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Common Fixed Point Results for α-ψ-Locally Contractive Type Mappings in Right Complete Dislocated Quasi G-Metric Spaces

Abdullah Shoaib[†], Fahim Uddin[‡], Muhammad Arshad[‡,1] and Muhammad Usman Ali[§]

[†]Department of Mathematics and Statistics, Riphah International University
Islamabad, Pakistan
e-mail : abdullahshoaib15@yahoo.com

[‡]Department of Mathematics, International Islamic University
Islamabad 44000, Pakistan
e-mail : fahamiiu@gmail.com (F. Uddin)
marshad_zia@yahoo.com (M. Arshad)

[§]Department of Mathematics, COMSATS Institute of
Information Technology, Attock Pakistan
e-mail : muh_usman_ali@yahoo.com

Abstract : We have introduced the concepts of right complete dislocated quasi G_d-metric spaces, G-α-admissible mapping with respect to η and α-ψ contractive type condition for a pair of such maps and have established common fixed point results in a closed ball in right complete dislocated quasi G_d-metric spaces. An example is also given which illustrate the superiority of our results. In the process we have generalized several well known, recent and classical results from the literature.

Keywords : common fixed point; right complete dislocated quasi G_d-metric Space; α-ψ-contractive mappings; G-α-admissible mapping with respect to η; closed ball.

[1]Corresponding author.

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1 Introduction

A point $x \in X$ is called a fixed point of an operator $T : X \to X$ if $x = Tx$. Many results appeared in literature related to the fixed point of mappings which are contractive on the whole domain, for example, see [1]–[32]. It is possible that $T : X \longrightarrow X$ is not a contraction but $T : Y \longrightarrow X$ is a contraction, where Y is a subset of X. One can obtain fixed point results for such mapping by using suitable conditions. Recently Arshad et. al. [6] proved a result concerning the existence of fixed points of a mapping satisfying a contractive conditions on closed ball in a complete dislocated metric space(see also [7, 8, 10, 11, 21, 28, 29, 32]). The notion of dislocated topologies have useful applications in the context of logic programming semantics (see [4, 14, 25]).On the other hand, Mustafa and Sims in [22] introduce the notion of a generalized metric space. Many useful results can be seen in [3, 12, 19, 20, 23, 24, 30, 31].

Samet et al [27] introduced the notions of α-ψ-contractive and α-admissible mappings in complete metric spaces. More recently Salimi et al. [26] modified the notion of α-ψ-contractive mappings and improved certain fixed point theorems for such mappings. In this paper we introduce the concepts of G-α-admissible mapping with respect to η and discuss common fixed point results for α-ψ-contractive type mappings in a closed ball in right complete dislocated quasi G_d-metric spaces. The existence of fixed points of α-ψ-contractive and α-admissible mappings in complete metric spaces has been studied by several researchers (see [15–17] and references there in).

The following definitions and results will be needed in the sequel.

Definition 1.1. Let X be a nonempty set and let $G_d : X \times X \times X \to R^+$ be a function satisfying the following axioms:

(i) If $G_d(x, y, z) = 0$, then $x = y = z$,

(ii) $G_d(x, y, z) \leq G_d(x, a, a) + G_d(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality).

Then the pair (X, G_d) is called the dislocated quasi G_d-metric space.

It is clear that if $x = y = z$ then $G_d(x, y, z)$ may not be 0. We say that (X, G_d) is a dislocated G_d-metric space if $G_d(x, y, z) = G_d(y, z, x) = G_d(z, x, y) = G_d(x, z, y) = G_d(y, x, z) = G_d(z, y, x)$ for all $x, y, z \in X$, satisfied in Definition 1.1. It is observed that if $x = y = z$ implies $G_d(x, y, z) = 0$ in dislocated G_d-metric space, then (X, G_d) becomes a G-metric space [22].

Example 1.2. If $X = R^+ \cup \{0\}$ then $G_d(x, y, z) = x + \max\{x, y, z\}$ defines a dislocated quasi G_d-metric on X.

Definition 1.3. Let (X, G_d) be a dislocated quasi G_d-metric space, and let $\{x_n\}$ be a sequence of points in X, a point x in X is said to be the limit of the sequence $\{x_n\}$ if $\lim_{m,n\to\infty} G_d(x, x_n, x_m) = 0$, and one says that sequence $\{x_n\}$ is G_d-convergent to x.

Definition 1.4. Let (X, G_d) be a dislocated quasi G_d-metric space. A sequence $\{x_n\}$ is called right G_d-Cauchy sequence if, for each $\epsilon > 0$ there exists a positive integer $n^\star \in N$ such that $G_d(x_n, x_m, x_m) < \epsilon$ for all $n \geq m \geq n^\star$; i.e. if $G_d(x_n, x_m, x_m) \to 0$ as $n, m \to \infty$.

Definition 1.5. A dislocated quasi G_d-metric space (X, G_d) is said to be right complete if every right G_d-Cauchy sequence in (X, G_d) is G_d-convergent in X.

Definition 1.6. Let (X, G_d) be a dislocated quasi G_d-metric space then for $x_0 \in X$, $r > 0$, the G_d-ball with centre x_0 and radius r is, $B(x_0, r) = \{y \in X : G_d(x_0, y, y) < r\}$. Also $\overline{B(x_0, r)} = \{y \in X : G_d(x_0, y, y) \leq r\}$ is a closed ball in X.

Lemma 1.7. *Every closed ball in a right complete dislocated quasi G_d-metric space is right complete.*

Definition 1.8. Let X be a nonempty set and $T, f : X \to X$. A point $y \in X$ is called point of coincidence of T and f if there exists a point $x \in X$ such that $y = Tx = fx$, here x is called coincidence point of T and f. The mappings T, f are said to be weakly compatible if they commute at their coincidence point (i.e. $Tfx = fTx$ whenever $Tx = fx$).

We require the following lemmas for subsequent use:

Lemma 1.9. *Let X be a nonempty set and $f : X \to X$ be a function. Then there exists $E \subset X$ such that $fE = fX$ and $f : E \to X$ is one to one.*

Lemma 1.10. [5] *Let X be a nonempty set and the mappings $S, T, f : X \to X$ have a unique point of coincidence v in X. If (S, f) and (T, f) are weakly compatible, then S, T, f have a unique common fixed point.*

Let Ψ denote the family of all nondecreasing functions $\psi : [0, +\infty) \to [0, +\infty)$ such that $\sum_{n=1}^{+\infty} \psi^n(t) < +\infty$ for all $t > 0$, where ψ^n is the n^{th} iterate of ψ.

Lemma 1.11. *If $\psi \in \Psi$, then $\psi(t) < t$ for all $t > 0$.*

Definition 1.12. [27] Let $f, g : X \to X$ be self-mappings and $\alpha : X \times X \to [0, \infty)$ be a mapping, then the mapping f is called α-admissible if, $x, y \in X$, $\alpha(x, y) \geq 1 \Rightarrow \alpha(fx, fy) \geq 1$. [2] The pair (f, g) is called α-admissible if, $x, y \in X$, $\alpha(x, y) \geq 1 \Rightarrow \alpha(fx, gy) \geq 1$ and $\alpha(gx, fy) \geq 1$. [26] Let $\alpha, \eta : X \times X \to [0, +\infty)$ be two functions. We say that f is α-admissible mapping with respect to η if $x, y \in X$ such that $\alpha(x, y) \geq \eta(x, y)$, then we have $\alpha(fx, fy) \geq \eta(fx, fy)$. [18] The pair (f, g) is called α-admissible with respect to η if, $x, y \in X$, $\alpha(x, y) \geq \eta(x, y) \Rightarrow \alpha(fx, gy) \geq \eta(fx, gy)$ and $\alpha(gx, fy) \geq \eta(gx, fy)$.

2 Main Results

We first introduce the concept of α-η-admissible mappings in G-metric space.

Definition 2.1. Let $S,T : X \to X$ and $\alpha, \eta : X \times X \times X \to R$ be two functions. We say that the pair (S,T) is G-α-admissible with respect to η, if $x,y,z \in X$ such that $\alpha(x,y,z) \geq \eta(x,y,z)$ then we have $\alpha(Sx,Ty,Tz) \geq \eta(Sx,Ty,Tz)$ and $\alpha(Tx,Sy,Sz) \geq \eta(Tx,Sy,Sz)$. Also, if we take $\eta(x,y,z) = 1$, then, (S,T) is called G-α-admissible, if we take, $\alpha(x,y,z) = 1$, then we say that the pair (S,T) is η-subadmissible mapping. If we take $S = T$, we say that S is G-α-admissible mapping with respect to η. If we take $S = T$ and $\eta(x,y,z) = 1$, we say that S is G-α-admissible mapping.

Theorem 2.2. *Let (X,G_d) be a right complete dislocated quasi G_d-metric space and $S,T : X \to X$ be two mappings. Suppose there exist two functions, $\alpha, \eta : X \times X \times X \to R$ such that (S,T) is G-α-admissible with respect to η. For $r > 0$, $x_0 \in \overline{B(x_0,r)}$ and $\psi \in \Psi$, assume that,*

$$\begin{aligned} x,y,z \quad &\in \quad \overline{B(x_0,r)}, \quad \alpha(x,y,z) \geq \eta(x,y,z) \\ &\Longrightarrow \quad \max\{G_d(Sx,Ty,Tz), G_d(Tx,Sy,Sz)\} \leq \psi(G_d(x,y,z)). \end{aligned} \tag{2.1}$$

and

$$\sum_{i=0}^{j} \psi^i(G_d(x_0,Sx_0,Sx_0)) \leq r, \text{ for all } j \in N \cup \{0\}. \tag{2.2}$$

Suppose that the following assertions hold:

(i) $\alpha(x_0,Sx_0,Sx_0) \geq \eta(x_0,Sx_0,Sx_0)$;

(ii) *for any sequence $\{x_n\}$ in $\overline{B(x_0,r)}$ such that $\alpha(x_n,x_{n+1},x_{n+1}) \geq \eta(x_n,x_{n+1},x_{n+1})$ for all $n \in N \cup \{0\}$ and $x_n \to u \in \overline{B(x_0,r)}$ as $n \to +\infty$ then $\alpha(u,x_n,,x_n) \geq \eta(u,x_n,,x_n)$ for all $n \in N \cup \{0\}$.*

Then, there exists a point x^* in $\overline{B(x_0,r)}$ such that $x^* = Sx^* = Tx^*$.

Proof. Let x_1 in X be such that $x_1 = Sx_0$ and $x_2 = Tx_1$. Continuing this process, we construct a sequence x_n of points in X such that,

$$x_{2i+1} = Sx_{2i}, \text{ and } x_{2i+2} = Tx_{2i+1}, \text{ where } i = 0,1,2,\cdots.$$

By assumption $\alpha(x_0,x_1,x_1) \geq \eta(x_0,x_1,x_1)$ and (S,T) is G-α-admissible with respect to η, we have, $\alpha(Sx_0,Tx_1,Tx_1) \geq \eta(Sx_0,Tx_1,Tx_1)$ from which we deduce that $\alpha(x_1,x_2,x_2) \geq \eta(x_1,x_2,x_2)$ which also implies that $\alpha(Tx_1,Sx_2,Sx_2) \geq \eta(Tx_1,Sx_2,Sx_2)$. Continuing in this way we obtain

$$\alpha(x_n,x_{n+1},x_{n+1}) \geq \eta(x_n,x_{n+1},x_{n+1})$$

for all $n \in N \cup \{0\}$. First we show that $x_n \in \overline{B(x_0, r)}$ for all $n \in N$. Using inequality (2.2), we have,

$$\sum_{i=0}^{n} \psi^i(G_d(x_0, Sx_0, Sx_0)) \leq r, \text{ for all } n \in N \cup \{0\}.$$

It follows that,

$$x_1 \in \overline{B(x_0, r)}.$$

Let $x_2, \cdots, x_j \in \overline{B(x_0, r)}$ for some $j \in N$. If $j = 2i + 1$, where $i = 0, 1, 2, \ldots \frac{j-1}{2}$ so using inequality (2.1), we obtain,

$$\begin{aligned} G_d(x_{2i+1}, x_{2i+2}, x_{2i+2}) &= G_d(Sx_{2i}, Tx_{2i+1}, Tx_{2i+1}) \\ &\leq \psi(G_d(x_{2i}, x_{2i+1}, x_{2i+1})) \\ &\leq \psi^2(G_d(x_{2i-1}, x_{2i}, x_{2i})) \\ &\leq \cdots \leq \psi^{2i+1}(G_d(x_0, x_1, x_1)). \end{aligned}$$

Thus we have,

$$G_d(x_{2i+1}, x_{2i+2}, x_{2i+2}) \leq \psi^{2i+1}(G_d(x_0, x_1, x_1)). \tag{2.3}$$

If $j = 2i + 2$, then as x_1, x_2 $..., x_j \in \overline{B(x_0, r)}$ where ($i = 0, 1, 2, \ldots, \frac{j-2}{2}$). We obtain,

$$G_d(x_{2i+2}, x_{2i+3}, x_{2i+3}) \leq \psi^{2(i+1)}(G_d(x_0, x_1, x_1)). \tag{2.4}$$

Thus from inequality (2.3) and (2.4), we have

$$G_d(x_j, x_{j+1}, x_{j+1}) \leq \psi^j(G_d(x_0, x_1, x_1)). \tag{2.5}$$

Now,

$$\begin{aligned} G_d(x_0, x_{j+1}, x_{j+1}) &= G_d(x_0, x_1, x_1) + ... + G_d(x_j, x_{j+1}, x_{j+1}) \\ &\leq \sum_{i=0}^{j} \psi^i(G_d(x_0, x_1, x_1)) \leq r \end{aligned}$$

Thus $x_{j+1} \in \overline{B(x_0, r)}$. Hence $x_n \in \overline{B(x_0, r)}$ for all $n \in N$. Now inequality (2.5) can be written as

$$G_d(x_n, x_{n+1} x_{n+1}) \leq \psi^n(G_d(x_0, x_1, x_1)), \text{ for all } n \in N. \tag{2.6}$$

Fix $\varepsilon > 0$ and let $n(\varepsilon) \in N$ such that $\sum \psi^n(G_d(x_0, x_1, x_1)) < \varepsilon$. Let $n, m \in N$ with $m > n > k(\varepsilon)$ using the triangular inequality, we obtain,

$$\begin{aligned} G_d(x_n, x_m, x_m) &\leq \sum_{k=n}^{m-1} G_d(x_k, x_{k+1} x_{k+1}) \leq \sum_{k=n}^{m-1} \psi^k(G_d(x_0, x_1, x_1)) \\ &\leq \sum_{n \geq n(\varepsilon)} \psi^k(G_d(x_0, x_1, x_1)) < \varepsilon \end{aligned}$$

Thus we proved that $\{x_n\}$ is a Cauchy sequence in $(\overline{B(x_0,r)}, G_d)$. As every closed ball in a right complete dislocated quasi G_d-metric space is right complete, so there exists $x^* \in \overline{B(x_0,r)}$ such that $x_n \to x^*$. Also

$$\lim_{n\to\infty} G_d(x^*, x_n, x_n) = 0. \tag{2.7}$$

On the other hand, from (ii), we have,

$$\alpha(x^*, x_n, x_n) \geq \eta(x^*, x_n, x_n) \text{ for all } n \in N \cup \{0\}. \tag{2.8}$$

Now using inequalities (2.1) and (2.8), we get

$$G_d(Sx^*, x_{2i+2}, x_{2i+2}) \leq \psi(G_d(x^*, x_{2i+1}, x_{2i+1})) < G_d(x^*, x_{2i+1}, x_{2i+1}).$$

Letting $i \to \infty$ and by using inequality (2.7), we obtain $G_d(Sx^*, x^*, x^*) < 0$. Hence $Sx^* = x^*$. Similarly by using

$$G_d(Tx^*, x_{2i+1}, x_{2i+1}) \leq \psi(G_d(x^*, x_{2i}, x_{2i})) < G_d(x^*, x_{2i}, x_{2i}),$$

we obtain $G_d(Tx^*, x^*, x^*) = 0$, that is, $Tx^* = x^*$. Hence S and T have a common fixed point in $\overline{B(x_0,r)}$. □

If $\eta(x,y,z) = 1$ for all $x,\ y,\ z \in X$ in Theorem 2.2, we obtain following result.

Corollary 2.3. *Let (X, G_d) be a right complete dislocated quasi G_d-metric space and $S, T : X \to X$, $r > 0$ and x_0 be an arbitrary point in X. Suppose there exists, $\alpha : X \times X \times X \to R$ such that the pair (S,T) is α-admissible. For $\psi \in \Psi$, assume that,*

$$\begin{aligned} x, y, z \quad &\in \quad \overline{B(x_0,r)},\ \alpha(x,y,z) \geq 1 \\ &\Longrightarrow \quad \max\{G_d(Sx,Ty,Tz), G_d(Tx,Sy,Sz)\} \leq \psi(G_d(x,y,z)). \end{aligned}$$

and

$$\sum_{i=0}^{j} \psi^i(G_d(x_0, Sx_0, Sx_0)) \leq r, \text{ for all } j \in N \cup \{0\}.$$

Suppose that the following assertions hold:

(i) $\alpha(x_0, Sx_0, Sx_0) \geq 1$,

(ii) for any sequence $\{x_n\}$ in $\overline{B(x_0,r)}$ such that $\alpha(x_n, x_{n+1}, x_{n+1}) \geq 1$ for all $n \in N \cup \{0\}$ and $x_n \to u \in \overline{B(x_0,r)}$ as $n \to +\infty$ then $\alpha(u, x_n, , x_n) \geq 1$ for all $n \in N \cup \{0\}$.

Then, there exists a point x^* in $\overline{B(x_0,r)}$ such that $x^* = Sx^* = Tx^*$.

If $\alpha(x,y,z) = 1$ for all $x,\ y, z \in X$ in Theorem 2.2, we obtain following result.

Corollary 2.4. *Let (X, G_d) be a right complete dislocated quasi G_d-metric space and $S, T : X \to X$ be two mappings. Suppose there exists, $\eta : X \times X \times X \to R$ such that the pair (S, T) is η-subadmissible. For $\psi \in \Psi$, assume that,*

$$\begin{aligned} x, y, z &\in \overline{B(x_0, r)},\ \eta(x, y, z) \leq 1 \\ &\Longrightarrow \max\{G_d(Sx, Ty, Tz), G_d(Tx, Sy, Sz)\} \leq \psi(G_d(x, y, z)). \end{aligned}$$

and

$$\sum_{i=0}^{j} \psi^i(G_d(x_0, Sx_0, Sx_0)) \leq r,\ \text{for all } j \in N \cup \{0\}.$$

Suppose that the following assertions hold:

(i) $\eta(x_0, Sx_0, Sx_0) \leq 1$;

(ii) for any sequence $\{x_n\}$ in $\overline{B(x_0, r)}$ such that $\eta(x_n, x_{n+1}, x_{n+1}) \leq 1$ for all $n \in N \cup \{0\}$ and $x_n \to u \in \overline{B(x_0, r)}$ as $n \to +\infty$ then $\eta(u, x_n, , x_n) \leq 1$ for all $n \in N \cup \{0\}$.

Then, there exists a point x^* in $\overline{B(x_0, r)}$ such that $x^* = Sx^* = Tx^*$.

Theorem 2.5. *Adding condition "if x^* is any common fixed point in $\overline{B(x_0, r)}$ of S and T, x be any fixed point of S or T in $\overline{B(x_0, r)}$, then $\alpha(x^*, x, x) \geq \eta(x^*, x, x)$" to the hypotheses of Theorem 2.2. Then S and T have a unique common fixed point x^* and $G_d(x^*, x^*, x^*) = 0$.*

Proof. By assumption, $\alpha(x^*, x^*, x^*) \geq \eta(x^*, x^*, x^*)$, then,

$$\begin{aligned} G_d(x^*, x^*, x^*) &= \max\{G_d(Sx^*, Tx^*, Tx^*), G_d(Tx^*, Sx^*, Sx^*)\} \\ &\leq \psi(G_d(x^*, x^*, x^*)). \end{aligned}$$

This implies that,

$$G_d(x^*, x^*, x^*) = 0.$$

Assume that y^* be another fixed point of T in $\overline{B(x_0, r)}$, then, by assumption, $\alpha(x^*, y^*, y^*) \geq \eta(x^*, y^*, y^*)$, also,

$$G_d(x^*, y^*, y^*) = G_d(Sx^*, Ty^*, Ty^*) \leq \psi(G_d(x^*, y^*, y^*))$$

A contradiction to the fact that for each $t > 0$, $\psi(t) < t$. So $x^* = y^*$. Hence T has no fixed point other than x^*. Similarly, S has no fixed point other than x^*. □

Example 2.6. Let $X = R^+ \cup \{0\}$ and be endowed with usual order and let $G_d : X \times X \to X$ be the right complete ordered dislocated quasi metric on X defined by,

$$G_d(x, y, z) = 2x + y + z \text{ for all } x, y, z \in X.$$

Let $S, T : X \to X$ be defined by,

$$Sx = \left\{ \begin{array}{c} \frac{x}{4} \text{ if } x \in [0,1] \\ 3x \text{ if } x \in (1,\infty) \end{array} \right\}$$

and

$$Tx = \left\{ \begin{array}{c} \frac{2x}{7} \text{ if } x \in [0,1] \\ 4x \text{ if } x \in (1,\infty). \end{array} \right.$$

Considering, $x_0 = 1$, $r = 4$, then $\overline{B(x_0, r)} = [0, 1]$. Define $\alpha(x, y, z) = 2x - y + z$, $\eta(x, y, z) = x - 2y$. Clearly, (S, T) is G-α-admissible with respect to η for all $x, y, z \in X$. Let $\psi(t) = \frac{t}{3}$. Now,

$$G_d(x_0, Sx_0, Sx_0) = G_d(1, S1, S1) = G_d(1, \frac{1}{4}, \frac{1}{4}) = \frac{5}{2}$$

$$\sum_{i=0}^{n} \psi^n(G_d(x_0, Sx_0, Sx_0)) = \frac{5}{2} \sum_{i=0}^{n} \frac{1}{3^n} < (\frac{5}{2})\frac{3}{2} < 4$$

Now if, $x, y, z \in (1, \infty)$, then

Case 1. If $\max\{G_d(Sx, Ty, Tz), G_d(Tx, Sy, Sz)\} = G_d(Sx, Ty, Tz)$ then, for $x, y, z \in (1, \infty)$, we have

$$\begin{aligned} G_d(Sx, Ty, Tz) &= G_d(3x, 4y, 4z) = 3x + 4y + 4z \\ &> \frac{2x}{3} + \frac{y}{3} + \frac{z}{3} = \psi(G_d(x, y, z)) \end{aligned}$$

Case 2. If $\max\{G_d(Sx, Ty, Tz), G_d(Tx, Sy, Sz)\} = G_d(Tx, Sy, Sz)$,

$$\begin{aligned} G_d(Tx, Sy, Sz) &= G_d(4x, 3y, 3z) = 4x + 3y + 3z \\ &> \frac{2x}{3} + \frac{y}{3} + \frac{z}{3} = \psi(G_d(x, y, z)) \end{aligned}$$

So the contractive condition does not hold on X,

Now if, $x, y, z \in \overline{B(x_0, r)}$, then

Case 3. If $\max\{G_d(Sx, Ty, Tz), G_d(Tx, Sy, Sz)\} = G_d(Sx, Ty, Tz)$.

$$\begin{aligned} G_d(Sx, Ty, Tz) &= G_d(\frac{x}{4}, \frac{2y}{7}, \frac{2z}{7}) \\ &= 2(\frac{x}{4}) + \frac{2y}{7} + \frac{2z}{7} \\ &\leq 2(\frac{x}{3}) + \frac{y}{3} + \frac{z}{3} \\ &= \psi(G_d(x, y, z)). \end{aligned}$$

Case 4. If $\max\{G_d(Sx, Ty, Tz), G_d(Tx, Sy, Sz)\} = G_d(Tx, Sy, Sz)$.

$$\begin{aligned} G_d(Tx, Sy, Sz) &= G_d(\frac{2x}{7}, \frac{y}{4}, \frac{z}{4}) \\ &= 2(\frac{2x}{7}) + \frac{y}{4} + \frac{z}{4} \\ &\leq 2(\frac{x}{3}) + \frac{y}{3} + \frac{z}{3} \\ &= \psi(G_d(x, y, z)). \end{aligned}$$

Then the contractive condition holds on $\overline{B(x_0, r)}$.

Therefore, all the conditions of Theorem 2.2 are satisfied and S and T have a common fixed point 0.

Now we apply our Theorem 2.5 to obtain unique common fixed point of three mappings on closed ball in right complete dislocated quasi G_d-metric space.

Theorem 2.7. *Let (X, G_d) be a dislocated quasi G_d-metric space, $S, T, f : X \to X$ such that $SX \cup TX \subset fX$, $r > 0$ and x_0 be an arbitrary point in X. Suppose there exist two functions, $\alpha, \eta : X \times X \times X \to R$ α-admissible with respect to η and $\psi \in \Psi$ such that,*

$$\max\{G_d(Sx, Ty, Tz), G_d(Tx, Sy, Sz)\} \leq \psi(G_d(fx, fy, fz)). \tag{2.9}$$

for all $fx, fy, fz \in \overline{B(fx_0, r)}$, $\alpha(fx, fy, fz) \geq \eta(fx, fy, fz)$ and,

$$\sum_{i=0}^{j} \psi^i(G_d(fx_0, Sx_0, Sx_0)) \leq r, \text{ for all } j \in N \cup \{0\}. \tag{2.10}$$

Suppose that,
(i) The pair (S, T) and f are G-α-admissible with respect to η.
(ii) $\alpha(fx_0, Sx_0, Sx_0) \geq \eta(fx_0, Sx_0, Sx_0)$.
(iii) if $\{x_n\}$ is a sequence in $\overline{B(fx_0, r)}$ such that $\alpha(x_n, x_{n+1}, x_{n+1}) \geq \eta(x_n, x_{n+1}, x_{n+1})$ for all n and $x_n \to u \in \overline{B(fx_0, r)}$ as $n \to +\infty$ then $\alpha(u, x_n, , x_n) \geq \eta(u, x_n, , x_n)$ for all $n \in N \cup \{0\}$.
(iv) fx , fy and fz be any fixed points in $\overline{B(fx_0, r)}$ of S or T, then $\alpha(fx, fy, fz) \geq \eta(fx, fy, fz)$.
(v) fX is right complete subspace of X and (S, f) and (T, f) are weakly compatible. Then S, T and f have a unique common fixed point fp in $\overline{B(fx_0, r)}$. Moreover $G_d(fp, fp, fp) = 0$.

Proof. By Lemma 1.9, there exists $E \subset X$ such that $fE = fX$ and $f : E \to X$ is one-to-one. Now since $SX \cup TX \subset fX$, we define two mappings $g, h : fE \to fE$ by $g(fx) = Sx$ and $h(fx) = Tx$ respectively. Since f is one-to-one on E, then g, h are well-defined. Now $fx_0 \in \overline{B(fx_0, r)} \subseteq fX$. Then $fx_0 \in fX$. Let $y_0 = fx_0$, choose a point y_1 in fX such that $y_1 = g(y_0)$ and let $y_2 = h(y_1)$. Continuing this process and having chosen y_n in fX such that

$$y_{2i+1} = g(y_{2i}) \text{ and } y_{2i+2} = h(y_{2i+1}), \text{ where } i = 0, 1, 2, ...,$$

We know that f is G-α-admissible then, $\alpha(x, y, z) \geq \eta(x, y, z)$ implies that $\alpha(fx, fy, fz) \geq \eta(fx, fy, fz)$ and also (S, T) is G-α-admissible then, $\alpha(x, y, z) \geq \eta(x, y, z)$ implies $\alpha(Sx, Ty, Tz) = \alpha(g(fx), h(fy), h(fz)) \geq \eta(g(fx), h(fy), h(fz))$ and $\alpha(h(fx), g(fy), g(fy)) \geq \eta(h(fx), g(fy), g(fz))$. This implies that the pair (g, h) is G-α-admissible. As $\alpha(y_0, y_1, y_1) \geq \eta(y_0, y_1, y_1)$ implies that $\alpha(gy_0, hy_1, hy_1)$

$\geq \eta(gy_0, hy_1, hy_1)$ and $\alpha(y_1, y_2, y_2) \geq \eta(y_1, y_2, y_2)$ implies that $\alpha(hy_1, gy_2, gy_2) \geq \eta(hy_1, gy_2, gy_2)$. Continuing this process, we have

$$\alpha(y_n, y_{n+1}, y_{n+1}) \geq \eta(y_n, y_{n+1}, y_{n+1}).$$

Following similar arguments of Theorem 2.2, $y_n \in \overline{B(fx_0, r)}$. Also by inequality (10).

$$\sum_{i=0}^{j} \psi^i(G_d(y_0, gy_0, gy_0) \leq r, \text{ for all } j \in N.$$

Note that for $fx,\ fy,\ fz \in \overline{B(fx_0, r)}$ and $\alpha(fx, fy, fz) \leq \eta(fx, fy, fy)$. Then by using inequality (2.9), we have,

$$\max\{G_d\left(g(fx), h(fy)\,, h(fz)\right), G_d(h(fx), g(fy), g(fz))\} \leq \psi(G_d(fx, fy, fz)).$$

As fX is a right complete space, all conditions of Theorem 2.5 are satisfied, we deduce that there exists a unique common fixed point $fp \in \overline{B(fx_0, r)}$ of g and h. Now $fp = g(fp) = h(fp)$ or $fp = Sp = Tp = fp$. Thus fp is the point of coincidence of S, T and f. Let $v \in \overline{B(fx_0, r)}$ be another point of coincidence of f, S and T then there exist $u \in \overline{B(fx_0, r)}$ such that $v = fu = Su = Tu$, which implies that $fu = g(fu) = h(fu)$. A contradiction as $fp \in \overline{B(fx_0, r)}$ is a unique common fixed point of g and h. Hence $v = fp$. Thus S, T and f have a unique point of coincidence $fp \in \overline{B(fx_0, r)}$. Now since (S, f) and (T, f) are weakly compatible, by Lemma 1.10 fp is a unique common fixed point of S, T and f. □

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