if and only if one of the following equivalent conditions hold.

1. $a_\gamma b' \in (H_\omega)_\delta$ for some $b' \in W_\gamma^\delta (b)$,
2. $b_\delta \alpha \in (H_\omega)_\gamma$ for some $b' \in W_\gamma^\delta (b)$,
3. $a'_\alpha \phi b \in (H_\omega)_\theta$ for some $a' \in W_\theta^\phi (b)$, and
4. $b \theta a' \in (H_\omega)_\phi$ for some $a' \in W_\theta^\phi (b)$.

Proof. (1) $\iff$ (3) Let $H$ be a full and weakly-conjugate family of $S$ and suppose that $a_\gamma b' \in (H_\omega)_\delta$ for some $b' \in W_\gamma^\delta (b)$ where $\alpha, \delta \in \Gamma$. Then there exist $h \in H_\alpha, \alpha \in \Gamma$ such that $h \alpha (a_\gamma b') \in H_\delta$. Let $a' \in W_\theta^\phi (a)$ for some $\theta, \phi \in \Gamma$. Then $a'_\phi (h \alpha a_\gamma b') \delta a \in H_\theta$ and $(a'_\phi h \alpha a_\gamma b') \theta a' \phi b = (a'_\phi h \alpha a_\gamma b') \delta a \theta a' \phi b \in H_\theta$. Therefore $a'_\phi b \in (H_\omega)_\theta$.

Suppose that $a'_\phi b \in (H_\omega)_\theta$ for some $a' \in W_\theta^\phi (a), \theta, \phi \in \Gamma$. Then there exist $h \in H_\beta, \beta \in \Gamma$ such that $h \beta (a'_\phi b) \in H_\theta$ and $a \theta (h \beta a'_\phi b) a' \in H_\phi$. Therefore for some $b' \in W_\gamma^\delta (b)$,

$$(a \theta h \beta a'_\phi b) a' \gamma b' = (a \theta h \beta a'_\phi b) a' \gamma b' \in H_\delta.$$ 

Therefore $a_\gamma b' \in (H_\omega)_\delta$.

To show (2) $\iff$ (4), let $b'_\delta \alpha \in (H_\omega)_\gamma$ for some $b' \in W_\gamma^\delta (b), \gamma, \delta \in \Gamma$. Then there exist $h \in H_\alpha, \alpha \in \Gamma$ such that $h \alpha (b'_\delta \alpha) \in H_\gamma$. Let $a' \in W_\theta^\phi (a)$ for some $\theta, \phi \in \Gamma$. By Definition 2.4(3), $a_\gamma (h \alpha b'_\delta \alpha) \theta a' \in H_\phi$ and $b'_\delta \alpha a' \phi b = (h \alpha b'_\delta \alpha) \theta a' \phi b \in H_\phi$. Therefore $a'_\phi b \in (H_\omega)_\phi$.

Now, $(a_\gamma (h \alpha b'_\delta \alpha) \phi (b \theta a')) = (a_\gamma (h \alpha b'_\delta \alpha) \phi (b \theta a')) \gamma b' \in H_\delta$.

Suppose that $b \theta a' \in (H_\omega)_\phi$ for some $a' \in W_\theta^\phi (a), \theta, \phi \in \Gamma$. Then there exist $h \in H_\alpha, \alpha \in \Gamma$ such that $h \alpha (b \theta a') \in H_\phi$. Let $b' \in W_\omega^\delta (b)$ for some $\gamma, \delta \in \Gamma$. Now, $(b'_\delta h \alpha b_\theta a') \gamma (b'_\delta \alpha) = (b'_\delta h \alpha b_\theta a') \gamma (b'_\delta \alpha)$. By Definition 2.4(3), $b'_\delta (h \alpha b_\theta a') \phi b \in H_\gamma$ and $b'_\delta h \alpha b \in H_\gamma, a'_\phi b_\gamma b'_\delta \alpha \in H_\theta$. Thus $(b'_\delta h \alpha b_\theta a') \gamma (b'_\delta \alpha) \in H_\gamma$, so $(b'_\delta h \alpha b_\theta a') \gamma (b'_\delta \alpha) \in H_\gamma$, hence $b'_\delta \alpha \in (H_\omega)_\gamma$.

To show (4) $\iff$ (1), let $b \theta a' \in (H_\omega)_\phi$ for some $a' \in W_\theta^\phi (a), \theta, \phi \in \Gamma$. Then there exist $h \in H_\alpha, \alpha \in \Gamma$ such that $h \alpha (b \theta a') \in H_\phi$. Let $b' \in W_\omega^\delta (b)$ for some $\gamma, \delta \in \Gamma$. By Definition 2.4(3), $b'_\delta (a_\gamma \phi a_\gamma b') \gamma b' \in H_\delta$ and $h \alpha (b \theta a' \phi a_\gamma b') \in H_\delta$. Now, $(h \alpha (b \theta a' \phi a_\gamma b')) = (h \alpha (b \theta a' \phi a_\gamma b')) \gamma b' \in H_\delta$. Therefore $a_\gamma b' \in (H_\omega)_\delta$.

Suppose that $a_\gamma b' \in (H_\omega)_\delta$ for some $b' \in W_\omega^\delta (b), \gamma, \delta \in \Gamma$. Then there exist $h \in H_\alpha, \alpha \in \Gamma$ such that $h \alpha (a_\gamma b') \in H_\delta$. Let $a' \in W_\theta^\phi (a)$ for some $\theta, \phi \in \Gamma$. Since $b'_\delta b \in E_\gamma \subseteq H_\gamma$ and by Definition 2.4(3), $a_\gamma (b'_\delta b) a' \in H_\phi$ and $h \alpha (a_\gamma b'_\delta b a') \in H_\phi$. Now $(h \alpha (a_\gamma b')) \delta (b \theta a') = (h \alpha (a_\gamma b'_\delta b a')) \in H_\phi$ for
some $\theta, \phi \in \Gamma$. Therefore $b\theta a' \in (H_\omega)_{\phi}$.

Moreover, the symmetric property of $\rho_H$ shows that $a\gamma b' \in (H_\omega)_\delta$ for some (all) $b' \in W^\delta_\omega(b)$ if and only if $b\theta a' \in (H_\omega)_{\phi}$ for some (all) $a' \in W^\phi_\theta(a)$. Therefore the proof is completed.

To prove the least $\Gamma$-group congruence on $E$-inversive $\Gamma$-group $S$ by using the smallest element of full and weakly-conjugate family of $S$. Now the following Lemma easily follows:

**Lemma 4.8.** Let $C$ be the collection of all full and weakly-conjugate families $H_i$ of $S, (i \in \Lambda)$ where $H_i = \{H_{i\alpha}, \alpha \in \Gamma\}$.

Let $U_\alpha := \bigcap_{i \in \Lambda} H_{i\alpha}$ and $U := \{U_\alpha \mid \alpha \in \Gamma\}$. Then $U$ is a full and weakly-conjugate family of $S$ and $U$ is the smallest element in $C$.

**Proof.** Clearly, $E_\alpha \subseteq U_\alpha$ for all $\alpha \in \Gamma$. Let $a \in U_\alpha$ and $b \in U_{\beta}, \alpha, \beta \in \Gamma$. Then $a \in H_{i\alpha}$ for all $i \in \Lambda$ and $b \in H_{i\beta}$ for all $i \in \Lambda$, and since $H_{i\alpha}, H_{i\beta} \in H_i$ for all $i \in \Lambda$, we get $aob \in H_{i\beta}$ and $a\beta b \in H_{i\alpha}$ for all $i \in \Lambda$, it implies $aob \in U_{\beta}$ and $a\beta b \in U_\alpha$.

If $a' \in W^\beta_\delta(a)$ and $c \in U_\gamma, \alpha, \beta, \gamma \in \Gamma$, then $c \in H_{i\gamma}$ for all $i \in \Lambda$. Thus $aoc\gamma a', a\gamma coa' \in H_{i\beta}$ for all $i \in \Lambda$ and $a'\beta c\gamma a, a'\gamma c\beta a \in H_{i\alpha}$ for all $i \in \Lambda$, hence $aoc\gamma a', a\gamma coa' \in \bigcap_{i \in \Lambda} H_{i\beta} = U_\beta$ and $a'\beta c\gamma a, a'\gamma c\beta a \in \bigcap_{i \in \Lambda} H_{i\alpha} = U_\alpha$.

Therefore $U$ is a full and weakly-conjugate family of $S$ and $U$ is the smallest element in $C$. $\Box$

**Theorem 4.9.** Let $S$ be an $E$-inversive $\Gamma$-semigroup. If $\sigma$ is a $\Gamma$-group congruence on $S$, then $\text{Ker} \sigma$ is closed, full and weakly-conjugate of $S$. Moreover $\sigma = \rho_{\text{Ker} \sigma}$.

**Proof.** Suppose that $\sigma$ is a $\Gamma$-group congruence on $S$ and let $K = \text{ker} \sigma := \{(\text{Ker} \sigma)_{\alpha}, \alpha \in \Gamma\} = \{K_{\alpha}, \alpha \in \Gamma\}$ where $K_{\alpha} := \{a \in S \mid e\sigma a \text{ for some } e \in E_\alpha, \alpha \in \Gamma\}$. Let $e \in E_\alpha, \alpha \in \Gamma$. Then $e\sigma e$ and so $e \in K_{\alpha}$ for all $\alpha \in \Gamma$. Thus $E_\alpha \subseteq K_{\alpha}$ for all $\alpha \in \Gamma$. Let $a \in K_{\alpha}$ and $b \in K_{\beta}$ for some $\alpha, \beta \in \Gamma$. Then there exist $e \in E_\alpha$ and $f \in E_\beta$ such that $e\sigma a$ and $f\sigma b$. Thus $(aob)\sigma = (a\sigma)\alpha(b\sigma) = (e\sigma)\alpha(f\sigma) = (e\sigma f)\sigma = e\sigma$, because $\sigma$ is $\Gamma$-group congruence. Then $(aob, f) \in \sigma$ where $f \in E_\beta$ and $aob \in K_{\beta}$. Thus $(a\beta b)\sigma = (a\sigma)\beta(b\sigma) = (e\sigma)\beta(f\sigma) = (e\beta b)\sigma = e\sigma$, because $\sigma$ is $\Gamma$-group congruence. Therefore $(a\beta b, e) \in \sigma$ where $e \in E_\alpha$, hence $a\beta b \in K_{\alpha}$.

Next, let $a' \in W^\beta_\omega(a)$ for some $\alpha, \beta \in \Gamma$ and $c \in K_{\gamma}, \gamma \in \Gamma$. Then there exists $g \in E_\gamma$ such that $(c, g) \in \sigma$. Thus $(aac\gamma a')\sigma = (a\sigma)\alpha(c\sigma)\gamma(a'\sigma) =
Let \((a, b)\) be \((g\sigma)\gamma(a'\sigma) = (a\sigma)\alpha(a'\sigma) = (aa\sigma')\sigma\) because \(\sigma\) is \(\Gamma\)-group congruence. Therefore \((aa\sigma'\gamma, aa\sigma')\) where \(a\sigma'\gamma a' \in EH\), so \(aa\sigma'\gamma a' \in E\). Similarly, we can show that \(a\gamma\alpha a' \in K\) and \(a'\beta\gamma a, a'\gamma c\beta a \in K\). Therefore \(K\) is full and weakly-conjugate family of \(S\).

To show that \(K_\gamma = (K_\omega)_\gamma\) for all \(r \in \Gamma\). Clearly, \(K_\gamma \subseteq (K_\omega)_\gamma\), by Definition 4.2 and 4.4. To show that \((K_\omega)_\gamma \subseteq K_\gamma\), let \(x \in (K_\omega)_\gamma\). Then there exist \(h \in K\), \(\alpha, \beta \in \Gamma\) such that \(hax \in K_\gamma\). Consequently, \((hax)\gamma = g\sigma\) where \(g \in E\gamma\) or \((h\sigma)\alpha(x\sigma) = g\sigma\). Since \(h \in K\), \(\alpha \in \Gamma\), we get \((h, e) \in \sigma\) where \(e \in E\), so \(h\sigma = e\sigma\) and \(e\sigma\) is an identity of \(S/\sigma\) for all \(\alpha, \sigma \in \Gamma\). Then \(g\sigma = (h\sigma)\alpha(x\sigma) = (e\sigma)\alpha(x\sigma) = x\sigma\) because \(\sigma\) is \(\Gamma\)-group congruence. Thus \(x \in K\), hence \(K_\gamma = (K_\omega)_\gamma\). To show that \(\sigma = \rho_K\), by Theorem 4.3 and \(K\) is full and weakly-conjugate family of \(S\), it follows that \(\rho_K := \{(a, b) \in S \times S | a\gamma b' \in (K_\omega)_\delta = K_\delta\}\) for some \(b' \in W_\omega(b), \gamma, \delta \in \Gamma\). Let \((a, b) \in \rho_K\). Then \(a\gamma b' \in K_\delta\) for some \(b' \in W_\omega(b), \gamma, \delta \in \Gamma\). It implies that \((a\gamma b', e) \in \sigma\) where \(e \in E\) and \((a\gamma b', e\delta b) \in \sigma\). Since \(b'\delta b \in E\), we get \(a\sigma = (a\sigma)\gamma(b'\delta b)\sigma = (e\sigma)\delta(b\sigma) = b\sigma\), so \((a, b) \in \sigma\) and \(\rho_K \subseteq \sigma\).

Finally, we shall show that \(\sigma \subseteq \rho_K\), let \((a, b) \in \sigma\) and \(b' \in W_\omega(b)\) for some \(\gamma, \delta \in \Gamma\). Then \((a\gamma b', b\gamma b') \in \sigma\). Since \(b\gamma b' \in E\), we get \(a\gamma b' \in E\subseteq K_\delta\). Thus \((a, b) \in \rho_K\). Therefore \(\sigma = \rho_K\).

**Theorem 4.10.** Let \(S\) be an \(E\)-inversive \(\Gamma\)-semigroup with \(H \in C\) and let \(\rho_H\) be defined as in Theorem 4.1. Then \(\rho_U\) is the least \(\Gamma\)-group congruence on \(S\) and \(\text{Ker}\rho_U = U_\omega\).

**Proof.** Let \(\sigma\) be an arbitrary \(\Gamma\)-group congruence on \(S\). By Theorem 4.9, we obtain \(\sigma = \rho_K\) where \(K = \text{Ker}\sigma\) and \(K\) is a full and weakly-conjugate family of \(S\). Since \(U\) is the smallest full and weakly-conjugate family of \(S\), we get that \(U \subseteq K\).

Let \((a, b) \in \rho_U\). Then there exist \(x \in U_\alpha \subseteq K_\alpha\), \(\alpha \in \Gamma\) and \(y \in U_\beta \subseteq K_\beta\), \(\beta \in \Gamma\) such that \(a\alpha x = y\beta b\). Thus \((a, b) \in \rho_K = \sigma\). Hence \(\rho_U\) is the least \(\Gamma\)-group congruence on \(S\). By Theorem 4.6, \(\text{Ker}\rho_U = U_\omega\).

Now, we obtain the following theorems for characterizations of \(\Gamma\)-group congruences on \(E\)-inversive \(\Gamma\)-semigroups as obtained for regular \(\Gamma\)-semigroup in [1].

**Theorem 4.11.** Let \(S\) be an \(E\)-inversive \(\Gamma\)-semigroup with \(\rho_H\) a \(\Gamma\)-group congruence on \(S\) where \(H\) is a full and weakly-conjugate family of \(S\). The
following statements are equivalent.

(1) \( a_\theta H_b \),
(2) \( a_\mu x_\gamma b' \in H_\delta \), for some \( x \in H_\mu, \mu \in \Gamma \) and for some (all) \( b' \in W_\delta^\gamma(b) \),
(3) \( a'_\phi x_\mu b \in H_\theta \), for some \( x \in H_\mu, \mu \in \Gamma \) and for some (all) \( a' \in W_\theta^\phi(a) \),
(4) \( b_\mu x_\theta a' \in H_\phi \), for some \( x \in H_\mu, \mu \in \Gamma \) and for some (all) \( a' \in W_\phi^\theta(a) \),
(5) \( b'_\delta x_\mu a \in H_\gamma \), for some \( x \in H_\mu, \mu \in \Gamma \) and for some (all) \( b' \in W_\delta^\gamma(b) \),
(6) \( a_\alpha x = y_\beta b \) for some \( \alpha, \beta \in \Gamma \) and for some \( x \in H_\alpha, y \in H_\beta \),
(7) \( x_\alpha a = b_\beta y \) for some \( \alpha, \beta \in \Gamma \) and for some \( x \in H_\alpha, y \in H_\beta \), and
(8) \( H_\beta \beta_\alpha a H_\alpha \cap H_\beta \beta_\alpha a H_\alpha \neq \emptyset \) for some \( \alpha, \beta \in \Gamma \).

**Proof.** (2) \( \Rightarrow \) (3) Suppose that \( a_\mu x_\gamma b' \in H_\delta \) for some \( x \in H_\mu \) and \( b' \in W_\gamma^\delta(b), \gamma, \delta, \mu \in \Gamma \). If \( a' \in W_\theta^\phi(a) \) for some \( \theta, \phi \in \Gamma \), then \( a' \phi a \in E_\theta \subseteq E_\theta \) and \( b' \delta x_\mu b \in H_\gamma \). Since \( (a_\mu x_\gamma b') \delta x \in H_\mu \) and \( x_\gamma (b' \delta x_\mu b) \in H_\mu \), we have \( a'_\phi (a_\mu x_\gamma b') \delta x = (a'_\phi a) \mu (x_\gamma (b' \delta x_\mu b)) \in H_\theta \).

(3) \( \Rightarrow \) (6) Let \( a'_\phi x_\mu b \in H_\theta \), for some \( a' \in W_\theta^\phi(a) \) and \( x \in H_\mu, \theta, \phi, \mu \in \Gamma \). Thus \( a_\theta a_\phi = (a_\theta a'_\phi x) \mu b \), where \( a'_\phi x_\mu b \in H_\theta \) and \( a_\theta a'_\phi x \in H_\mu \). Hence (6) holds.

(6) \( \Rightarrow \) (8) Let \( a_\alpha x = y_\beta b \) for some \( \alpha, \beta \in \Gamma \) and \( x, y \in H_\alpha, y \in H_\beta \). Then \( y_\beta b(a_\alpha x) = (y_\beta b)x_\alpha \). Since \( y_\beta b(a_\alpha x) \in H_\beta \beta_\alpha a H_\alpha \) and \( y_\beta b(a_\alpha x) \in H_\beta \beta b_\alpha a H_\alpha \), we get that \( H_\beta \beta_\alpha a H_\alpha \cap H_\beta \beta b_\alpha a H_\alpha \neq \emptyset \) for some \( \alpha, \beta \in \Gamma \).

(8) \( \Rightarrow \) (2) Let \( H_\beta \beta_\alpha a H_\alpha \cap H_\beta \beta b_\alpha a H_\alpha \neq \emptyset \) for some \( \alpha, \beta \in \Gamma \). Then \( x_\beta a_\alpha y = x_1 \beta b_\alpha y_1 \) for some \( x, y \in H_\beta, y \in H_\alpha \). Let \( a' \in W_\theta^\phi(a) \) for some \( a_\theta \in E_\theta \subseteq E_\theta \) and \( b' \in W_\gamma^\delta(b) \) for some \( b'_\delta \in \Gamma \). Then \( a'_\phi x_\beta a_\alpha y \in H_\theta \) and \( (a'_\phi x_\beta a_\alpha y) \alpha \gamma = \gamma \beta \gamma \alpha \). Since \( a_\theta a'_\phi x_1 \in H_\beta \) and \( b_\alpha y_1 \gamma b'_\delta \in H_\delta \), we get that \( (a_\theta a'_\phi x_1)(b_\alpha y_1 \gamma b'_\delta) \in H_\delta \). Then \( \alpha \phi (a'_\phi x_\beta a_\alpha y) b'_\delta = (a_\theta a'_\phi x_1 b_\alpha y_1 \gamma b'_\delta) = (a_\theta a'_\phi x_1 \beta b_\alpha y_1 \gamma b'_\delta) \in H_\delta \), hence (2), (3), (6) and (8) are equivalent.

(2) \( \Rightarrow \) (4) Suppose that \( a_\mu x_\gamma b' \in H_\delta \) for some \( x \in H_\mu \) and \( b' \in W_\gamma^\delta(b), \gamma, \delta, \mu \in \Gamma \). Let \( b' \in W_\gamma^\delta(b) \) for some \( \gamma, \delta \in \Gamma \). Now, \( b_\mu (x_\alpha a_\phi x_\mu b' \delta a) = (b_\mu x_\alpha a_\phi x_\mu b' \delta a) \in H_\phi \). By Definition 2.4 (3), we have \( a'_\phi (a_\phi x_\mu b' \delta a) \in H_\theta \), so \( x_\theta (a'_\phi x_\mu b' \delta a) \in H_\mu \). Since \( a' \theta a' \in H_\phi \) and (8) (3), we have \( b_\mu x_\theta a'_\phi x_\mu b' \delta a \in H_\phi \), then \( (b_\mu x_\theta a'_\phi x_\mu b' \delta a) \in H_\phi \). Hence \( b_\mu (x_\alpha a_\phi x_\mu b' \delta a) \in H_\phi \).

(4) \( \Rightarrow \) (5) Suppose that \( b_\mu x_\theta a' \in H_\phi \) for some \( x \in H_\mu \) and \( a' \in W_\phi^\theta(a), \theta, \phi, \mu \in \Gamma \). Let \( b' \in W_\gamma^\delta(b) \) for some \( \gamma, \delta \in \Gamma \). Now, \( b'_\delta (b_\mu x_\theta a' \phi x) \mu a = (b'_\delta b) x_\alpha a_\phi x \mu a \). Since \( (b_\mu x_\theta a') \phi x \in H_\mu \) and \( b'_\delta b \in H_\gamma, a'_\phi x_\mu a \in H_\theta \), we get that \( (b'_\delta b) x_\alpha a_\phi x \mu a \in H_\gamma \). Hence
\( b'\delta(bx\theta a'\phi x)\mu a \in H_\gamma. \)

(5) \( \Rightarrow \) (7) Let \( b'\delta x\mu a \in H_\gamma \) for some \( b' \in W^\delta(b), x \in H_\mu \) and \( \gamma, \delta, \mu \in \Gamma \). Now, \( (b'\gamma\delta x)\mu a = b\gamma(b'\delta x\mu a) \). Since \( b\gamma b'\delta x \in H_\mu \) and \( b'\delta x\mu a \in H_\gamma \), we have (7).

(7) \( \Rightarrow \) (1) Let \( xa = b\beta y \) for some \( \alpha, \beta \in \Gamma \) and \( x \in H_\alpha, y \in H_\beta \). Let \( a' \in W^\phi(a) \) for some \( \theta, \phi \in \Gamma \) and \( b' \in W^\delta(b) \). Now, \( a\theta(a'\phi x\alpha a\gamma b'\delta b) = (a\theta a'\phi b\beta y\gamma b')\delta b \). Since \( b'\delta b \in H_\gamma \) and \( a'\phi x\alpha a \in H_\theta \), we have \( (a\phi x\alpha a)\gamma(b'\delta b) \in H_\theta \). Since \( a\theta a' \in H_\phi \) and \( b\beta y\gamma b' \in H_\delta \), we have \( (a\theta a')\phi(b\beta y\gamma b') \in H_\delta \). Then \( (a, b) \in \rho_H \). Hence (2), (4), (5) and (7) are equivalent.

Also (1) \( \iff \) (6) by Theorem 4.1.

\[ \square \]

References


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