Edge-Chromatic Numbers of Glued Graphs

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Abstract: Let $G_1$ and $G_2$ be any two graphs. Assume that $H_1 \subseteq G_1$ and $H_2 \subseteq G_2$ are connected, not a single vertex and such that $H_1 \cong H_2$ with an isomorphism $f$. The glued graph of $G_1$ and $G_2$ at $H_1$ and $H_2$ with respect to $f$, denoted by $G_1 \triangleright \triangleleft H_1 \cong H_2 G_2$, is the graph that results from combining $G_1$ with $G_2$ by identifying $H_1$ and $H_2$ with respect to the isomorphism $f$ between $H_1$ and $H_2$. We give upper bounds of the edge-chromatic numbers of glued graphs; one is in terms of the edge-chromatic numbers of their original graphs where we give a characterization of graphs satisfying its equality. We further obtain a better upper bound of the chromatic numbers of glued graphs when the original graphs are line graphs.

Keywords: Graph coloring; Glued graph.

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1 Introduction

Let $G_1$ and $G_2$ be any graphs, $H_1 \subseteq G_1$ and $H_2 \subseteq G_2$ be connected, not a single vertex and such that $H_1 \cong H_2$ with an isomorphism $f$. The glued graph of $G_1$ and $G_2$ at $H_1$ and $H_2$ with respect to $f$, denoted by $G_1 \triangleleft H_1 \cong H_2 G_2$, is the graph that results from combining $G_1$ with $G_2$ by identifying $H_1$ and $H_2$ with respect to the isomorphism $f$. If $H$ is the copy of $H_1$ and $H_2$ in the glued graph, $H$ is referred as its clone, and $G_1$ and $G_2$ are referred as its original graphs. The glued graph $G_1 \triangleleft G_2$ at the clone $H$ means that there exist a subgraph $H$ of $G_1$, a subgraph $H_2$ of $G_2$, and an isomorphism $f$ such that $G_1 \triangleleft G_2$ and $H$ is the copy of $H_1$ and $H_2$ in the resulting graph. Unless we define specifically, we denote $G_1 \triangleleft G_2$ as an arbitrary graph resulting from gluing $G_1$ and $G_2$.

A $k$-edge-coloring of a graph $G$ is a labelling $f: E(G) \to S$, where $|S| = k$. The labels are colors; the edges of one color form a color class. A $k$-edge-coloring is proper if incident edges have different labels. A graph is $k$-edge-colorable if it has a proper $k$-edge-coloring. The edge-chromatic number of a loopless graph $G$, $\chi'(G)$, is the least $k$ such that $G$ is $k$-edge-colorable.

Let $\Delta(G)$ be the maximum degree of a graph $G$. Since all edges incident to a vertex with maximum degree cannot be labelled by the same color, $\chi'(G) \leq \Delta(G)$. For simple graph $G$, a well-known result was independently proved by Vizing [5]...
and Gupta [1] that
\[ \chi'(G) \leq \Delta(G) + 1. \]
We referred to it as Vizing and Gupta’s upper bound. Then we denote that G is
Class 1 if \( \chi'(G) = \Delta(G) \) and G is Class 2 if \( \chi'(G) = \Delta(G) + 1 \). Nevertheless, Vizing and Gupta’s upper bound is not satisfied by loopless non-simple graphs, Shannon [4] proved that
\[ \chi'(G) \leq \frac{3}{2}\Delta(G) \]
which we refer to as Shannon’s upper bound. The sharpness of this bound is provided by the fat triangles; the loopless triangles with multiple edges similar to the graph in Figure 1.

![Figure 1: A fat triangle](image)

We note few facts that the copy of both original graphs are subgraphs of their glued graphs. The graph gluing does not create an edge. Also, a glued graph of simple graphs may not be simple. Some interesting properties of glued graphs and the chromatic numbers of glued graphs are studied in [3]. Here we investigate the edge-chromatic numbers of glued graphs. In section 3, we apply our result to obtain a better upper bound of the chromatic numbers of glued graphs when original graphs are line graphs. The notation \( C_n(v_1, \ldots, v_n) \) denotes a cycle of \( n \) vertices on the vertex set \( \{v_1, \ldots, v_n\} \).

## 2 Bounds of the Edge-Chromatic Numbers of Glued Graphs

For any glued graph \( G_1 \leftrightarrow G_2 \), since \( G_1 \) and \( G_2 \) are subgraphs \( G_1 \cong G_2 \), the edge-chromatic number of \( G_1 \leftrightarrow G_2 \) is at least \( \chi'(G_1) \) and \( \chi'(G_2) \). We therefore get a lower bound for any graphs \( G_1 \) and \( G_2 \) that
\[ \chi'(G_1 \leftrightarrow G_2) \geq \max\{\chi'(G_1), \chi'(G_2)\}. \]

An upper bound of the edge-chromatic number of a glued graph is in terms of the sum of the edge-chromatic numbers of their original graphs. This result is shown next along with its sharpness.
Remark 2.1 Since the graph gluing does not identify vertices of the original graphs, non-incident edges in original graphs are still non-incident in a glued graph.

Theorem 2.2 For any graph $G_1$ and $G_2$,

$$
\chi'(G_1 \bowtie G_2) \leq \chi'(G_1) + \chi'(G_2).
$$

Proof. Let $G_1$ and $G_2$ be graphs and let $G_1 \bowtie H \bowtie G_2$ be a glued graph of $G_1$ and $G_2$ at arbitrary clone $H$. There are proper edge-colorings $f : E(G_1) \rightarrow S_1$ and $g : E(G_2) \rightarrow S_2$ of $G_1$ and $G_2$, respectively, where $S_1$ and $S_2$ are sets of colors such that $|S_1| = \chi'(G_1)$, $|S_2| = \chi'(G_2)$ and $S_1 \cap S_2 = \phi$. Define $\alpha : E(G_1 \bowtie_H G_2) \rightarrow S_1 \cup S_2$ by for all $e \in E(G_1 \bowtie_H G_2)$,

$$
\alpha(e) = \begin{cases} 
    f(e) & \text{if } e \in E(G_1), \\
    g(e) & \text{if } e \in E(G_2 \setminus H).
\end{cases}
$$

To prove that $\alpha$ is proper, let $e_1$ and $e_2$ be incident edges in $G_1 \bowtie_H G_2$.

Case 1. $e_1 \in E(G_1)$ and $e_2 \in E(G_2 \setminus H)$: Because $S_1 \cap S_2 = \phi$, we have $\alpha(e_1) \neq \alpha(e_2)$.

Case 2. $e_1$ and $e_2$ are edges in $G_1$: By Remark 2.1, $e_1$ and $e_2$ are incident in $G_1$ and hence $\alpha(e_1) = f(e_1) \neq f(e_2) = \alpha(e_2)$.

Case 3. $e_1$ and $e_2$ are edges in $G_2 \setminus H$: Similar to case 2, we have that $\alpha(e_1) = g(e_1) \neq g(e_2) = \alpha(e_2)$.

Therefore $\alpha$ is proper and hence $\chi'(G_1 \bowtie G_2) \leq \chi'(G_1) + \chi'(G_2)$. $\square$

Figure 2: The sharpness of Theorem 2.2

Consider graphs $G_1$ and $G_2$ with proper 6-edge-colorings in Figure 2. Note that both graphs have the maximum degree six. Thus $\chi'(G_1) = 6 = \chi'(G_2)$. We glue $G_1$ and $G_2$ with the isomorphism $f$ defined by $f(a) = m$, $f(b) = n$, $f(c) = o$, $f(d) = p$, $f(e) = q$, $f(f) = r$, $f(g) = s$, $f(h) = t$, $f(i) = u$, $f(j) = v$, $f(k) = w$, $f(l) = x$, $f(m) = y$, $f(n) = z$, $f(o) = a$, $f(p) = b$, $f(q) = c$, $f(r) = d$, $f(s) = e$, $f(t) = f$, $f(u) = g$, $f(v) = h$, $f(w) = i$, $f(x) = j$, $f(y) = k$, $f(z) = l$.
f(d) = p, f(e) = q and f(h) = r. The glued graph \( G_1 \triangledown H_1 \sim H_2 G_2 \) with a proper 12-edge-coloring is shown as in Figure 2. Since a fat triangle with the maximum degree 8 is subgraph of \( G_1 \triangledown H_1 \sim H_2 G_2 \), we have \( \chi'(G_1 \triangledown H_1 \sim H_2 G_2) \geq \frac{3}{2}(8) = 12. \)

Hence \( \chi'(G_1 \triangledown H_1 \sim H_2 G_2) = 12. \) Therefore \( \chi'(G_1 \triangledown H_1 \sim H_2 G_2) = \chi'(G_1) + \chi'(G_2), \) and hence the upper bound of the edge-chromatic number in Theorem 2.2 is sharp.

Now consider another upper bound of the edge-chromatic number of any glued graph. It shall be expressed in terms of the maximum degree of its original graphs and the minimum degree of its clone. Let \( \delta(G) \) and \( \Delta(G) \) denote the minimum and maximum degrees of a graph \( G \), respectively.

**Lemma 2.3** Let \( G_1 \) and \( G_2 \) be graphs and let \( H \) be the clone of a glued graph \( G_1 \triangledown H \sim G_2 \). Then

\[
\Delta(G_1 \triangledown H \sim G_2) \leq \Delta(G_1) + \Delta(G_2) - \delta(H).
\]

**Proof.** Let \( G_1 \) and \( G_2 \) be graphs and let \( H \) be the clone of a glued graph \( G_1 \triangledown H \sim G_2 \). For convenience, let \( G = G_1 \triangledown H \sim G_2 \). Let \( v \) be a vertex with maximum degree of \( G \).

If \( v \) is not in \( H \), then \( \deg_G(v) = \max\{\Delta(G_1), \Delta(G_2)\} \leq \Delta(G_1) + \Delta(G_2) - \delta(H). \)

Suppose that \( v \) is in \( H \). So \( v \) is in both \( G_1 \) and \( G_2 \). Since each edge which is incident to \( v \) in \( H \) contributes twice in the degree sum,

\[
\deg_G(v) = \deg_{G_1}(v) + \deg_{G_2}(v) - \deg_H(v).
\]

Since \( v \in H \), we get that \( \deg_H(v) \geq \delta(H) \). Hence

\[
\deg_G(v) = \deg_{G_1}(v) + \deg_{G_2}(v) - \deg_H(v) \leq \Delta(G_1) + \Delta(G_2) - \delta(H).
\]

\[\square\]

**Remark 2.4** Consequently from Lemma 2.3, since \( \delta(H) \geq 1 \), \( \Delta(G_1 \triangledown H \sim G_2) \leq \Delta(G_1) + \Delta(G_2) - 1. \)

**Theorem 2.5** Let \( G_1 \) and \( G_2 \) be graphs and let \( G_1 \triangledown H \sim G_2 \) be a glued graph of \( G_1 \) and \( G_2 \) at a clone \( H \). Then

\[
\chi'(G_1 \triangledown H \sim G_2) \leq \frac{3}{2}(\Delta(G_1) + \Delta(G_2) - \delta(H)).
\]

In particular, if \( G_1 \triangledown H \sim G_2 \) is a simple graph, then

\[
\chi'(G_1 \triangledown H \sim G_2) \leq \Delta(G_1) + \Delta(G_2) - \delta(H) + 1.
\]
Proof. Let $G_1$ and $G_2$ be graphs and let $G_1 \triangledown_H G_2$ be a glued graph of $G_1$ and $G_2$ at a clone $H$. Following from Shannon’s upper bound and Lemma 2.3, we have that $\chi'(G_1 \triangledown_H G_2) \leq \frac{3}{2} (\Delta(G_1) + \Delta(G_2) - \delta(H))$. If $G_1 \triangledown_H G_2$ is a simple graph, by Vizing and Gupta’s upper bound and Lemma 2.3, we obtain that $\chi'(G_1 \triangledown_H G_2) \leq \Delta(G_1) + \Delta(G_2) - \delta(H) + 1$. □

We now show the sharpness of Theorem 2.5. In Figure 3, consider $H_1 = C_9(u_1, u_2, \ldots, u_9)$ and $H_2 = C_9(v_1, v_2, \ldots, v_9)$. We glue $G_1$ and $G_2$ at $H_1$ and $H_2$ by isomorphism $f$ defined by $f(u_i) = v_i$ for all $i = 1, 2, \ldots, 9$. So we have $G_1 \triangledown_{H_1 \cong f, H_2} G_2$ which is isomorphic to $K_9$. Note that $\chi'(K_n) = n$ when $n$ is odd. [6]

Hence $\chi'(G_1 \triangledown_{H_1 \cong f, H_2} G_2) = 9 = 6 + 4 - 2 + 1 = \Delta(G_1) + \Delta(G_2) - \delta(H) + 1$.

For non-simple glued graphs, consider graphs $G_1$ and $G_2$ with maximum degree four in Figure 4. Gluing $G_1$ and $G_2$ at edge sets $\{a, b, c\}$ and $\{1, 2, 3\}$ with isomorphism $f$ such that $f(a) = 1$, $f(b) = 2$ and $f(c) = 3$ yields the glued graph $G_1 \triangledown_{H_1 \cong f, H_2} G_2$ as shown in Figure 4. Hence we have $\chi'(G_1 \triangledown_{H_1 \cong f, H_2} G_2) = 9 = \frac{3}{2} (4 + 4 - 2) = \frac{3}{2} (\Delta(G_1) + \Delta(G_2) - \delta(H))$.

We next discuss a characterization of graphs with the edge-chromatic number
is precisely the sum of the edge-chromatic number of the original graphs, that is, it satisfies the equality in Theorem 2.2.

**Corollary 2.6** If $G_1$, $G_2$ and $G_1 \perp G_2$ are simple graphs, $\chi'(G_1 \perp G_2) = \chi'(G_1) + \chi'(G_2)$ if and only if $G_1$ and $G_2$ are Class 1, $G_1 \perp G_2$ is Class 2, and $\Delta(G_1 \perp G_2) = \Delta(G_1) + \Delta(G_2) - 1$.

**Proof.** Necessity. Assume $\chi'(G_1 \perp G_2) = \chi'(G_1) + \chi'(G_2)$. By Vizing and Gupta’s upper bound and Remark 2.4, we have that

\[\chi'(G_1 \perp G_2) \leq \Delta(G_1 \perp G_2) + 1 \leq \Delta(G_1) + \Delta(G_2) \leq \chi'(G_1) + \chi'(G_2).\]

Therefore, $\chi'(G_1 \perp G_2) = \Delta(G_1 \perp G_2) + 1$, $\Delta(G_1 \perp G_2) = \Delta(G_1) + \Delta(G_2) - 1$, $\chi'(G_1) = \Delta(G_1)$ and $\chi'(G_2) = \Delta(G_2)$.

Sufficiency. All conditions in the right hand side yield that

\[\chi'(G_1 \perp G_2) = \Delta(G_1 \perp G_2) + 1 = (\Delta(G_1) + \Delta(G_2) - 1) + 1 = \chi'(G_1) + \chi'(G_2).\]

\[\square\]

Determining whether a graph is Class 1 or Class 2 is generally hard [2, 6]. Gluing Class 1 graphs may get a Class 2 glued graph and vice versa. It is an open problem to determine conditions that forbid or guarantee $\Delta(G_1 \perp G_2)$-edge-colorability.

3 The Chromatic Numbers of Glued Line Graphs

The line graph $L(G)$ of a connected graph $G$ is the graph generated from $G$ by $V(L(G)) = E(G)$ and for any two vertices $e, f \in V(L(G))$, vertex $e$ and vertex $f$ are adjacent in $L(G)$ if and only if edge $e$ and edge $f$ share a common vertex in $G$. If $H$ is the line graph of $G$, we call $G$ the root graph of $H$. All graphs have their line graphs, but not all graphs are line graphs. For example, there is no graph $G$ such that $L(G) = K_{1,3}$. So the $K_{1,3}$ is not a line graph.

A $k$-coloring of a graph $G$ is a labelling $f : V(G) \to S$, where $|S| = k$. The labels are colors; the vertices of one color form a color class. A $k$-coloring is proper if adjacent vertices have different labels. A graph is $k$-colorable if it has a proper $k$-coloring. The chromatic number of graph $G$, $\chi(G)$, is the least $k$ such that $G$ is $k$-colorable.

Ones may intuitively believe that $\chi(G_1 \perp G_2) \leq \chi(G_1) + \chi(G_2)$. However, we proved in [3] and showed its sharpness that $\chi(G_1 \perp G_2) \leq \chi(G_1)\chi(G_2)$ for any graphs $G_1$ and $G_2$. Here we shall show by using Theorem 2.2 that if $G_1$ and $G_2$ are line graphs, $G_1 \perp G_2$ has a proper $(\chi(G_1) + \chi(G_2))$-coloring.

**Remark 3.1** For any subgraph $H$ of a graph $G$, $L(H) \subseteq L(G)$.

**Remark 3.2** For any graph $G$, $\chi'(G) = \chi(L(G))$. 
Lemma 3.3 Let $G_1$ and $G_2$ be graphs. $L(G_1) \triangle L(G_2) \subseteq L(G_1 \triangle G_2)$.

**Proof.** Since $G_1$ and $G_2$ are subgraphs of $G_1 \triangle G_2$, the line graphs $L(G_1)$ and $L(G_2)$ are subgraphs of $L(G_1 \triangle G_2)$. So $L(G_1) \cup L(G_2) \subseteq L(G_1 \triangle G_2)$. Because for each vertex and edge in $L(G_1) \triangle L(G_2)$ are in $L(G_1) \cup L(G_2)$ which is a subgraph of $L(G_1 \triangle G_2)$, so $L(G_1) \triangle L(G_2) \subseteq L(G_1 \triangle G_2)$. □

Theorem 3.4 Let $G_1$ and $G_2$ be graphs. If $G_1$ and $G_2$ are line graphs, then $\chi(G_1 \triangle G_2) \leq \chi(G_1) + \chi(G_2)$.

**Proof.** Let $G_1$ and $G_2$ be graphs. Assume that $G_1$ and $G_2$ are line graphs. So there are graphs $G_1^*$ and $G_2^*$ such that $L(G_1^*) = G_1$ and $L(G_2^*) = G_2$. By lemma 3.3, we have that $L(G_1^*) \triangle L(G_2^*) \subseteq L(G_1 \triangle G_2)$. This yields $\chi(L(G_1^*) \triangle L(G_2^*)) \leq \chi(L(G_1^* \triangle G_2^*))$. Hence

$$\chi(G_1 \triangle G_2) = \chi(L(G_1^*) \triangle L(G_2^*)) \leq \chi(L(G_1^* \triangle G_2^*))$$

$$= \chi(G_1^* \triangle G_2^*)$$

$$\leq \chi(G_1^*) + \chi(G_2^*) \quad \text{(by Theorem 2.2.)}$$

$$= \chi(L(G_1^*)) + \chi(L(G_2^*)) = \chi(G_1) + \chi(G_2).$$

□

References


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