Rate of Convergence of P-Iteration and S-Iteration for Continuous Functions on Closed Intervals

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Abstract: In this paper, we first give a necessary and sufficient condition for convergence of P-iteration to a fixed point of continuous functions on an arbitrary interval and prove equivalence of P-iteration and S-iteration. We also compare the rate of convergence between P-iteration and S-iteration. Some numerical examples for comparing the rate of convergence of those two methods are also given.

Keywords: rate of convergence; P-iteration; S-iteration; continuous function; closed interval.

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1 Introduction

Let $E$ be a closed interval on the real line and $f : E \to E$ be a continuous function. A point $p \in E$ is a fixed point of $f$ if $f(p) = p$. We denote by $F(f)$ the set of fixed points of $f$. It is known that if $E$ also bounded, then $F(f)$ is nonempty. The Mann iteration (see [1]) is defined by $u_1 \in E$ and

$$u_{n+1} = (1 - \alpha_n) u_n + \alpha_n f(u_n)$$

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for all \( n \geq 1 \), where \( \{\alpha_n\}_{n=1}^{\infty} \) is a sequence in \([0,1]\), and will be denoted by \( M(u_1, \alpha_n, f) \). The Ishikawa iteration (see [2]) is defined by \( s_1 \in E \) and

\[
\begin{align*}
  t_n &= (1 - \beta_n) s_n + \beta_n f(s_n) \\
  s_{n+1} &= (1 - \alpha_n) s_n + \alpha_n f(t_n)
\end{align*}
\]

(1.2)

for all \( n \geq 1 \), where \( \{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty} \) are sequences in \([0,1]\), and will be denoted by \( I(s_1, \alpha_n, \beta_n, f) \). The S-iteration (see [3]) is defined by \( q_1 \in E \) and

\[
\begin{align*}
  r_n &= (1 - \beta_n) q_n + \beta_n f(q_n) \\
  q_{n+1} &= (1 - \alpha_n) f(q_n) + \alpha_n f(r_n)
\end{align*}
\]

(1.3)

for all \( n \geq 1 \), where \( \{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty} \) are sequences in \([0,1]\), and will be denoted by \( S(q_1, \alpha_n, \beta_n, f) \).

It was shown in [4] that the Mann and Ishikawa iterations are equivalent for the class of Zamfirescu operators. In 2006, Babu and Prasad [5] showed that the Mann iteration converges faster than the Ishikawa iteration for the class of operators. Two years later, Qing and Rhoades [6] provided an example to show that the claim of Babu and Prasad is false. In 2013, Kosol [3] showed that the S-iteration converges faster than the Ishikawa iteration on an arbitrary interval. In 2011, Phuengrattana and Suantai [7] introduced a new three-step iteration, called SP-iteration, and showed that it converges faster than Mann, Ishikawa, Noor-iterations.

Motivated by the above results, we modify S and SP-iterations for construction a new iteration as follows: The P-iteration is defined by \( x_1 \in E \) and

\[
\begin{align*}
  z_n &= (1 - \gamma_n) x_n + \gamma_n f(x_n) \\
  y_n &= (1 - \beta_n) z_n + \beta_n f(z_n) \\
  x_{n+1} &= (1 - \alpha_n) f(z_n) + \alpha_n f(y_n)
\end{align*}
\]

(1.4)

for all \( n \geq 1 \), where \( \{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty} \) and \( \{\gamma_n\}_{n=1}^{\infty} \) are sequences in \([0,1]\), and will be denoted by \( P(x_1, \alpha_n, \beta_n, \gamma_n, f) \).

In this paper, we give a necessary and sufficient condition for the convergence of the P-iteration of continuous non-decreasing functions on an arbitrary interval. We also prove that if the S-iteration converges, then the P-iteration converges and converges faster than the S-iteration for the class of continuous and non-decreasing functions. Moreover, we present the numerical examples for the P-iteration to compare with the Ishikawa and the S-iterations.

### 2 Preliminaries

In this section we recall some lemmas, definition, theorems and known results in the existing literature on this concept.
Lemma 2.1 (3). Let $E$ be a closed interval on the real line and $f : E \to E$ be a continuous function. Let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ be sequences in $[0, 1]$. For $q_1 \in E$, let $\{q_n\}_{n=1}^{\infty}$ be the sequence defined by (1.3). Then the following hold:

(i) If $f(q_1) < q_1$, then $f(q_n) \leq q_n$ for all $n \geq 1$ and $\{q_n\}_{n=1}^{\infty}$ is non-increasing.

(ii) If $f(q_1) > q_1$, then $f(q_n) \geq q_n$ for all $n \geq 1$ and $\{q_n\}_{n=1}^{\infty}$ is non-decreasing.

Proposition 2.2 (3). Let $E$ be a closed interval on the real line and $f : E \to E$ be a continuous and non-decreasing function such that $F(f)$ is nonempty and bounded with $q_1 \geq \sup \{p \in E : p = f(p)\}$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0, 1]$. If $f(q_1) > q_1$, then the sequence $\{q_n\}$ defined by S-iteration does not converge to a fixed point of $f$.

Proposition 2.3 (3). Let $E$ be a closed interval on the real line and $f : E \to E$ be a continuous and non-decreasing function such that $F(f)$ is nonempty and bounded with $q_1 \leq \sup \{p \in E : p = f(p)\}$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0, 1]$. If $f(q_1) < q_1$, then the sequence $\{q_n\}$ defined by S-iteration does not converge to a fixed point of $f$.

Definition 2.4 (7). Let $E$ be a closed interval on the real line and $f : E \to E$ be a continuous function. Suppose that $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are two iterations which converge to the fixed point $p$ of $f$. Then $\{x_n\}_{n=1}^{\infty}$ is said to converge faster than $\{y_n\}_{n=1}^{\infty}$ if $|x_n - p| \leq |y_n - p|$ for all $n \geq 1$.

Theorem 2.5 (3). Let $E$ be a closed interval on the real line and $f : E \to E$ be a continuous and non-decreasing function such that $F(f)$ is nonempty and bounded. For $s_1 = q_1 \in E$, let $\{s_n\}$ and $\{q_n\}$ be the sequences defined by (1.2) and (1.3), respectively. If the Ishikawa iteration $\{S_n\}$ converges to $p \in F(f)$, then the S-iteration $\{q_n\}$ converges to $p$. Moreover, the S-iteration converges faster than the Ishikawa iteration.

3 Main Results

We first give some useful facts for our main results.

Lemma 3.1. Let $E$ be a closed interval on the real line and $f : E \to E$ be a continuous and non-decreasing function. Let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ be sequences in $[0, 1]$. For $x_1 \in E$, let $\{x_n\}_{n=1}^{\infty}$ be defined by P-iteration. Then the following hold:

(i) If $f(x_1) < x_1$, then $f(x_n) \leq x_n$ for all $n \geq 1$ and $\{x_n\}_{n=1}^{\infty}$ is non-increasing.

(ii) If $f(x_1) > x_1$, then $f(x_n) \geq x_n$ for all $n \geq 1$ and $\{x_n\}_{n=1}^{\infty}$ is non-decreasing.
Since induction, we can conclude that \( f^n(y) \leq z_k \). Since \( f \) is non-decreasing, we have \( f(z_k) \leq y_k \leq z_k \). By (1.4), we get \( f(z_k) \leq y_k \leq z_k \). Since \( f \) is non-decreasing, we have \( f(y_k) \leq f(z_k) \leq y_k \leq z_k \).

It follows from (1.4), that \( f(y_k) \leq x_{k+1} \leq f(z_k) \). This implies \( x_{k+1} \leq f(z_k) \leq y_k \).

Since \( f \) is non-decreasing, we have \( f(x_{k+1}) \leq f(y_k) \). Thus \( f(x_{k+1}) \leq x_{k+1} \).

By induction, we can conclude that \( f(x_n) \leq x_n \) for all \( n \geq 1 \). This together with (1.4), we have \( y_n \leq z_n \leq x_n \) for all \( n \geq 1 \). Since \( f \) is non-decreasing, we have \( f(y_n) \leq f(z_n) \leq f(x_n) \) for all \( n \leq 1 \). It follows that \( x_{n+1} = (1 - \alpha_n)f(z_n) + \alpha_n f(y_n) \leq f(z_n) \leq f(x_n) \leq x_n \) for all \( n \geq 1 \). Thus \( \{x_n\} \) is non-increasing.

(ii) By using the same argument as in (i), We obtain the desired result.

**Theorem 3.2.** Let \( E \) be a closed interval on the real line and \( f : E \to E \) be a continuous and non-decreasing function. For \( x_1 \in E \), let \( \{x_n\}_{n=1}^\infty \) be defined by (1.4), where \( \{\alpha_n\}_{n=1}^\infty \), \( \{\beta_n\}_{n=1}^\infty \) and \( \{\gamma_n\}_{n=1}^\infty \) are sequences in \([0,1]\) and \( \lim_{n \to \infty} \beta_n = \lim_{n \to \infty} \gamma_n = 0 \). Then \( \{x_n\}_{n=1}^\infty \) is bounded if and only if \( \{x_n\}_{n=1}^\infty \) converges to a fixed point of \( f \).

**Proof.** If \( \{x_n\} \) is convergent, then it is bounded. Now, assume that \( \{x_n\} \) is bounded. We will show that \( \{x_n\} \) is convergent. If \( f(x_1) = x_1 \), by (1.4) we have

\[
  z_1 = (1 - \gamma_1)x_1 + \gamma_1 f(x_1) = x_1 \\
  y_1 = (1 - \beta_1)z_1 + \beta_1 f(z_1) = x_1 \\
  x_2 = (1 - \alpha_1)f(z_1) + \alpha_1 f(y_1) = x_1.
\]

We can show by induction that \( x_n = x_1 \) for all \( n \geq 1 \). Thus \( \{x_n\} \) is convergent.

Suppose that \( f(x_1) \neq x_1, f(x_1) < x_1 \) or \( f(x_1) > x_1 \). By Lemma 3.1, we obtain that \( \{x_n\} \) is non-increasing or non-decreasing. Since \( \{x_n\} \) is bounded, it implies that \( \{x_n\} \) is convergent. Next, we prove that \( \{x_n\} \) converges to a fixed point of \( f \). Let \( \lim_{n \to \infty} x_n = p \) for some \( p \in E \). By continuity of \( f \) and \( \{x_n\} \) is bounded, we have \( \{f(x_n)\} \) is bounded. By (1.4), we obtain \( z_n = (1 - \gamma_n)x_n + \gamma_n f(x_n) = x_n + \gamma_n(f(x_n) - x_n) \). Since \( \lim_{n \to \infty} \gamma_n = 0 \), we have \( \lim_{n \to \infty} z_n = \lim_{n \to \infty} x_n = p \).

By continuity of \( f \) and \( \{x_n\} \) is bounded, we have \( \{z_n\} \) and \( \{f(z_n)\} \) are bounded.

By (1.4), we get \( y_n = (1 - \beta_n)z_n + \beta_n f(z_n) = z_n + \beta_n(f(z_n) - z_n) \).

Since \( \lim_{n \to \infty} \beta_n = 0 \), we have \( \lim_{n \to \infty} y_n = \lim_{n \to \infty} z_n = p \).

By continuity of \( f \), we have \( \lim_{n \to \infty} (f(y_n) - f(z_n)) = f(p) - f(p) = 0 \).

From \( x_{n+1} = f(z_n) + \alpha_n(f(y_n) - f(z_n)) \) and continuity of \( f \), we have

\[
  p = \lim_{n \to \infty} x_{n+1} \\
  = \lim_{n \to \infty} f(z_n) + \lim_{n \to \infty} \alpha_n(f(y_n) - f(z_n)) \\
  = \lim_{n \to \infty} f(z_n) \\
  = f(p).
\]

Hence \( p \) is a fixed point of \( f \) and \( \{x_n\} \) converge to \( p \).
Lemma 3.3. Let $E$ be a closed interval on the real line and $f : E \to E$ be a continuous and non-decreasing function. For $x_1 \in E$, let $\{x_n\}_{n=1}^{\infty}$ be the $P$-iteration defined by (1.4), where $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ are sequences in $[0,1]$. Then we have the following:

(i) If $p \in F(f)$ with $x_1 > p$, then $x_n \geq p$ for all $n \geq 1$.

(ii) If $p \in F(f)$ with $x_1 < p$, then $x_n \leq p$ for all $n \geq 1$.

Proof. (i) Suppose that $p \in F(f)$ and $x_1 > p$. Since $f$ is non-decreasing, we have $f(x_1) \geq f(p) = p$. By (1.4), we get

$$z_1 = (1 - \gamma_1)x_1 + \gamma_1 f(x_1) \geq (1 - \gamma_1)p + (\gamma_1)p = p.$$ 

Thus $f(z_1) \geq f(p) = p$. From (1.4), we have

$$y_1 = (1 - \beta_1)z_1 + \beta_1 f(z_1) \geq (1 - \beta_1)p + (\beta_1)p = p.$$ 

Thus $f(y_1) \geq f(p) = p$. Again (1.4), implies that

$$x_2 = (1 - \alpha_1)f(z_1) + \alpha_1 f(y_1) \geq (1 - \alpha_1)p + (\alpha_1)p = p.$$ 

Assume that $x_k \geq p$ for $k > 2$. Thus $f(x_k) \geq f(p) = p$.

By (1.4), we have

$$z_k = (1 - \gamma_k)x_k + \gamma_k f(x_k) \geq (1 - \gamma_k)p + (\gamma_k)p = p.$$ 

Thus $f(z_k) \geq f(p) = p$. This implies

$$y_k = (1 - \beta_k)z_k + \beta_k f(z_k) \geq (1 - \beta_k)p + \beta_k p = p.$$ 

Hence $f(y_k) \geq f(p) = p$. It follows that

$$x_{k+1} = (1 - \alpha_k)f(z_k) + \alpha_k f(y_k) \geq (1 - \alpha_k)p + \alpha_k p = p.$$ 

By induction, we can conclude that $x_n \geq p$ for all $n \geq 1$.

(ii) By using the same argument as in (i), we can show that $x_n \leq p$ for all $n \geq 1$.

Lemma 3.4. Let $E$ be a closed interval on the real line and $f : E \to E$ be a continuous and non-decreasing function. For $x_1 \in E$, let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ be sequences in $[0,1]$. For $x_1 = q_1 \in E$, let $\{q_n\}_{n=1}^{\infty}$ and $\{x_n\}_{n=1}^{\infty}$ be sequences defined by (1.3) and (1.4) respectively. Then we have the following:

(i) If $f(q_1) < q_1$, then $x_n \leq q_n$ for all $n \geq 1$.

(ii) If $f(q_1) > q_1$, then $x_n \geq q_n$ for all $n \geq 1$.

Proof. (i) Let $f(q_1) < q_1$. Since $x_1 = q_1$, we get $f(x_1) < x_1$. First, we show that $x_n \leq q_n$ for all $n \geq 1$.

From (1.4), we get $f(x_1) \leq z_1 \leq x_1$. Since $f$ is non-decreasing, we have $f(z_1) \leq f(x_1) \leq z_1 \leq x_1$.

By (1.4), we have $f(y_1) \leq y_1 \leq z_1$. Since $f$ is non-decreasing, we obtain

$$f(y_1) \leq f(z_1) \leq f(x_1) \leq z_1 \leq x_1.$$
From (1.3) and (1.4), we get $z_1 - q_1 = (1 - \gamma_1)x_1 + \gamma_1 f(x_1) - q_1 = \gamma_1(f(x_1) - x_1) \leq 0$. Thus $z_1 \leq q_1$. Since $f$ is non-decreasing, we have $f(z_1) \leq f(q_1)$.

By (1.3) and (1.4), we get $y_1 = (1 - \beta_1)(z_1 - q_1) + \beta_1(f(z_1) - f(q_1)) \leq 0$.

Thus $y_1 \leq r_1$. Since $f$ is non-decreasing, we have $f(y_1) \leq f(r_1)$. By (1.3) and (1.4), it follows that

$$x_2 - q_2 = (1 - \alpha_1)[f(z_1) - f(q_1)] + \alpha_1[f(y_1) - f(r_1)] \leq 0.$$ 

Thus $x_2 \leq q_2$. Assume that $x_k \leq q_k$. Thus $f(x_k) \leq f(q_k)$. By Lemma 2.1 $f(q_k) \leq q_k$ and by (1.4), Lemma 3.1 $f(x_k) \leq x_k$. This implies $f(x_k) \leq z_k \leq x_k \leq q_k$. Since $f$ is non-decreasing, we have $f(z_k) \leq f(q_k)$. By (1.3) and (1.4), it follows that

$$y_k - r_k = (1 - \beta_k)(z_k - q_k) + \beta_k(f(z_k) - f(q_k)) \leq 0.$$ 

Thus $y_k \leq r_k$. Since $f$ is non-decreasing, we have $f(y_k) \leq f(r_k)$ it follows that

$$x_{k+1} - q_{k+1} = (1 - \alpha_k)[f(z_k) - f(q_k)] + \alpha_k[f(y_k) - f(r_k)] \leq 0.$$ 

By Mathematical induction, we obtain $x_n \leq q_n$ for all $n \geq 1$.

(ii) By using the same argument as in (i), we obtain the desired result. 

The next two propositions show that convergence of P-iteration depends on how far the initial point from the fixed point set.

**Proposition 3.5.** Let $E$ be a closed interval on the real line and $f : E \to E$ be a continuous and non-decreasing function such that $F(f)$ is nonempty and bounded with $x_1 < \inf\{p \in E : p = f(p)\}$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $[0, 1]$. If $f(x_1) < x_1$, then the sequence $\{x_n\}$ defined by P-iteration does not converge to a fixed point of $f$.

**Proof.** By Lemma 3.1(i), we have that $\{x_n\}$ is non-increasing. Since the initial point $x_1 < \inf\{p \in E : p = f(p)\}$, it follows that $\{x_n\}$ does not converge to a fixed point of $f$. 

**Proposition 3.6.** Let $E$ be a closed interval on the real line and $f : E \to E$ be a continuous and non-decreasing function such that $F(f)$ is nonempty and bounded with $x_1 > \sup\{p \in E : p = f(p)\}$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $[0, 1]$. If $f(x_1) > x_1$, then the sequence $\{x_n\}$ defined by P-iteration does not converge to a fixed point of $f$.

**Proof.** By Lemma 3.1(ii), we have that $\{x_n\}$ is non-decreasing. Since the initial point $x_1 > \sup\{p \in E : p = f(p)\}$, it follows that $\{x_n\}$ does not converge to a fixed point of $f$. 

**Theorem 3.7.** Let $E$ be a closed interval on the real line and $f : E \to E$ be a continuous and non-decreasing function such that $F(f)$ is nonempty and bounded. For $q_1 = x_1 \in E$, let $\{q_n\}$ and $\{x_n\}$ be the sequences defined by (1.3) and (1.4), respectively. If the S-iteration $\{q_n\}$ converges to $p \in F(f)$, then the P-iteration $\{x_n\}$ converges to $p$. Moreover, the P-iteration converges faster than the S-iteration.

**Proof.** Suppose the S-iteration $\{q_n\}$ converges to $p \in F(f)$. Put $l = \inf\{x \in E : x = f(x)\}$ and $u = \sup\{x \in E : x = f(x)\}$. We devide our proof into the following three cases:
Case 1: \( q_1 = x_1 > u \). By Proposition 2.2 and Proposition 3.6, we get \( f(q_1) < q_1 \) and \( f(x_1) < x_1 \). By Lemma 3.4 (i), we have \( x_n \leq q_n \) for all \( n \geq 1 \). By continuity of \( f \), we have \( f(u) = u \), so \( u = f(u) \leq f(x_1) < x_1 \). This implies by (1.4) that \( f(x_1) \leq z_1 \leq x_1 \), so \( u \leq z_1 \leq x_1 \). Since \( f \) is non-decreasing, we have \( u = f(u) \leq f(z_1) \leq f(x_1) \leq z_1 \leq x_1 \). It follows by (1.4), that \( y_1 = (1-\beta_1)z_1 + \beta_1 f(z_1) \leq z_1 \). Since \( f \) is non-decreasing, we have \( u \leq f(y_1) \leq f(z_1) \leq f(x_1) \leq z_1 \leq x_1 \) and \( u \leq f(y_1) \leq f(z_1) \). By mathematical induction, we can show that \( u \leq x_n \) for all \( n \geq 1 \). Hence, we have \( p \leq x_n \leq q_n \) for all \( n \geq 1 \), which implies \( |x_n - p| \leq |q_n - p| \) for all \( n \geq 1 \). Thus \( x_n \to p \) and the P-iteration converges to \( p \) faster than the S-iteration.

Case 2: \( q_1 = x_1 < l \). By Proposition 2.3 and Proposition 3.5, we get \( f(q_1) > q_1 \) and \( f(x_1) > x_1 \). By Lemma 3.4 (ii), we have \( x_n \geq q_n \) for all \( n \geq 1 \). We note that \( x_1 < l \), by (1.4) and mathematical induction, we can show that \( x_n < l \) for all \( n \geq 1 \). So \( q_n \leq x_n \leq p \) for all \( n \geq 1 \). Hence \( |x_n - p| \leq |q_n - p| \). It follows that \( x_n \to p \) and the P-iteration converges to \( p \) faster than the S-iteration.

Case 3: \( l < q_1 = x_1 < u \). Suppose that \( f(x_1) \neq x_1 \). If \( f(x_1) < x_1 \), by Lemma 2.1 (i), we have that \( \{q_n\} \) is non-increasing. It follows that \( p \leq q_n \) for all \( n \geq 1 \). By Lemma 3.3 (i) and Lemma 3.4 (i), we get \( p \leq x_n \leq q_n \) for all \( n \geq 1 \). This implies \( |x_n - p| \leq |q_n - p| \). It follows that \( x_n \to p \) and the P-iteration converges to \( p \) faster than the S-iteration.

If \( f(x_1) > x_1 \), by Lemma 2.1 (ii), we have that \( \{q_n\} \) is non-decreasing. This implies \( q_n \leq p \) for all \( n \geq 1 \). By Lemma 3.3 (ii) and Lemma 3.4 (ii), we get \( q_n \leq x_n \leq p \) for all \( n \geq 1 \). It follows that \( |x_n - p| \leq |q_n - p| \) for all \( n \geq 1 \). Hence \( x_n \to p \) and the P-iteration converges to \( p \) faster than the S-iteration.

**Example 3.8.** Let \( f : [0, \infty) \to [0, \infty) \) be defined by \( f(x) = \frac{x^2 + 3}{4} \). Then \( f \) is a continuous and non-decreasing function. The comparisons of the convergence of the Ishikawa iteration, S-iteration and the P-iteration to the exact fixed point \( p = 1 \) are given in Table 1, with the initial point \( x_1 = q_1 = s_1 = 2 \) and \( \alpha_n = \frac{1}{n} \), \( \beta_n = \gamma_n = \frac{1}{4n} \).

| \( n \) | \( n \) | \( q_n \) | \( x_n \) | \( |f(x_n) - x_n| \) |
|---|---|---|---|---|
| 3 | 1.540872070 | 1.228210401 | 1.150245305 | 0.129225591 |
| ... | ... | ... | ... | ... |
| 26 | 1.364330605 | 1.000000032 | 1.000000002 | 2.56567E-09 |
| 27 | 1.361571178 | 1.000000016 | 1.000000001 | 1.23348E-09 |
| 28 | 1.358927101 | 1.000000008 | 1.000000001 | 5.93889E-09 |

Table 1

Comparison of rate of convergence of the Ishikawa iteration, S-iteration and P-iteration for the given function in Example 3.8. From Table 1, we see that the P-iteration converges to \( p = 1 \) faster than the Ishikawa and S-iterations.
Example 3.9. Let \( f : [0, 5] \to [0, 5] \) be defined by \( f(x) = \sqrt{x^2 + 4} \). Then \( f \) is a continuous and non-decreasing function. The comparisons of the convergence of the Ishikawa iteration, S-iteration and the P-iteration to the exact fixed point \( p = 2 \) are given in Table 2, with the initial point \( x_1 = q_1 = s_1 = 3 \) and \( \alpha_n = \beta_n = \gamma_n = \frac{1}{n} \).

| \( n \) | \( s_n \) | \( q_n \) | \( x_n \) | \( |f(x_n) - x_n| \) |
|---|---|---|---|---|
| 3  | 2.055372105 | 2.010415225 | 2.001802129 | 0.005000393 |
| ... | ... | ... | ... | ... |
| 12 | 2.021489268 | 2.0000000462 | 2.000000033 | 6.9311E-08 |
| 13 | 2.020359903 | 2.0000000153 | 2.000000010 | 2.17191E-09 |
| 14 | 2.019368059 | 2.0000000051 | 2.000000003 | 6.84134E-09 |
| 15 | 2.018488772 | 2.0000000017 | 2.000000001 | 2.16447E-09 |

Table 2

Comparison of rate of convergence of the Ishikawa iteration, S-iteration and P-iteration for the given function in Example 3.9. From Table 2, we see that the P-iteration converges to \( p = 2 \) faster than the Ishikawa and S-iterations.

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References


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