On $\varphi$-$\mathcal{T}$-Symmetric ($\varepsilon$)-Para Sasakian Manifolds

Punam Gupta

Department of Mathematics & Statistics, School of Mathematical & Physical Sciences, Dr. Harisingh Gour University
Sagar-470 003, M.P., India
e-mail : punam_2101@yahoo.co.in

Abstract : The purpose of the present paper is to study the globally and locally $\varphi$-$\mathcal{T}$-symmetric ($\varepsilon$)-para Sasakian manifold. The globally $\varphi$-$\mathcal{T}$-symmetric ($\varepsilon$)-para Sasakian manifold is either Einstein manifold or has a constant scalar curvature. The necessary and sufficient condition for Einstein manifold to be globally $\varphi$-$\mathcal{T}$-symmetric is given. A 3-dimensional ($\varepsilon$)-para Sasakian manifold is locally $\varphi$-$\mathcal{T}$-symmetric if and only if the scalar curvature $r$ is constant. A 3-dimensional ($\varepsilon$)-para Sasakian manifold with $\eta$-parallel Ricci tensor is locally $\varphi$-$\mathcal{T}$-symmetric. In the last, an example of 3-dimensional locally $\varphi$-$\mathcal{T}$-symmetric ($\varepsilon$)-para Sasakian manifold is given.

Keywords : $\mathcal{T}$-curvature tensor; ($\varepsilon$)-para Sasakian manifold; globally and locally $\varphi$-$\mathcal{T}$-symmetric manifold; $\eta$-parallel Ricci tensor.

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1 Introduction

Let $M$ be an $m$-dimensional semi-Riemannian manifold and $\nabla$ the Levi-Civita connection on $M$. A semi-Riemannian manifold $M$ is said to recurrent [1] if the Riemann curvature tensor $R$ satisfies the relation

$$(\nabla_U R)(X, Y, Z, V) = \alpha(U) R(X, Y, Z, V), \quad X, Y, Z, V, U \in TM,$$

where $\alpha$ is 1-form. If $\alpha = 0$, then $M$ is called symmetric in the sense of Cartan [2].

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In 1977, Takahashi [3] introduced the notion of locally \( \varphi \)-symmetry on a Sasakian manifold, which is weaker than the local symmetry. A Sasakian manifold is said to have locally \( \varphi \)-symmetry if it satisfies

\[
\varphi^2 ((\nabla U R) (X, Y) Z) = 0,
\]

where \( X, Y, Z, U \) are horizontal vector fields. If \( X, Y, Z, U \) are arbitrary vector fields, then it is known as globally \( \varphi \)-symmetric Sasakian manifold. A \( \varphi \)-symmetric space condition is weak condition for a Sasakian manifold in comparison to the symmetric space condition. Local symmetry is a very strong condition for the class of \( K \)-contact or Sasakian manifolds. Indeed, such spaces must have constant curvature equal to 1 [4, 5]. On the other hand, local symmetry is also a very strong condition for the class of \( (\varepsilon) \)-para Sasakian manifold. Such spaces must have constant curvature equal to \( -\varepsilon \) [6]. In 2010, Tripathi et al. [6] proved that the condition of semi-symmetry \( (R \cdot R = 0) \), symmetry and have a constant curvature \( -\varepsilon \) is equivalent for \( (\varepsilon) \)-para Sasakian manifold.

Three-dimensional locally \( \varphi \)-symmetric Sasakian manifold is studied by Watanabe [7]. Many authors like De [8], De et al. [9], De and Pathak [10], Shaikh and De [11] have extended this notion to 3-dimensional Kenmotsu, trans-Sasakian and LP-Sasakian manifolds. Yildiz et al. [12] studied the case for 3-dimensional \( \alpha \)-Sasakian manifolds and gave the example for locally \( \varphi \)-symmetric 3-dimensional \( \alpha \)-Sasakian manifolds. De and De [13] studied the \( \varphi \)-concircularly symmetric Kenmotsu manifold and gave the example of such manifold in dimension 3. De et al. [14] studied the 3-dimensional globally and locally \( \varphi \)-quasiconformally symmetric Sasakian manifolds and also gave the example.

In the present work, globally and locally \( \varphi \)-\( T \)-symmetric \( (\varepsilon) \)-para Sasakian manifold are studied. The paper is organized as follows: Section 2 and 3 is devoted to the study of \( T \)-curvature tensor and \( (\varepsilon) \)-para Sasakian manifold, respectively. In section 4, the definition of globally and locally \( \varphi \)-\( T \)-symmetric manifold are given. Globally \( \varphi \)-\( T \)-symmetric \( (\varepsilon) \)-para Sasakian manifold is either Einstein or has a constant scalar curvature under some condition. The necessary and sufficient condition for locally \( \varphi \)-\( T \)-symmetric 3-dimensional \( (\varepsilon) \)-para Sasakian manifold to be locally \( \varphi \)-symmetric is given. In section 5, the definition of \( \eta \)-parallel \( (\varepsilon) \)-para Sasakian manifold is given. A 3-dimensional \( (\varepsilon) \)-para Sasakian manifold with \( \eta \)-parallel Ricci tensor is locally \( \varphi \)-\( T \)-symmetric. In the last section, the example of a locally \( \varphi \)-\( T \)-symmetric in 3-dimensional \( (\varepsilon) \)-para Sasakian manifold is given.

## 2 \( T \)-Curvature Tensor

The definition of \( T \)-curvature tensor [15] is given by

**Definition 2.1.** In an \( m \)-dimensional semi-Riemannian manifold \((M, g)\), the \( T \)-
curvature tensor of type \( (1,3) \) defined by
\[
\mathcal{T}(X, Y) Z = a_0 R(X, Y) Z + a_1 S(Y, Z) X + a_2 S(X, Z) Y + a_3 S(X, Y) Z \\
+ a_4 g(Y, Z) QX + a_5 g(X, Z) QY + a_6 g(X, Y) QZ \\
+ a_7 r(g(Y, Z) X - g(X, Z) Y),
\]
(2.1)
for all \( X, Y, Z \in TM \), where \( a_0, \ldots, a_7 \) are some constants; and \( R, S, Q \) and \( r \) are the curvature tensor, the Ricci tensor, the Ricci operator of type \( (1,1) \) and the scalar curvature respectively.

In particular, the \( \mathcal{T} \)-curvature tensor is reduced to
1. the \textit{Riemann curvature tensor} \( R \) if
   \[
a_0 = 1, \quad a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = a_7 = 0,
\]
2. the \textit{quasiconformal curvature tensor} \( C \) \cite{16} if
   \[
a_1 = -a_2 = a_4 = -a_5, \quad a_3 = a_6 = 0, \quad a_7 = -\frac{1}{m} \left( \frac{a_0}{m-1} + 2a_1 \right),
\]
3. the \textit{conformal curvature tensor} \( C \) \cite{17, p. 90} if
   \[
a_0 = 1, \quad a_1 = -a_2 = a_4 = -a_5 = -\frac{1}{m-2}, \quad a_3 = a_6 = 0,
\]
   \[
a_7 = \frac{1}{(m-1)(m-2)},
\]
4. the \textit{conharmonic curvature tensor} \( \mathcal{L} \) \cite{18} if
   \[
a_0 = 1, \quad a_1 = -a_2 = a_4 = -a_5 = -\frac{1}{m-2}, \quad a_3 = a_6 = 0, \quad a_7 = 0,
\]
5. the \textit{concircular curvature tensor} \( \mathcal{V} \) \cite{19, 20, p. 87} if
   \[
a_0 = 1, \quad a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = 0, \quad a_7 = -\frac{1}{m(m-1)},
\]
6. the \textit{pseudo-projective curvature tensor} \( \mathcal{P} \) \cite{21} if
   \[
a_1 = -a_2, \quad a_3 = a_4 = a_5 = a_6 = 0, \quad a_7 = -\frac{1}{m} \left( \frac{a_0}{m-1} + a_1 \right),
\]
7. the \textit{projective curvature tensor} \( \mathcal{P} \) \cite{20, p. 84} if
   \[
a_0 = 1, \quad a_1 = -a_2 = -\frac{1}{(m-1)}, \quad a_3 = a_4 = a_5 = a_6 = a_7 = 0,
\]
8. the $M$-projective curvature tensor [22] if
\[
a_0 = 1, \quad a_1 = -a_2 = a_4 = -a_5 = -\frac{1}{2(m-1)}, \quad a_3 = a_6 = a_7 = 0,
\]

9. the $W_0$-curvature tensor [22] eq (1.4) if
\[
a_0 = 1, \quad a_1 = -a_5 = -\frac{1}{(m-1)}, \quad a_2 = a_3 = a_4 = a_6 = a_7 = 0,
\]

10. the $W_0^*$-curvature tensor [22] eq (1.4) if
\[
a_0 = 1, \quad a_1 = -a_5 = \frac{1}{(m-1)}, \quad a_2 = a_3 = a_4 = a_6 = a_7 = 0,
\]

11. the $W_1$-curvature tensor [22] if
\[
a_0 = 1, \quad a_1 = -a_2 = \frac{1}{(m-1)}, \quad a_3 = a_4 = a_5 = a_6 = a_7 = 0,
\]

12. the $W_1^*$-curvature tensor [22] if
\[
a_0 = 1, \quad a_1 = -a_2 = -\frac{1}{(m-1)}, \quad a_3 = a_4 = a_5 = a_6 = a_7 = 0,
\]

13. the $W_2$-curvature tensor [23] if
\[
a_0 = 1, \quad a_4 = -a_5 = -\frac{1}{(m-1)}, \quad a_1 = a_2 = a_3 = a_6 = a_7 = 0,
\]

14. the $W_3$-curvature tensor [22] if
\[
a_0 = 1, \quad a_2 = -a_4 = -\frac{1}{(m-1)}, \quad a_1 = a_3 = a_5 = a_6 = a_7 = 0,
\]

15. the $W_4$-curvature tensor [22] if
\[
a_0 = 1, \quad a_5 = -a_6 = \frac{1}{(m-1)}, \quad a_1 = a_2 = a_3 = a_4 = a_7 = 0,
\]

16. the $W_5$-curvature tensor [24] if
\[
a_0 = 1, \quad a_2 = -a_5 = -\frac{1}{(m-1)}, \quad a_1 = a_3 = a_4 = a_6 = a_7 = 0,
\]

17. the $W_6$-curvature tensor [24] if
\[
a_0 = 1, \quad a_1 = -a_6 = -\frac{1}{(m-1)}, \quad a_2 = a_3 = a_4 = a_5 = a_7 = 0,
\]
18. the $W_7$-curvature tensor $[24]$ if

$$
a_0 = 1, \quad a_1 = -a_4 = -\frac{1}{(m-1)}, \quad a_2 = a_3 = a_5 = a_6 = a_7 = 0,
$$

19. the $W_8$-curvature tensor $[24]$ if

$$
a_0 = 1, \quad a_1 = -a_3 = -\frac{1}{(m-1)}, \quad a_2 = a_4 = a_5 = a_6 = a_7 = 0,
$$

20. the $W_9$-curvature tensor $[24]$ if

$$
a_0 = 1, \quad a_3 = -a_4 = \frac{1}{(m-1)}, \quad a_1 = a_2 = a_5 = a_6 = a_7 = 0.
$$

3 $(\varepsilon)$-Para Sasakian Manifold

A manifold $M$ is said to admit an almost paracontact structure if it admit a tensor field $\varphi$ of type $(1,1)$, a vector field $\xi$ and a 1-form $\eta$ satisfying

$$
\varphi^2 = I - \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \varphi \xi = 0, \quad \eta \circ \varphi = 0.
$$

(3.1)

Let $g$ be a semi-Riemannian metric with index($g$) = $\nu$ such that

$$
g(\varphi X, \varphi Y) = g(X, Y) - \varepsilon \eta(X) \eta(Y), \quad X, Y \in TM,
$$

(3.2)

where $\varepsilon = \pm 1$. Then $M$ is called an $(\varepsilon)$-almost paracontact metric manifold equipped with an $(\varepsilon)$-almost paracontact metric structure $(\varphi, \xi, \eta, g, \varepsilon)$. In particular, if index($g$) = 1, then an $(\varepsilon)$-almost paracontact metric manifold is said to be a Lorentzian almost paracontact manifold. In particular, if the metric $g$ is positive definite, then an $(\varepsilon)$-almost paracontact metric manifold is the usual almost paracontact metric manifold $[25]$.

The equation (3.2) is equivalent to

$$
g(X, \varphi Y) = g(\varphi X, Y)
$$

(3.3)

along with

$$
g(X, \xi) = \varepsilon \eta(X).
$$

(3.4)

From (3.1) and (3.4) it follows that

$$
g(\xi, \xi) = \varepsilon.
$$

(3.5)

**Definition 3.1.** An $(\varepsilon)$-almost paracontact metric structure is called an $(\varepsilon)$-para Sasakian structure if

$$
(\nabla_X \varphi) Y = -g(\varphi X, \varphi Y)\xi - \varepsilon \eta(Y) \varphi^2 X, \quad X, Y \in TM,
$$

(3.6)

where $\nabla$ is the Levi-Civita connection with respect to $g$. A manifold endowed with an $(\varepsilon)$-para Sasakian structure is called an $(\varepsilon)$-para Sasakian manifold $[6]$. 

For $\varepsilon = 1$ and $g$ Riemannian, $M$ is the usual para Sasakian manifold [26, 27]. For $\varepsilon = -1$, $g$ Lorentzian and $\xi$ replaced by $-\xi$, $M$ becomes a Lorentzian para Sasakian manifold [28].

For $(\varepsilon)$-para Sasakian manifold, it is easy to prove that

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X,$$

(3.7)

$$R(\xi,X)Y = \eta(Y)X - \varepsilon g(X,Y)\xi,$$

(3.8)

$$R(\xi,X)\xi = X - \eta(X)\xi,$$

(3.9)

$$R(X,Y,Z,\xi) = \eta(Y)g(X,Z) - \eta(X)g(Y,Z),$$

(3.10)

$$\eta(R(X,Y)Z) = \varepsilon(\eta(Y)g(X,Z) - \eta(X)g(Y,Z)),$$

(3.11)

$$S(X,\xi) = -(m-1)\eta(X),$$

(3.12)

$$Q\xi = -\varepsilon(m-1)\xi,$$

(3.13)

$$S(\xi,\xi) = -(m-1),$$

(3.14)

$$S(\varphi X,\varphi Y) = S(Y,Z) + (m-1)\eta(X)\eta(Y),$$

(3.15)

$$\nabla X\xi = \varepsilon \varphi X.$$  

(3.16)

For detail study of $(\varepsilon)$-para Sasakian manifold, see [6].

4 $\varphi$-$T$-Symmetric $(\varepsilon)$-Para Sasakian Manifold

We begin with the following definition.

**Definition 4.1.** An $(\varepsilon)$-para Sasakian manifold is said to be locally $\varphi$-$T$-symmetric manifold if

$$\varphi^2((\nabla_W T)(X,Y)Z) = 0,$$

(4.1)

for arbitrary vector fields $X, Y, Z, W$ orthogonal to $\xi$. If $X, Y, Z, W$ are arbitrary vector fields, then it is known as globally $\varphi$-$T$-symmetric manifold.

This notion of locally $\varphi$-symmetric was introduced by Takahashi for Sasakian manifolds [3].

**Theorem 4.2.** Let $M$ be a $m$-dimensional globally $\varphi$-$T$-symmetric $(\varepsilon)$-para Sasakian manifold. Then

(i) $M$ is Einstein manifold if $a_0 + (m-1)a_1 + a_2 + a_6 \neq 0$.

(ii) $M$ has constant scalar curvature if $a_0 + (m-1)a_1 + a_2 + a_6 = 0$ and $a_4 + (m-1)a_7 \neq 0$. 

Proof. Let $M$ be a $m$-dimensional globally $\varphi-T$-symmetric ($\varepsilon$)-para Sasakian manifold. Then by using (3.1) and (4.1), we have

$$(\nabla_w T)(X,Y)Z - \eta((\nabla_w T)(X,Y)Z)\xi = 0,$$

from which it follows that

$$g((\nabla_w T)(X,Y)Z, U) - \eta((\nabla_w T)(X,Y)Z)g(\xi, U) = 0. \quad (4.2)$$

Using (2.1) in (4.2), we obtain

$$0 = a_0 (\nabla_w R)(X,Y,Z,U) + a_1 (\nabla_w S)(Y,Z)g(X,U) + a_2 (\nabla_w S)(X,Z)g(Y,U) + a_3 (\nabla_w S)(X,Y)g(Z,U) + a_4 (\nabla_w S)(X,U)g(Y,Z) + a_5 (\nabla_w S)(Y,U)g(X,Z) + a_6 (\nabla_w S)(Z,U)g(X,Y) + a_7 (\nabla_w r)(g(Y,Z)g(X,U) - g(X,Z)g(Y,U)) + \eta(U) (a_0 (\nabla_w R)(X,Y,Z,\xi) + a_1 (\nabla_w S)(Y,Z)g(X,\xi) + a_2 (\nabla_w S)(X,Z)g(X,\xi) + a_3 (\nabla_w S)(X,Y)g(Z,\xi) + a_4 g(Y,Z) (\nabla_w S)(X,\xi) + a_5 g(X,Z) (\nabla_w S)(Y,\xi) + a_6 g(X,Y) (\nabla_w S)(Z,\xi) + a_7 (\nabla_w r)(g(Y,Z)g(X,\xi) - g(X,Z)g(Y,\xi))). \quad (4.3)$$

Let $\{e_i\}, i = 1, \ldots, m$ be an orthonormal basis of tangent space at any point of the manifold. Taking $X = U = e_i$ in (4.3), we get

$$0 = (a_0 + (m-1)a_1 + a_2 + a_3 + a_5 + a_6) (\nabla_w S)(Y,Z) + a_0 \varepsilon \sum_{i=1}^{m} (\nabla_w R)(e_i, Y,Z,\xi) g(e_i, \xi) + (a_4 + (m-1)a_7) (\nabla_w r)g(Y,Z) + a_7 (\nabla_w r)(g(Y,Z) - \varepsilon \eta(Y)\eta(Z)) - (a_2 + a_6) (\nabla_w S)(Z,\xi)\eta(Y) - (a_3 + a_5) (\nabla_w S)(Y,\xi)\eta(Z). \quad (4.4)$$

Putting $Z = \xi$ in (4.4), we have

$$0 = (a_0 + (m-1)a_1 + a_2 + a_6) (\nabla_w S)(Y,\xi) + a_0 \varepsilon \sum_{i=1}^{m} (\nabla_w R)(e_i, Y,\xi,\xi) g(e_i, \xi) + (a_4 + (m-1)a_7) (\nabla_w r)g(Y,\xi) - (a_2 + a_6) (\nabla_w S)(\xi,\xi)\eta(Y). \quad (4.5)$$

Since, we have

$$(\nabla_w R)(e_i, Y,\xi,\xi) = g((\nabla_w R)(e_i, Y)\xi, \xi) = g(\nabla_w R(e_i, Y)\xi, \xi) - g(R(\nabla_w e_i, Y)\xi, \xi) - g(R(e_i, Y)\nabla_w \xi, \xi), \quad (4.6)$$

at any point $p \in M$. We know that $\{e_i\}$ is an orthonormal basis, therefore $\nabla_w e_i = 0$ at $p$. Using (3.4) and (3.7) in (4.6), we have

$$(\nabla_w R)(e_i, Y,\xi,\xi) = g(\nabla_w R(e_i, Y)\xi, \xi) - g(R(e_i, Y)\nabla_w \xi, \xi). \quad (4.7)$$
By using the property of curvature tensor
\[ g(R(e_i, Y)\xi, \xi) = -g(R(\xi, \xi)Y, e_i) = 0, \]
we have
\[ g(\nabla W R(e_i, Y)\xi, \xi) + g(R(e_i, Y)\xi, \nabla W \xi) = 0. \tag{4.8} \]
By (4.7) and (4.8), we get
\[ (\nabla W R) (e_i, Y, \xi, \xi) = 0. \tag{4.9} \]
We know that
\[ (\nabla W S)(Y, \xi) = \nabla W S(Y, \xi) - S(\nabla W Y, \xi) - S(Y, \nabla W \xi). \tag{4.10} \]
Using (3.12), (3.16) in (4.10), we get
\[ (\nabla W S)(Y, \xi) = \nabla W (-m-1)\eta(Y) + (m-1)\eta (\nabla W Y, \varphi W - \varepsilon S(Y, \varphi W) = -(m-1)\varepsilon g(Y, \varphi W) - \varepsilon S(Y, \varphi W). \tag{4.11} \]
By (4.11), we have
\[ (\nabla W S)(\xi, \xi) = 0. \tag{4.12} \]
Using (4.9), (4.11), (4.12) in (4.5), we have
\[ 0 = (a_0 + (m-1)a_1 + a_2 + a_6) (-m-1)g(Y, \varphi W - \varepsilon S(Y, \varphi W)) + \varepsilon (a_4 + (m-1)a_7) (\nabla W r)\eta(Y). \tag{4.13} \]
Replacing \( Y \) by \( \varphi Y \) in (4.13) and using (3.2), (3.15), we get
\[ S(Y, W) = -\varepsilon (m-1)g(Y, W), \quad a_0 + (m-1)a_1 + a_2 + a_6 \neq 0. \]
If \( a_0 + (m-1)a_1 + a_2 + a_6 = 0 \) and \( a_4 + (m-1)a_7 \neq 0 \), then by (4.5), we have \( \nabla W r = 0 \), that is, \( r = \text{constant}. \)

**Remark 4.3.** The first condition of Theorem 4.2 is satisfied if \( T \in \{R, C, V, P, \mathcal{P}, \mathcal{M}, W^*_0, W_1, W^*_1, W_2, \ldots, W_6, W_9\} \) and second condition is satisfied if \( T \in \{\mathcal{C}, W_0, W_8\} \). None of the condition is satisfied.

**Theorem 4.4.** An Einstein manifold is globally \( \varphi-T \)-symmetric iff it is globally \( \varphi \)-symmetric and \( a_0 \neq 0 \).

**Proof.** By using (2.1) and (4.1), we have the result.

**Remark 4.5.** For all known curvature tensors \( a_0 \neq 0 \).
5 3-Dimensional Locally $\varphi$-$\mathcal{T}$-Symmetric $(\varepsilon)$-Para Sasakian Manifold

It is well known that in a 3-dimensional semi-Riemannian manifold the conformal curvature tensor $C$ vanishes, therefore

$$R(X, Y)Z = g(Y, Z)QX - g(X, Z)QY + S(Y, Z)X - S(X, Z)Y - \frac{r}{2}(g(Y, Z)X - g(X, Z)Y).$$

(5.1)

Take $Z = \xi$ in (5.1) and using (3.4), (3.7), (3.12), we get

$$\left(\frac{\varepsilon r}{2} + 1\right)(\eta(Y)X - \eta(X)Y) = \varepsilon(\eta(Y)QX - \eta(X)QY).$$

(5.2)

Putting $Y = \xi$ in (5.2) and using (3.13), we get

$$QX = \left(\frac{r}{2} + \varepsilon\right)X - \left(\frac{r}{2} + 3\varepsilon\right)\eta(X)\xi.$$  

(5.3)

Then by (5.3), we easily obtain

$$S(X, Y) = \left(\frac{r}{2} + \varepsilon\right)g(X, Y) - \left(\frac{\varepsilon r}{2} + 3\right)\eta(X)\eta(Y)$$

(5.4)

and

$$R(X, Y)Z = \left(\frac{r}{2} + 2\varepsilon\right)(g(Y, Z)X - g(X, Z)Y) + \left(\frac{\varepsilon r}{2} + 3\right)(\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X) + \left(\frac{r}{2} + 3\varepsilon\right)(g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi).$$

(5.5)

Lemma 5.1. A 3-dimensional $(\varepsilon)$-para Sasakian manifold is a manifold of constant curvature $-\varepsilon$ if and only if $r = -6\varepsilon$.

Corollary 5.2. Let $M$ be a 3-dimensional $(\varepsilon)$-para Sasakian manifold. Then

$$T(X, Y)Z = \left(\frac{r}{2} + \varepsilon\right)(a_0 + a_1 + a_4 + a_7r + \varepsilon a_0)g(Y, Z)X - \left(\frac{r}{2} + \varepsilon\right)(a_0 - a_2 - a_5 + a_7r + \varepsilon a_0)g(X, Z)Y + \left(\frac{\varepsilon r}{2} + 3\right)(a_0 + a_1)\eta(Y)\eta(Z)X + \left(\frac{\varepsilon r}{2} + 3\right)(a_0 + a_2)\eta(X)\eta(Z)Y + \left(\frac{r}{2} + 3\varepsilon\right)(a_0 - a_5)g(X, Z)\eta(Y)\xi - \left(\frac{r}{2} + 3\varepsilon\right)a_6g(X, Y)\eta(Z)\xi - \left(\frac{r}{2} + 3\varepsilon\right)(a_0 + a_4)g(Y, Z)\eta(X)\xi.$$  

(5.6)
Theorem 5.3. Let $M$ be a 3-dimensional $(\varepsilon)$-para Sasakian manifold. $M$ is locally $\phi$-T-symmetric manifold if and only if the scalar curvature $r$ is constant.

Proof. Let $M$ be a 3-dimensional $(\varepsilon)$-para Sasakian manifold. Differentiate covariantly on both sides of (5.6), we have

\[
(\nabla_{W}T)(X,Y)Z = \frac{\nabla_{W}r}{2}(a_{0} + a_{1} + a_{4} + 2a_{7})g(Y,Z)X
\]

\[
- \frac{\nabla_{W}r}{2}(a_{0} - a_{2} - a_{5} + 2a_{7})g(X,Z)Y
\]

\[
+ \frac{\nabla_{W}r}{2}(a_{3} + a_{6})g(X,Y)Z
\]

\[
- \frac{\nabla_{W}r}{2}a_{3}\eta(X)\eta(Y)Z
\]

\[
- \left(\frac{\varepsilon r}{2} + 3\right)a_{3}(\nabla_{W}\eta)(X)\eta(Y)Z
\]

\[
- \left(\frac{\varepsilon r}{2} + 3\right)a_{3}\eta(X)(\nabla_{W}\eta)(Y)Z
\]

\[
- \frac{\nabla_{W}r}{2}(a_{0} + a_{1})\eta(Y)\eta(Z)X
\]

\[
- \left(\frac{\varepsilon r}{2} + 3\right)(a_{0} + a_{1})(\nabla_{W}\eta)(Y)\eta(Z)X
\]

\[
- \left(\frac{\varepsilon r}{2} + 3\right)(a_{0} + a_{1})\eta(Y)(\nabla_{W}\eta)(Z)X
\]

\[
+ \frac{\nabla_{W}r}{2}(a_{0} - a_{2})\eta(X)\eta(Z)Y
\]

\[
+ \left(\frac{\varepsilon r}{2} + 3\right)(a_{0} - a_{2})(\nabla_{W}\eta)(X)\eta(Z)Y
\]

\[
+ \left(\frac{\varepsilon r}{2} + 3\right)(a_{0} - a_{2})\eta(X)(\nabla_{W}\eta)(Z)Y
\]

\[
+ \frac{\nabla_{W}r}{2}(a_{0} - a_{5})g(X,Z)\eta(Y)\xi
\]

\[
+ \left(\frac{r}{2} + 3\varepsilon\right)(a_{0} - a_{5})g(X,Z)(\nabla_{W}\eta)(Y)\xi
\]

\[
+ \left(\frac{r}{2} + 3\varepsilon\right)(a_{0} - a_{5})g(X,Z)\eta(Y)\nabla_{W}\xi
\]

\[
- \left(\frac{r}{2} + 3\varepsilon\right)(a_{0} + a_{4})g(Y,Z)(\nabla_{W}\eta)(X)\xi
\]

\[
- \left(\frac{r}{2} + 3\varepsilon\right)(a_{0} + a_{4})g(Y,Z)\eta(X)\nabla_{W}\xi
\]

\[
- \left(\frac{r}{2} + 3\varepsilon\right)a_{6}g(X,Y)\eta(Z)\nabla_{W}\xi
\]

\[
- \frac{\nabla_{W}r}{2}(a_{0} + a_{4})g(Y,Z)\eta(X)\xi
\]

\[
- \frac{\nabla_{W}r}{2}a_{6}g(X,Y)\eta(Z)\xi
\]

\[
- \left(\frac{r}{2} + 3\varepsilon\right)a_{6}g(X,Y)(\nabla_{W}\eta)(Z)\xi. \quad (5.7)
\]
Applying $\varphi^2$ on both sides of (5.7), we have

$$\varphi^2(\nabla_W T)(X,Y)Z = \frac{\nabla_W r}{2} (a_0 + a_1 + a_4 + 2a_7) g(Y,Z)(X - \eta(X)\xi)$$
$$- \frac{\nabla_W r}{2} (a_0 - a_2 - a_5 + 2a_7) g(X,Z)(Y - \eta(Y)\xi)$$
$$+ \frac{\nabla_W r}{2} (a_3 + a_6) g(X,Y)(Z - \eta(Z)\xi)$$
$$- \frac{\nabla_W r}{2} a_3 \eta(X)\eta(Y)(Z - \eta(Z)\xi)$$
$$- \left( \frac{\xi r}{2} + 3 \right) a_3 (\nabla_W \eta)(X)\eta(Y)(Z - \eta(Z)\xi)$$
$$- \left( \frac{\xi r}{2} + 3 \right) a_3 \eta(X)(\nabla_W \eta)(Y)(Z - \eta(Z)\xi)$$
$$- \left( \frac{\xi r}{2} + 3 \right) a_3 \eta(Y)(\nabla_W \eta)(Z)(X - \eta(X)\xi)$$
$$+ \frac{\nabla_W r}{2} (a_0 - a_2) \eta(X)\eta(Z)(Y - \eta(Y)\xi)$$
$$+ \left( \frac{\xi r}{2} + 3 \right) a_0 (\nabla_W \eta)(X)\eta(Y)(X - \eta(X)\xi)$$
$$+ \left( \frac{\xi r}{2} + 3 \right) a_0 \eta(X)(\nabla_W \eta)(Z)(Y - \eta(Y)\xi)$$
$$+ \left( \frac{\xi r}{2} + 3 \right) a_0 \eta(Y)(\nabla_W \eta)(X)(Z - \eta(Z)\xi)$$
$$+ \frac{\xi r}{2} (a_0 - a_3) g(X,Z)\eta(Y)\varphi^2 \nabla_W \xi$$
$$- \frac{\xi r}{2} (a_0 + a_4) g(Y,Z)\eta(X)\varphi^2 \nabla_W \xi.$$  \hspace{1cm} (5.8)

Using the fact that $X$, $Y$, $Z$ are horizontal vector fields in (5.8), we get

$$\varphi^2(\nabla_W T)(X,Y)Z = \frac{\nabla_W r}{2} (a_0 + a_1 + a_4 + 2a_7) g(Y,Z)X$$
$$- \frac{\nabla_W r}{2} (a_0 - a_2 - a_5 + 2a_7) g(X,Z)Y$$
$$+ \frac{\nabla_W r}{2} (a_3 + a_6) g(X,Y)Z.$$ \hspace{1cm} (5.9)

If one of them $a_0 + a_1 + a_4 + 2a_7$, $a_0 - a_2 - a_5 + 2a_7$ and $a_3 + a_6$ is not equal to zero, then by using (4.1), we get the result. \hfill $\square$

**Remark 5.4.** One of them $a_0 + a_1 + a_4 + 2a_7$, $a_0 - a_2 - a_5 + 2a_7$ and $a_3 + a_6$ is not equal to zero, for all the known curvature tensors.
6 \(\eta\)-Parallel Ricci Tensor

**Definition 6.1.** The Ricci tensor \(S\) of an \((\varepsilon)\)-para-Sasakian manifold is called \(\eta\)-parallel Ricci tensor if it satisfies

\[
(\nabla_X S)(\varphi Y, \varphi Z) = 0
\]

for all vector fields \(X, Y\) and \(Z\).

**Theorem 6.2.** In a 3-dimensional \((\varepsilon)\)-para Sasakian manifold with \(\eta\)-parallel Ricci tensor, the scalar curvature \(r\) is constant.

**Proof.** By equation (5.4), we get

\[
S(\varphi Y, \varphi Z) = \left(\frac{r^2 + \varepsilon}{2}\right) \left(g(Y, Z) - \varepsilon \eta(Y)\eta(Z) \right).
\]

Differentiating (6.1) covariantly with respect to \(X\), we get

\[
(\nabla_X S)(\varphi Y, \varphi Z) = \nabla_X r^2 \left(g(Y, Z) - \varepsilon \eta(Y)\eta(Z) \right) - \varepsilon \left(\frac{r^2 + \varepsilon}{2}\right) \left(\nabla_X \eta\right)(Y)\eta(Z) + \eta(Y) \left(\nabla_X \eta\right)(Z).
\]

Suppose the Ricci tensor is \(\eta\)-parallel. Then from the above, we obtain

\[
\frac{\nabla_X r^2}{2} \left(g(Y, Z) - \varepsilon \eta(Y)\eta(Z) \right) = \varepsilon \left(\frac{r^2 + \varepsilon}{2}\right) \left(\nabla_X \eta\right)(Y)\eta(Z) + \eta(Y) \left(\nabla_X \eta\right)(Z).
\]

(6.2)

Let \(\{e_i\}, i = 1, 2, 3\) be the orthonormal basis of tangent space at each point of the manifold. Taking \(Y = e_i = Z\) in (6.2), we have \(\nabla_X r = 0\). Hence scalar curvature \(r\) is constant.

From Theorems 5.3 and 6.2, we can state the following:

**Corollary 6.3.** A 3-dimensional \((\varepsilon)\)-para Sasakian manifold with \(\eta\)-parallel Ricci tensor is locally \(\varphi-\mathcal{T}\)-symmetric.

7 Example of a Locally \(\varphi-\mathcal{T}\)-Symmetric \((\varepsilon)\)-Para Sasakian Manifold of Dimension 3

Consider the 3-dimensional manifold \(M = \{(x, y, z) \in \mathbb{R}^3, z \neq 0\}\), where \((x, y, z)\) are the standard coordinates of \(\mathbb{R}^3\). The vector fields

\[
e_1 = z \frac{\partial}{\partial x}, \quad e_2 = z \frac{\partial}{\partial y}, \quad e_3 = -z \frac{\partial}{\partial z}\]

are linearly independent at each point of \(M\). Let \(g\) be the semi-Riemannian metric defined by

\[
g(e_1, e_3) = 0, \quad g(e_1, e_2) = 0, \quad g(e_2, e_3) = 0, \quad g(e_1, e_1) = 1, \quad g(e_2, e_2) = 1, \quad g(e_3, e_3) = \varepsilon.
\]
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where $\varepsilon = \pm 1$. Let $\eta$ be the 1-form defined by $\eta(Z) = \varepsilon g(Z,e_3)$ for any $Z \in TM$. Let $\varphi$ be the $(1,1)$-tensor field defined by

$$
\varphi e_1 = \varepsilon e_1, \quad \varphi e_2 = \varepsilon e_2, \quad \varphi e_3 = 0.
$$

Using the linearity of $\varphi$ and $g$, we have

$$
\varphi^2 X = X - \eta(X)e_3,
$$

$$
\eta(e_3) = 1,
$$

$$
g(\varphi X, \varphi Y) = g(X,Y) - \varepsilon \eta(X)\eta(Y),
$$

$$
(\nabla_X \varphi) Y = -g(\varphi X, \varphi Y)e_3 - \varepsilon \eta(Y)\varphi^2 X,
$$

for any $X, Y \in TM$. Then for $\xi = e_3$, the structure $(\varphi, \xi, \eta, g, \varepsilon)$ defines an ($\varepsilon$)-para Sasakian structure on $M$. Let $\nabla$ be the Levi-Civita connection with respect to the metric $g$. Then we have

$$
[e_1, e_2] = 0, \quad [e_1, e_3] = e_1, \quad [e_1, e_2] = e_2.
$$

The Koszul’s formula for the Riemannian connection $\nabla$ of the metric $g$ is given by

$$
2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y)
$$

$$
- g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]).
$$

By using Koszul’s formula, we have

$$
\nabla_{e_1} e_1 = -\varepsilon e_3, \quad \nabla_{e_2} e_1 = 0, \quad \nabla_{e_3} e_1 = -e_1,
$$

$$
\nabla_{e_1} e_2 = 0, \quad \nabla_{e_2} e_2 = -\varepsilon e_3, \quad \nabla_{e_3} e_2 = -e_2,
$$

$$
\nabla_{e_1} e_3 = e_1, \quad \nabla_{e_2} e_3 = e_2, \quad \nabla_{e_3} e_3 = 0.
$$

From the above results, it is easy to check that equations (3.1), (3.2), (3.3), (3.4), (3.5) and (3.6) hold. Hence the manifold is an ($\varepsilon$)-para Sasakian manifold.

Using the above results, it is easy to find out the following results

$$
R(e_1, e_2)e_1 = \varepsilon e_2, \quad R(e_2, e_3)e_1 = 0, \quad R(e_1, e_3)e_1 = 2\varepsilon e_3,
$$

$$
R(e_1, e_2)e_2 = -\varepsilon e_1, \quad R(e_2, e_3)e_2 = 2\varepsilon e_3, \quad R(e_1, e_3)e_2 = 0,
$$

$$
R(e_1, e_2)e_3 = 0, \quad R(e_2, e_3)e_3 = 0, \quad R(e_1, e_3)e_3 = 0.
$$

Then

$$
S(e_1, e_1) = -(\varepsilon + 2), \quad S(e_2, e_2) = -(\varepsilon + 2), \quad S(e_3, e_3) = 0,
$$

and

$$
r = -2(\varepsilon + 2).
$$

Hence the scalar curvature $r$ is constant. From Theorem 5.3 $M$ is a 3-dimensional locally $\varphi$-$T$-symmetric ($\varepsilon$)-para Sasakian manifold.
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