A Note on Representable Autometrized Algebras

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Abstract : In this paper, two remarkable results are obtained in a Representable Autometrized Algebra $A = (A, +, \leq, *)$ satisfying the condition $[R] :$

$$a * (a \land b) + a \land b = a$$

for all $a, b \in A.$

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1 Introduction

Representable Autometrized Algebras were introduced and studied by B. V. Subba Rao (113). In this paper we obtained two results regarding Representable Autometrized Algebra $A = (A, +, \leq, *)$ satisfying the condition $[R] : a * (a \land b) + a \land b = a$ for all $a, b \in A.$

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In the first theorem, if there exists \( 1 \in A \) such that \( a + (1 * a) = 1 + 1 \), for all \( a \in A \), then we proved that it is unique.

In the second theorem, if \( 1 \in A \) such that \( a + (1 * a) = 1 + 1 \), for all \( a \in A \), then we proved the following.

(1) If \( A \) has least element say \( \alpha \), then \( \alpha = 0 \) and

(2) If \( A \) has greatest element say \( \beta \), then \( \beta = 1 \).

2 Lattice Ordered Autometrized Algebra and Representable Autometrized Algebra

Here we recall the definition of (i) Lattice Ordered Autometrized algebra (Swamy [4, 5]), (ii) Representable Autometrized Algebra, introduced and studied by Subba Rao ([1–3]).

Definition 2.1. A system \( A = (A, +, \leq, *) \) is called a “Lattice Ordered Autometrized Algebra” if and only if

(i) \( (A, +, \leq) \) is a commutative lattice ordered semigroup with \( '0' \), and

(ii) \( * \) is a metric operation on \( A \).

i.e., \( * \) is a mapping from \( A \times A \) into \( A \) satisfying the formal properties of distance, namely,

\[
(M_1) \quad a * b \geq 0 \text{ for all } a, b \in A, \text{ equality, if and only if } a = b,
\]

\[
(M_2) \quad a * b = b * a \text{ for all } a, b \in A, \text{ and}
\]

\[
(M_3) \quad a * b \leq a * c + c * b \text{ for all } a, b, c \in A.
\]

Definition 2.2. A lattice ordered autometrized algebra \( A = (A, +, \leq, *) \) is called a “Representable Autometrized Algebra”, if and only if, \( A \) satisfies the following conditions:

\[
(L_1) \quad A = (A, +, \leq, *) \text{ is a semiregular autometrized algebra,}
\]

i.e., \( a \in A \) and \( a \geq 0 \) implies \( a * 0 = a \), and

\[
(L_2) \quad \text{For every } a \text{ in } A \text{ all the mapping } x \mapsto a + x, x \mapsto a \lor x, x \mapsto a \land x \text{ and}
\]

\[
\mapsto a * x \text{ are contractions (i.e., if } \theta \text{ denotes any one of the operations } +, \lor, \land \text{ and } * \text{, then, for each } a \text{ in } A, (a \theta x) * (a \theta y) \leq x * y \text{ for all } x, y \text{ in } A).
\]

Theorem 2.3. Let \( A = (A, +, \leq, *) \) be a Representable Autometrized Algebra satisfying the condition \( [R] a * (a \land b) + a \land b = a \), for all \( a, b \in A \). If there exists \( 1 \in A \) such that \( a + (1 * a) = 1 + 1 \) \( \forall a \in A \), then, it is unique.

Proof. Let \( A = (A, +, \leq, *) \) be a Representable Autometrized Algebra satisfying \( (R) \) mentioned in the theorem. Let \( 1 \in A \) and \( 1^1 \in A \) such that

\[
a + (1 * a) = 1 + 1 \text{ for all } a \in A \quad (2.1)
\]
and
\[ a + (1^1 * a) = 1^1 + 1^1 \text{ for all } a \in A. \]  \tag{2.2}

Since \(1 \in A\), by (2.1) we have
\[ 1 + (1 * 1) = 1 + 1, \]
i.e.,
\[ 1 + 0 = 1 + 1. \]
So,
\[ 1 = 1 + 1. \]  \tag{2.3}

In a similar manner since \(1^1 \in A\), from (2.2), we get
\[ 1^1 = 1^1 + 1^1. \]  \tag{2.4}

Putting \(a = 1^1\) in (2.1), we get
\[ 1^1 + (1^1 * 1^1) = 1^1 + 1 = 1 + 1 \quad \text{(from (2.3)).} \]  \tag{2.5}

Also putting \(a = 1\) in (2.2), we get
\[ 1 + (1^1 * 1) = 1^1 + 1^1 = 1^1 \quad \text{(from (2.4)).} \]  \tag{2.6}

Therefore we get
\[ 1 + 1^1 = 1^1 + 1 \]
\[ = 1^1 + [1^1 + (1^1 * 1)] \]
\[ = (1^1 + 1^1) + (1^1 * 1) \]
\[ = 1^1 + (1^1 * 1) \]
\[ = 1. \]

Again from (2.6), we get
\[ 1 + 1^1 = 1 + [1 + (1^1 * 1)] \]
\[ = 1 + 1 + (1^1 * 1) \]
\[ = 1 + (1 * 1) \]
\[ = 1^1, \quad \text{(from (2.6) since } 1 * 1^1 = 1^1 * 1) \]
\[ \Rightarrow 1 = 1^1. \]

Thus, there exists a unique \(1 \in A\) such that \(a + (1 * a) = 1 + 1\) for all \(a \in A\), if it exists.

**Theorem 2.4.** Let \(A = (A,+,\leq,*)\) be a Representable Autometrized Algebra satisfying the condition \([R]\) \(a * (a \land b) + a \land b = a\), for all \(a,b \in A\). If there exists \(1 \in A \ni a + (1 * a) = 1 + 1 \forall a \in A\), then, we have

(i) If \(A\) has least element \(\alpha\) then \(\alpha = 0\),

(ii) If \(A\) has greatest element \(\beta\) then \(\beta = 1\).
Proof. Let $A = (A, +, \leq, *)$ be Representable Autometrized Algebra satisfying the condition $(R)$ mentioned in the theorem.

Let $1 \in A$ such that $a + (1 * a) = 1 + 1 \ \forall \ a \in A. \quad (2.7)$

Therefore, by Theorem 2.3 above, 1 is unique.

(i) Assume that $A$ has a least element $\alpha$.
Therefore $\alpha \leq x \ \forall x \in A$.
In particular $\alpha \leq 0$.
Therefore

\[ 0 = (0 * \alpha) + \alpha \quad \text{by} \quad (R) \]
\[ = \alpha + (0 * \alpha). \quad (2.8) \]

We have $2\alpha \leq \alpha$ (since $\alpha \leq 0$)
but $\alpha \leq 2\alpha$ (since $\alpha$ is the least element of $A$).
Therefore $2\alpha = \alpha$.
By induction, it follows that $n\alpha = \alpha$ for every positive integer $n$.
Therefore

\[ \alpha * 0 = (\alpha + 0) * 0 \]
\[ = [\alpha + (\alpha + (0 * \alpha))] * [\alpha + (0 * \alpha)] \text{ from } (2.8) \]
\[ = [2\alpha + (0 * \alpha)] * [\alpha + (0 * \alpha)] \]
\[ \leq 2\alpha * \alpha \quad \text{(since } x \mapsto a + x \text{ is a contraction, } \forall a \in A) \]
\[ = 0 \quad \text{(since } 2\alpha = \alpha), \]

i.e., $\alpha * 0 \leq 0$,
but $\alpha * 0 \geq 0$,
therefore $\alpha * 0 = 0$.
Hence $\alpha = 0$.
Thus, if $\alpha$ is the least element of $A$, then it must be 0, the additive identity of $A$.

(ii) Assume that $A$ has a greatest element, $\beta$, say.
So, $x \leq \beta \ \forall \ x \in A$.
In particular, we have $1 \leq \beta$ and $0 \leq \beta$.
Therefore, by $(R)$, we have $(\beta + 1) + 1 = \beta$,
and by (2.7), we have $\beta + (1 * \beta) = 1 + 1$.
By (2.7) we also have
\[ 1 + (1 * 1) = 1 + 1 \]
\[ \Rightarrow 1 + 0 = 1 + 1 \]
\[ \Rightarrow 1 = 1 + 1. \]
Therefore $\beta + (1 * \beta) = 1$.
Since $0 \leq \beta$ we have $\beta \leq 2\beta$,
but $\beta \geq 2\beta$ (since $\beta$ is the greatest element in $A$).
Therefore $2\beta = \beta$. 

By induction, it follows that \( n \beta = \beta \) for every positive integer \( n \).

Now using the properties of contraction mappings we have

\[
0 = \beta \ast \beta \geq (\beta \ast 1) \ast (\beta \ast 1) \\
\geq [(\beta \ast 1) + \beta] \ast [(\beta \ast 1) + \beta] \\
\geq 1 \ast [(\beta \ast 1) + \beta] \\
\geq (1 + 1) \ast [(\beta \ast 1) + 1 + \beta] \\
= 1 \ast (\beta + \beta) \\
= 1 \ast 2\beta.
\]

So, \( 2\beta \ast 1 \leq 0 \).

But \( 2\beta \ast 1 \geq 0 \), therefore \( 2\beta \ast 1 = 0 \)

\[ \Rightarrow 2\beta = 1 \]

\[ \Rightarrow \beta = 1 \] (since \( 2\beta = \beta \)).

Thus, if there exists a greatest element \( \beta \) in \( A \), then \( \beta = 1 \).

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