On Characterizations of General Helices for Ruled Surfaces in the Pseudo-Galilean Space $G^1_3$-(PART II)

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Abstract: In this paper, we obtained characterizations of a curve with respect to the Frenet frame of Ruled surfaces in the 3-dimensional Pseudo-Galilean space $G^1_3$.

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1 Introduction

T. Ikawa obtained in [6] the following characteristic ordinary differential equation

$$\nabla X \nabla X \nabla X X - K \nabla X X = 0, \quad K = k^2 - \tau^2$$

for the circular helix which corresponds to the case that the curvatures $k$ and $\tau$ of a time-like curve $\alpha$ on the Lorentzian manifold $M$ are constant.

N. Ekmekçi and H. H. Hacisalihoglu generalized in [4]. T. Ikawa’s this result, i.e. $k$ and $\tau$ are variable, but $\frac{k}{\tau}$ is constant.

Recently, N. Ekmekçi and K. İlarslan obtained characterizations of timelike null helices in terms of principal normal or binormal vector fields [5].

Furthermore, M. Bektas [1] obtained characterizations of a curve with respect to the Frenet frame of ruled surfaces in the 3-dimensional pseudo-Galilean space $G^1_3$.

In this paper, we obtained characterizations of helices in terms of principal normal vector fields and another two characterizations for a curve with respect to the Frenet frame of Ruled surfaces in the 3-dimensional Pseudo-Galilean space $G^1_3$.

2 Preliminaries

We will use the same notations and terminologies as in [3] unless otherwise stated. The pseudo-Galilean geometry is one of the real Cayley-Klein geometries.
(of projective signature $(0,0,+,-)$). The absolute of the pseudo-Galilean geometry is an ordered triple \( \{w, f, I\} \) where \( w \) is the ideal (absolute) plane, \( f \) is a line in \( w \) and \( I \) is the fixed hyperbolic involution of points of \( f \) [2].

A vector \( X(x, y, z) \) is said to \textit{non isotropic} if \( x \neq 0 \). All unit non-isotropic vectors are of the form \((1, y, z)\). For isotropic vectors \( x = 0 \) holds. There are four types of isotropic vectors: space-like \((y^2 - z^2 > 0)\), time-like \((y^2 - z^2 < 0)\) and two types of lightlike \((y = \pm z)\) vectors. A non-lightlike isotropic vector is a unit vector if \( y^2 - z^2 = \pm 1 \).

A trihedron \((T_0; e_1, e_2, e_3)\) with a proper origin \(T_0(x_0, y_0, z_0) \sim (1 : x_0 : y_0 : z_0)\), is orthonormal in pseudo-Galilean sense iff the vectors \( e_1, e_2, e_3 \) are of following form
\[ e_1 = (1, y, z), \quad e_2 = (0, y_2, z_2), \quad e_3 = (0, \epsilon z_2, y_2), \] with \( y_2^2 - z_2^2 = \delta \), where \( \epsilon, \delta \) is +1 or -1.

Such trihedron \((T_0; e_1, e_2, e_3)\) is called \textit{positively oriented} if for its vectors \( \det(e_1, e_2, e_3) = 1 \) holds i.e. if \( y_2^2 - z_2^2 = \epsilon \).

\section{Ruled Surfaces in the Galilean Space}

A general equation of a ruled surface \( G^3 \) is
\[ x(u, v) = r(u) + va(u), \quad v \in \mathbb{R}; \ r, a \in \mathbb{C}^3 \] (3.1)
where the curve \( r \) does not line in a pseudo-Euclidean plane and is called a \textit{directrix}. The curve \( r \) is given by
\[ r(u) = (u, y(u), z(u)). \] (3.2)
This means that the curve \( r \) is parametrized by the pseudo-Galilean arc length. Further, the generator vector field is of the form
\[ a(u) = (1, a_2(u), a_3(u)). \] (3.3)

Notice that under the given assumptions all tangent planes of ruled surfaces are isotropic.

According to the absolute figure, we distinguish two types of ruled surfaces in \( G^3 \). More about ruled surface in \( G^3 \) can be found in [3].

**Type I**: The equation of a ruled surface of type I in \( G^3 \) is
\[ \begin{cases} \ x(u, v) = (u, y(u), z(u)) + v(1, a_2(u), a_3(u)), \\ y, z, a_2, a_3 \in \mathbb{C}^3, \quad u \in I \subseteq \mathbb{R}, \quad v \in \mathbb{R}. \end{cases} \] (3.4)
The ruled surfaces of type I are non-conoidal and conoidal surfaces whose directional straight line at infinity is not the absolute line. The striction curve of these surfaces does not lie in a pseudo-Euclidean space.
The associated trihedron of a ruled surface of type I in $G_3^1$ is defined by

$$T(u) = a(u), \quad N(u) = \frac{1}{k(u)}a'(u), \quad B(u) = \frac{1}{k(u)}(0, a_2'(u), a_3'(u)).$$

The curvature is given by $k(x) = \sqrt{|a_2''^2 - a_3''^2|}$.

**Type II:** The equation of ruled surface of type II in $G_3^1$ is

$$x(u, v) = (0, y(u), z(u)) + v(1, a_2(u), a_3(u)),$$

$$y, z, a_2, a_3 \in C^3, \quad u \in I \subseteq \mathbb{R}, \quad v \in \mathbb{R},$$

$$|y''^2 - z'^2| = 1, \quad y'a_2' - z'a_3' = 0.$$  \hspace{1cm} (3.5)

A ruled surface of type II is a surface whose striction curve lies in a pseudo-Euclidean plane.

The associated trihedron of ruled surface of type II in $G_3^1$ is defined by

$$T(u) = a(u) = (1, a_2, a_3),$$

$$N(u) = (0, z'(u), y'(u)),$$

$$B(u) = (0, y'(u), z'(u)),$$

where

$$k(u) = \frac{a_2(u)}{z'(u)}, \quad \tau(u) = \frac{y''(u)}{z'(u)}.$$  

The Frenet formulas are in type I or type II as follows.

$$\nabla_{T(u)} T(u) = k(u)N(u),$$

$$\nabla_{N(u)} N(u) = \tau(u)B(u),$$

$$\nabla_{B(u)} B(u) = \tau(u)N(u).$$  \hspace{1cm} (3.6)

### 4 The Characterizations of Curves on Ruled Surfaces

**Definition 4.1** Let $\alpha$ be a curve of a ruled surface of type I or II and $\{T(u), N(u), B(u)\}$ be the Frenet frame on ruled surface of type I or II along $\alpha$. If $k$ and $\tau$ are positive constants along $\alpha$, then $\alpha$ is called a **circular helix** with respect to the Frenet frame.

**Definition 4.2** Let $\alpha$ be a curve of a ruled surface of type I or II and $\{T(u), N(u), B(u)\}$ be the Frenet frame on ruled surface of type I or II along $\alpha$. A curve $\alpha$ such that

$$\frac{k(u)}{\tau(u)} = \text{const}$$

is called a **general helix** with respect to Frenet Frame.
Theorem 4.3 Let $\alpha$ be a curve of a ruled surface of type I or II in pseudo-Galilean space $G^1_3$. $\alpha$ is a general helix with respect to the Frenet Frame $\{T(u), N(u), B(u)\}$ if and only if

$$\nabla_{T(u)} \nabla_{T(u)} \nabla_{T(u)} N(u) - K(u) \nabla_{T(u)} N(u) = \frac{3}{\lambda} \tau'(u) \nabla_{T(u)} T(u)$$  (4.1)

where $K(u) = \frac{\tau'(u)}{\tau(u)} + \tau^2(u)$.

Proof. Suppose that $\alpha$ is general helix with respect to the Frenet Frame $\{T(u), N(u), B(u)\}$. Then from (3.6), we have

$$\nabla_{T(u)} \nabla_{T(u)} \nabla_{T(u)} N(u) = (\tau''(u) + \tau^3(u)) B(u) + (3\tau(u)\tau'(u)) N(u).$$  (4.2)

Now, since $\alpha$ is general helix with respect to the Frenet Frame

$$\frac{k(u)}{\tau(u)} = \lambda = \text{const.}$$  (4.3)

If we substitute the equations

$$N(u) = \frac{1}{k(u)} \nabla_{T(u)} T(u),$$  (4.4)

$$B(u) = \frac{1}{\tau(u)} \nabla_{T(u)} N(u)$$  (4.5)

and (4.5) in (4.2), we obtain (4.1).

Conversely let us assume that the equation (4.1) holds. We show that the curve $\alpha$ is a general helix. Differentiating covariantly (4.5) we obtain

$$\nabla_{T(u)} B(u) = -\frac{\tau'(u)}{\tau^2(u)} \nabla_{T(u)} N(u) + \frac{1}{\tau(u)} \nabla_{T(u)} \nabla_{T(u)} N(u)$$  (4.6)

and so

$$\nabla_{T(u)} \nabla_{T(u)} B(u) = \left(-\frac{\tau'(u)}{\tau^2(u)}\right)' \nabla_{T(u)} N(u) - \frac{2}{\tau(u)} \nabla_{T(u)} \nabla_{T(u)} N(u)$$

$$+ \frac{1}{\tau(u)} \nabla_{T(u)} \nabla_{T(u)} \nabla_{T(u)} N(u).$$  (4.7)

If we use (4.1) in (4.7) and make some calculations, we have

$$\nabla_{T(u)} \nabla_{T(u)} B(u) = \left\{ \left(-\frac{\tau'(u)}{\tau^2(u)}\right)' + \frac{K(u)}{\tau(u)} \right\} \nabla_{T(u)} N(u) - \frac{2}{\tau(u)} \nabla_{T(u)} \nabla_{T(u)} N(u)$$

$$+ \frac{3}{\lambda} \frac{\tau'(u)k(u)}{\tau(u)} N(u).$$  (4.8)

Also we obtain

$$\nabla_{T(u)} \nabla_{T(u)} B(u) = \tau^2(u) B(u) + \tau'(u) N(u)$$  (4.9)

since (4.8) and (4.9) are equal, routine calculations show that $\alpha$ is a general helix. □
Corollary 4.4 Let $\alpha$ be a curve of ruled surface of type I or II in pseudo-Galilean space $G_3^1$. $\alpha$ is a circular helix with respect to the Frenet Frame \{T(u), N(u), B(u)\}, if and only if
\[
\nabla T(u) \nabla T(u) \nabla T(u) N(u) = \tau^2(u) \nabla T(u) N(u). \quad (4.10)
\]

Proof. From the hypothesis of corollary 4.4 and since $\alpha$ is a circular helix, we can show easily (4.10).

Theorem 4.5 Let $\alpha$ be a curve of a ruled surface of type I or II in pseudo-Galilean space $G_3^1$. $\alpha$ is a general helix with respect to the Frenet Frame \{T(u), N(u), B(u)\} if and only if $\nabla T(u) T(u)$ and $\nabla T(u) B(u)$ are linear independent.

Proof. Suppose that $\alpha$ is a general helix with respect to the Frenet Frame \{T(u), N(u), B(u)\}. Then from (4.3), we have
\[
k(u) = \lambda \tau(u). \quad (4.11)
\]
If we product n with (4.11) equation and consider (3.6), we obtain
\[
\nabla T(u) T(u) = \lambda \nabla T(u) B(u). \quad (4.12)
\]
Conversely let us assume that the equation (4.12) holds. We show that the curve $\alpha$ is a general helix. From (4.12), we obtain
\[
k(u) \over \tau(u) = \lambda = \text{const}
\]
That is $\alpha$ is a general helix.

Theorem 4.6 Let $\alpha$ be a curve of a ruled surface of type I or II in pseudo-Galilean space $G_3^1$. $\alpha$ is a general helix with respect to the Frenet Frame \{T(u), N(u), B(u)\} if and only if $\nabla T(u) \nabla T(u) T(u)$ and $\nabla T(u) \nabla T(u) B(u)$ are linear independent.

Proof. It is similar to the proof of Theorem 4.5.

References


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