On the Least (Ordered) Semilattice Congruence in Ordered Γ-Semigroups

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Abstract : In this paper, we firstly characterize the relationship between the (ordered) filters, (ordered) s-prime ideals and (ordered) semilattice congruences in ordered Γ-semigroups. Finally, we give some characterizations of semilattice congruences and ordered semilattice congruences on ordered Γ-semigroups and prove that
1. $\mathcal{N}$ is the least semilattice congruence,
2. $\mathcal{N}$ is the least ordered semilattice congruence,
3. $\mathcal{N}$ is not the least semilattice congruence in general.

Keywords : Ordered Γ-semigroup; (ordered) filter; (ordered) s-prime ideal; Least (ordered) semilattice congruence.

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1 Preliminaries

In 1998, Gao [8] gives some characterizations of semilattice congruences and ordered semilattice congruences on ordered semigroups. Now we also characterize the semilattice congruences and ordered semilattice congruences on ordered Γ-semigroups and give some characterizations of semilattice congruences and ordered semilattice congruences on ordered Γ-semigroups analogous to the characterizations of semilattice congruences and ordered semilattice congruences on ordered semigroups.

Let $M$ and $\Gamma$ be any two nonempty sets. $M$ is called a $\Gamma$-semigroup [3,4] if there exists a mapping $M \times \Gamma \times M \longrightarrow M$, written as $((a, \gamma, b) \mapsto a\gamma b$, satisfying the following identity $(aab)\beta c = a(a(b)c)$ for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$. A Γ-semigroup $M$ is called a commutative $\Gamma$-semigroup if $a\gamma b = b\gamma a$ for all $a, b \in M$ and $\gamma \in \Gamma$. A nonempty subset $K$ of a Γ-semigroup $M$ is called a sub-$\Gamma$-semigroup of $M$ if $a\gamma b \in K$ for all $a, b \in K$ and $\gamma \in \Gamma$.

For examples of Γ-semigroups, see [1,3,4].

A partially ordered Γ-semigroup $M$ is called an ordered $\Gamma$-semigroup (po-Γ-semigroup) if for any $a, b, c \in M$ and $\gamma \in \Gamma$, $a \leq b$ implies $a\gamma c \leq b\gamma c$ and $c\gamma a \leq c\gamma b$. 
Example 1. For \( a, b \in [0,1] \), let \( M = [0,a] \) and \( \Gamma = [0,b] \). Then \( M \) is an ordered \( \Gamma \)-semigroup under usual multiplication and usual partial order relation.

Example 2. Fix \( m \in \mathbb{Z} \), let \( M \) be the set of all integers of the form \( mn + 1 \) and \( \Gamma \) denote the set of all integers of the form \( mn + m - 1 \) where \( n \) is an integer. Then \( M \) is an ordered \( \Gamma \)-semigroup under usual addition and usual partial order relation.

Throughout this paper, \( M \) stands for an ordered \( \Gamma \)-semigroup. For nonempty subsets \( A \) and \( B \) of \( M \) and a nonempty subset \( \Gamma' \) of \( \Gamma \), let \( A\Gamma'B := \{a\gamma b : a \in A, b \in B \text{ and } \gamma \in \Gamma' \} \). If \( A = \{a\} \), then we also write \( \{a\}\Gamma'B \) as \( a\Gamma'B \), and similarly if \( B = \{b\} \) or \( \Gamma' = \{\gamma \} \). A nonempty subset \( A \) of \( M \) is called a left (right) ideal of \( M \) \([7]\) if \( M\Gamma A \subseteq A \) (\( A\Gamma M \subseteq A \) ). \( A \) is called an ideal of \( M \) if it is both a left ideal and a right ideal of \( M \). A left ideal (right ideal, ideal) \( A \) of \( M \) is called an ordered left ideal (right ideal, ideal) of \( M \) if for any \( b \in M \) and \( a \in A, b \leq a \) implies \( b \in A \).

The following definitions in this paper are introduced analogous some definitions in \([5, 7, 8]\).

A left ideal (right ideal, ideal) \( A \) of \( M \) is called an \( s \)-prime left ideal (right ideal, ideal) of \( M \) if for any \( a, b \in M \) and \( \gamma \in \Gamma, a\gamma b \in A \) implies \( a \in A \) or \( b \in A \). Equivalently, for any subsets \( B \) and \( C \) of \( M \) and \( \gamma \in \Gamma, B\gamma C \subseteq A \) implies \( B \subseteq A \) or \( C \subseteq A \). An \( s \)-prime left ideal (right ideal, ideal) \( A \) of \( M \) is called an ordered \( s \)-prime left ideal (right ideal, ideal) of \( M \) if \( A \) is an ordered left ideal (right ideal, ideal) of \( M \). Let 

\[
SP(M) := \{A : A \text{ is an } s\text{-prime ideal of } M\},
\]

\[
OSP(M) := \{A : A \text{ is an ordered } s\text{-prime ideal of } M\}.
\]

Then \( \emptyset \neq OSP(M) \subseteq SP(M) \).

For a subset \( H \) of \( M \) and \( a \in M \), denote \((H) := \{t \in M : t \leq h \text{ for some } h \in H\}\), \([H] := \{t \in M : h \leq t \text{ for some } h \in H\}\) and \( a \cup H := \{a\} \cup H \). For \( H = \{a\} \), we also write \((\{a\})\) as \( (a) \). Clearly, \( H \subseteq [H] \subseteq (H) \). For any subsets \( A \) and \( B \) of \( M \) with \( A \subseteq B \), we have \((A) \subseteq (B) \). A sub-\( \Gamma \)-semigroup \( F \) of \( M \) is called a left (right) filter of \( M \) if for any \( a, b \in M \) and \( \gamma \in \Gamma, a\gamma b \in F \) implies \( b \in F \) (\( a \in F \)). \( F \) is called a filter of \( M \) if it is both a left filter and a right filter of \( M \). A left filter (right filter, filter) \( F \) of \( M \) is called an ordered left filter (right filter, filter) of \( M \) if for any \( b \in M \) and \( a \in F, a \leq b \) implies \( b \in F \). The intersection of all filters (ordered filters) of \( M \) containing a nonempty subset \( A \) of \( M \) is the filter (ordered filter) of \( M \) generated by \( A \). For \( A = \{x\} \), let 

\[
n(x) \text{ denote the filter of } M \text{ generated by } \{x\},
\]

\[
N(x) \text{ denote the ordered filter of } M \text{ generated by } \{x\}.
\]

An equivalence relation \( \sigma \) on \( M \) is called a congruence \([2]\) if for any \( a, b, c \in M \) and \( \gamma \in \Gamma, (a, b) \in \sigma \) implies \((a\gamma c, b\gamma c) \in \sigma \) and \((c\gamma a, c\gamma b) \in \sigma \). A congruence \( \sigma \)
on $M$ is called a semilattice congruence [6] if for all $a, b \in M$ and $\gamma \in \Gamma$, $(a\gamma a, a) \in \sigma$ and $(a\gamma b, b\gamma a) \in \sigma$. A semilattice congruence $\sigma$ on $M$ is called an ordered semilattice congruence if for any $a, b \in M$ and $\gamma \in \Gamma$, $a \leq b$ implies $(a, a\gamma b) \in \sigma$. 

Now, let

$$SC(M) := \{ \sigma : \sigma \text{ is a semilattice congruence on } M \},$$

$$OSC(M) := \{ \sigma : \sigma \text{ is an ordered semilattice congruence on } M \}.$$

Then $\emptyset \neq OSC(M) \subseteq SC(M)$.

For a nonempty subset $A$ of $M$, define equivalence relations on $M$ as follows:

$$\sigma_A := \{(x, y) \in M \times M : x, y \in A \text{ or } x, y \notin A\},$$

$$n := \{(x, y) \in M \times M : n(x) = n(y)\},$$

$$N := \{(x, y) \in M \times M : N(x) = N(y)\}.$$ 

We note here that $\sigma_A = \sigma_M \setminus A$.

For any congruence $\sigma$ on $M$ and $x \in M$, let

$$f(x)_\sigma \text{ denote the filter of } M \text{ generated by } $\sigma$-class } (x)_\sigma,$$

$$t \text{ denote the filter of } M \text{ generated by } \bigcup_{y \in (x)_\sigma} n(y),$$

$$F(x)_\sigma \text{ denote the ordered filter of } M \text{ generated by } $\sigma$-class } (x)_\sigma,$$

$$T \text{ denote the ordered filter of } M \text{ generated by } \bigcup_{y \in (x)_\sigma} N(y).$$

The following results are also necessary for our considerations.

**Theorem 1.1.** Let $F$ be a nonempty subset of $M$. Then $F$ is a left filter of $M$ if and only if $M \setminus F = \emptyset$ or $M \setminus F$ is an s-prime left ideal of $M$.

**Proof.** Assume that $F$ is a left filter of $M$ and $M \setminus F \neq \emptyset$. First to show that $M \setminus F$ is a left ideal of $M$, let $x, y \in M \setminus F$ and $\gamma \in \Gamma$. Since $F$ is a left filter of $M$ and $y \notin F$, $x\gamma y \in M \setminus F$. Thus $M \setminus F$ is a left ideal of $M$. Next, let $x, y \in M$ and $\gamma \in \Gamma$ be such that $x\gamma y \in M \setminus F$. Since $F$ is a sub-$\Gamma$-semigroup of $M$, $x \in M \setminus F$ or $y \in M \setminus F$. Thus $M \setminus F$ is an s-prime left ideal of $M$.

Conversely, if $M \setminus F = \emptyset$, then $F = M$. Hence $F$ is a left filter of $M$. Assume that $M \setminus F$ is an s-prime left ideal of $M$. First to show that $F$ is a sub-$\Gamma$-semigroup of $M$, let $x, y \in F$ and $\gamma \in \Gamma$. Then $x\gamma y \in F$ because $M \setminus F$ is an s-prime left ideal of $M$. Thus $F$ is a sub-$\Gamma$-semigroup of $M$. Next, let $x, y \in M$ and $\gamma \in \Gamma$ be such that $x\gamma y \in F$. Then $y \in F$ because $M \setminus F$ is a left ideal of $M$, so $F$ is a left filter of $M$. \hfill \square

A similar result holds if we replace the word “left” by “right”. Then we get the following.
Corollary 1.2. Let $F$ be a nonempty subset of $M$. Then $F$ is a filter of $M$ if and only if $M \setminus F = \emptyset$ or $M \setminus F$ is an $s$-prime ideal of $M$.

Theorem 1.3. Let $F$ be a nonempty subset of $M$. Then $F$ is an ordered left filter of $M$ if and only if $M \setminus F = \emptyset$ or $M \setminus F$ is an ordered $s$-prime left ideal of $M$.

Proof. Assume that $F$ is an ordered left filter of $M$ and $M \setminus F \neq \emptyset$. By Theorem 1.1, $M \setminus F$ is an $s$-prime left ideal of $M$. Now, let $x \in M$ and $y \in M \setminus F$ be such that $x \leq y$. Then $x \in M \setminus F$ because $F$ is an ordered left filter of $M$, so $M \setminus F$ is an ordered $s$-prime left ideal of $M$.

Conversely, if $M \setminus F = \emptyset$, then $F = M$. Hence $F$ is an ordered left filter of $M$. Assume that $M \setminus F$ is an ordered $s$-prime left ideal of $M$. By Theorem 1.1, $F$ is a left filter of $M$. Now, let $x \in M$ and $y \in F$ be such that $y \leq x$. Then $x \in F$ because $M \setminus F$ is an ordered left ideal of $M$, so $F$ is an ordered left filter of $M$. □

Corollary 1.4. Let $F$ be a nonempty subset of $M$. Then $F$ is an ordered filter of $M$ if and only if $M \setminus F = \emptyset$ or $M \setminus F$ is an ordered $s$-prime ideal of $M$.

2 Semilattice Congruences and Ordered Semilattice Congruences

In this section, we characterize the relationship between the semilattice congruences, filters, and $s$-prime ideals in ordered $\Gamma$-semigroups. Likewise, the relationship between the ordered semilattice congruences, ordered filters, and ordered $s$-prime ideals in ordered $\Gamma$-semigroups are characterized.

The following lemmas are necessary for the main results and the first two lemmas are easy to verify.

Lemma 2.1. An equivalence relation $\sigma$ on $M$ is a congruence if and only if for any $a, b, c, d \in M$ and $\gamma \in \Gamma$, $(a, b) \in \sigma$ and $(c, d) \in \sigma$ imply $(a \gamma c, b \gamma d) \in \sigma$.

Lemma 2.2. If $\sigma \in \text{SC}(M)$, then the following statements hold.

(a) For each $x \in M$, the $\sigma$-class $(x)_\sigma$ is a sub-$\Gamma$-semigroup of $M$.

(b) The set $M/\sigma := \{(x)_\sigma : x \in M\}$ is a commutative $\Gamma$-semigroup under the multiplication defined by $(x)_\sigma \gamma (y)_\sigma = (x \gamma y)_\sigma$ for all $(x)_\sigma, (y)_\sigma \in M/\sigma$ and $\gamma \in \Gamma$.

Lemma 2.3. Let $A$ be a subset of $M$ and $\sigma_A \in \text{SC}(M)$. If $x \in M \setminus A$ and $a \in A$ with $x \mu a \notin A$ (resp. $a \mu x \notin A$) for some $\mu \in \Gamma$, then $x \gamma a \notin A$ (resp. $a \gamma x \notin A$) for all $\gamma \in \Gamma$.

Proof. Assume that $x \in M \setminus A$, $a \in A$ and $x \mu a \notin A$ for some $\mu \in \Gamma$. Then $(x, x \mu a) \in \sigma_A$, so $(x)_\sigma = (x \mu a)_{\sigma_A}$. Suppose that there exists $\gamma \in \Gamma$ such that
$x\gamma a \in A$. Then $(a, x\gamma a) \in \sigma_A$. Thus $(a)_{\sigma_A} = (x\gamma a)_{\sigma_A}$. By Lemma 2.2 (b),
$(x)_{\sigma_A} = (x\mu a)_{\sigma_A} = (x\mu a\gamma a)_{\sigma_A} = (x\gamma a)_{\sigma_A} = (a)_{\sigma_A}$. Thus $(x, a) \in \sigma_A$, so $a \notin A$. This is a contradiction. Therefore $x\gamma a \notin A$ for all $\gamma \in \Gamma$. □

As a consequence of this result, we obtain

**Lemma 2.4.** Let $A$ be a nonempty subset of $M$. Then $\sigma_A \in SC(M)$ if and only if one of $A$ or $M \setminus A$ is an $s$-prime ideal of $M$.

**Proof.** Assume that $\sigma_A \in SC(M)$. If $A = M$, then $A \in SP(M)$. Suppose that $A \subset M$. Then $M \setminus A \neq \emptyset$. First to show that $A$ and $M \setminus A$ are sub-$\Gamma$-semigroups of $M$, let $x, y \in A$ and $\gamma \in \Gamma$. Then $(x\gamma y, y\gamma y) \in \sigma_A$ and $(y\gamma y, y) \in \sigma_A$ because $(x, y) \in \sigma_A$. Thus $(x, y) \in \sigma_A$. Hence $x\gamma y \in A$, so $A$ is a sub-$\Gamma$-semigroup of $M$. The same argument applies to $M \setminus A$, we have $M \setminus A$ is a sub-$\Gamma$-semigroup of $M$.

Next, consider the following two cases:

**Case 1:** $M\Gamma A \subseteq A$. Then $\forall \gamma \in \Gamma$ such that $(x\gamma a, a\gamma x) \in \sigma_A$ and $x\gamma a \in A$ for all $\gamma \in \Gamma$. Hence $A$ is an ideal of $M$.

**Case 2:** $M\Gamma A \nsubseteq A$. Then there exist $x \in M, a \in A, \mu \in \Gamma$ but $x\mu a \notin A$. Since $A$ is a sub-$\Gamma$-semigroup of $M$, $x\gamma a \notin A \notin A$. By Lemma 2.3, $x\gamma a \notin A$. For all $\gamma \in \Gamma$. Thus $(x, x\gamma a) \in \sigma_A$ for all $\gamma \in \Gamma$. By Lemma 2.2 (b), $(x)_{\sigma_A} = (x\gamma a)_{\sigma_A} = (x)_{\sigma_A}(a)_{\sigma_A}$ for all $\gamma \in \Gamma$. Obviously, $M \setminus A = (x)_{\sigma_A}$ and $A = (a)_{\sigma_A}$, so $M \setminus A = (M \setminus A)\gamma A$ for all $\gamma \in \Gamma$. This implies that

$$M \setminus A = \bigcup_{\gamma \in \Gamma} (M \setminus A)\gamma A = (M \setminus A)\Gamma A.$$ 

Therefore

$$(M \setminus A)\Gamma M = (M \setminus A)\Gamma(A \cup (M \setminus A)) \subseteq ((M \setminus A)\Gamma A) \cup (M \setminus A) = M \setminus A,$$

so $M \setminus A$ is a right ideal of $M$. Since $(x\mu a, a\mu x) \in \sigma_A$ and $x\mu a \notin A, a\mu x \notin A$. By symmetry, $M \setminus A$ is a left ideal of $M$. This proves that $M \setminus A$ is an ideal of $M$.

Assume that $A$ is an ideal of $M$. Let $x, y \in M$ and $\gamma \in \Gamma$ be such that $x\gamma y \in A$. If $x, y \notin A$, then $(x, y) \in \sigma_A$. Thus $(x\gamma x, x) \in \sigma_A$ and $(x\gamma x, x\gamma y) \in \sigma_A$, so $(x, x\gamma y) \in \sigma_A$. Thus $x\gamma y \notin A$, which is impossible. Hence $A \in SP(M)$.

Similarly, we can show that if $M \setminus A$ is an ideal of $M$, then $M \setminus A \in SP(M)$.

Conversely, assume that $A \in SP(M)$. Now, let $x, y \in M$ be such that $(x, y) \in \sigma_A, a \in M$ and $\gamma \in \Gamma$. Then we have the following two cases:

**Case 1:** $x, y \in A$. Then $c\gamma x, c\gamma y, c\gamma x, c\gamma c \in A$ because $A$ is an ideal of $M$. Thus $(c\gamma x, c\gamma y) \in \sigma_A$ and $(c\gamma c, y\gamma c) \in \sigma_A$.

**Case 2:** $x, y \notin A$. Then $c\gamma x \in A$ if and only if $c\gamma y \in A$. Thus $(c\gamma x, c\gamma y) \in \sigma_A$.

By symmetry, $(c\gamma c, y\gamma c) \in \sigma_A$.

Hence $\sigma_A$ is a congruence on $M$. Next, let $a, b \in M$ and $\gamma \in \Gamma$. Then $a \in A$ if and only if $a\gamma a \in A$, so $(a, a\gamma a) \in \sigma_A$. Similarly, we have $a\gamma b \in A$ if and only if $b\gamma a \in A$, so $(a, a\gamma b, b\gamma a) \in \sigma_A$. This proves that $\sigma_A \in SC(M)$. Similarly, we can show that if $M \setminus A \in SP(M)$, then $\sigma_A \in SC(M)$.

Hence the proof is completed. □
Lemma 2.5. If \( A \) is a nonempty subset of \( M \), then the following statements are equivalent.

(a) \( \sigma_A \in OSC(M) \).

(b) One of \( A \) or \( M \setminus A \) is an ordered \( s \)-prime ideal of \( M \).

Proof. Assume that \( \sigma_A \in OSC(M) \). By Lemma 2.4, \( A \in SP(M) \) or \( M \setminus A \in SP(M) \). Assume that \( A \in SP(M) \). Now, let \( x \in M \) and \( a \in A \) be such that \( x \leq a \) and \( \gamma \in \Gamma \). Then \( (x, x \gamma a) \in \sigma_A \), so \( x \in A \) because \( x \gamma a \in A \). Hence \( A \in OSP(M) \).

Similarly, we can show that if \( M \setminus A \in SP(M) \), then \( M \setminus A \in OSP(M) \).

Conversely, assume that \( A \in OSP(M) \). Then \( \sigma_A \in SC(M) \) by Lemma 2.4. Now, let \( a, b \in M \) be such that \( a \leq b \) and \( \gamma \in \Gamma \). If \( a \in A \), then \( a \gamma b \in A \). If \( a \notin A \), then \( b \notin A \) and so \( a \gamma b \notin A \). Hence \( (a, a \gamma b) \in \sigma_A \), so \( \sigma_A \in OSC(M) \). Similarly, we can show that if \( M \setminus A \in OSP(M) \), then \( \sigma_A \in OSC(M) \).

Hence the proof is completed. \( \square \)

Lemma 2.6. If \( x \in M \) and \( \sigma \in SC(M) \), then the following statements hold.

(a) \( f(x)_\sigma = \{ a \in M : a \in (x)_\sigma \text{ or } u \gamma a \in (x)_\sigma \text{ for some } u \in f(x)_\sigma \text{ and } \gamma \in \Gamma \} \).

(b) \( f(x)_\sigma = t \).

(c) If \( b \in f(x)_\sigma \), then \( f(b)_\sigma \subseteq f(x)_\sigma \).

(d) \( \sigma = \{(x, y) \in M \times M : f(x)_\sigma = f(y)_\sigma \} \).

Proof. (a) Let

\[
N := \{ a \in M : a \in (x)_\sigma \text{ or } u \gamma a \in (x)_\sigma \text{ for some } u \in f(x)_\sigma \text{ and } \gamma \in \Gamma \}.
\]

It is clear that \((x)_\sigma \subseteq N \subseteq f(x)_\sigma \). Conversely, to show that \( N \) is a filter of \( M \), let \( a, b \in N \) and \( \gamma \in \Gamma \). If \( u_1 \gamma_1 a, u_2 \gamma_2 b \in (x)_\sigma \) for some \( u_1, u_2 \in f(x)_\sigma \) and \( \gamma_1, \gamma_2 \in \Gamma \), then \( u_1 \gamma_1 a \gamma u_2 \gamma_2 b \in (x)_\sigma \) by Lemma 2.2 (a). It follows from Lemma 2.2 (b) that

\[
(x)_\sigma = (u_1 \gamma_1 a \gamma u_2 \gamma_2 b)_\sigma = (u_1 \gamma_1 a \gamma b \gamma_2 u_2)_\sigma = (u_1 \gamma_1 u_2 \gamma a \gamma b)_\sigma.
\]

Thus \( a \gamma b \in N \) because \( u_1 \gamma_1 u_2 \in f(x)_\sigma \). Similarly, it is easy to verify in the remain cases that \( a \gamma b \in N \). Hence \( N \) is a sub-\( \Gamma \)-semigroup of \( M \). We note here that for any \( a, b \in M \) and \( \gamma \in \Gamma \), \( a \gamma b \in N \) implies \( b \gamma a \in N \). Next, let \( a, b \in M \) and \( \gamma \in \Gamma \) be such that \( a \gamma b \in N \). Since \( N \subseteq f(x)_\sigma \), we have \( a, b \in f(x)_\sigma \). Since \( a \gamma b \in N \), \( a \gamma b \in (x)_\sigma \) or \( u \gamma a \gamma b \in (x)_\sigma \) for some \( u \in f(x)_\sigma \) and \( \gamma \in \Gamma \). Thus \( b \in N \). Since \( b \gamma a \in N \), \( a \in N \). Hence \( N \) is a filter of \( M \), so \( f(x)_\sigma \subseteq N \). Therefore \( N = f(x)_\sigma \).

(b) From the fact that \((x)_\sigma \subseteq \bigcup_{y \in (x)_\sigma} n(y), \) we get \( f(x)_\sigma \subseteq t \). On the other hand, we have \( n(y) \subseteq f(x)_\sigma \) for all \( y \in (x)_\sigma \). Thus \( \bigcup_{y \in (x)_\sigma} n(y) \subseteq f(x)_\sigma \), so \( t \subseteq f(x)_\sigma \). Therefore \( f(x)_\sigma = t \).
(c) Let \( b \in f(x)_\sigma \). By (a), we have \( b \in (x)_\sigma \) or \( uab \in (x)_\sigma \) for some \( u \in f(x)_\sigma \) and \( \alpha \in \Gamma \). Thus \((x)_\sigma = (b)_\sigma \) or \((x)_{\sigma} = (uab)_\sigma \) which implies that \((b)_\sigma \subseteq f(x)_\sigma \). Therefore \( f(b)_\sigma \subseteq f(x)_\sigma \).

(d) Let \( \sigma \) is a congruence on \( M \). 

\[ \tau := \{(x, y) \in M \times M : f(x)_\sigma = f(y)_\sigma \}. \]

It is clear that \( \sigma \subseteq \tau \). Conversely, let \( x, y \in M \) be such that \((x, y) \in \tau \). Then \( f(x)_\sigma = f(y)_\sigma \), so \( x \in f(y)_\sigma \) and \( y \in f(x)_\sigma \). By (a), if \( x \in (y)_\sigma \) or \( y \in (x)_\sigma \), then \((x)_\sigma = (y)_\sigma \). Therefore \((x)_\sigma = (y)_\sigma \). Let \( u_1 \gamma_1 x \in (y)_\sigma \) and \( u_2 \gamma_2 y \in (x)_\sigma \) for some \( u_1, u_2 \in f(x)_\sigma \). Thus \( u_1 \gamma_1 x = u_2 \gamma_2 y \). Hence \((x, y) \in \tau \), so \( \sigma = \tau \).

Immediately from Lemma 2.6, we have

**Corollary 2.7.** If \( x \in M \) and \( \sigma \in SC(M) \), then \( f(x)_\sigma = \{a \in M : a \in (x)_\sigma \) or \( u \gamma a \in (x)_\sigma \) or \( u \mu \gamma v \in (x)_\sigma \) for some \( u, v \in f(x)_\sigma \) and \( \gamma, \mu \in \Gamma \} \).

**Corollary 2.8.** If \( x \in M \), then the following statements hold.

(a) \( n \in SC(M) \).

(b) \( f(x)_n = n(x) \).

(c) \( n(x) = \{a \in M : a \in (x)_n \) or \( u \gamma a \in (x)_n \) for some \( u \in n(x) \) and \( \gamma \in \Gamma \} \).

**Proof.** (a) Let \( a, b \in M \) be such that \((a, b) \in n, c \in M \) and \( \gamma \in \Gamma \). Then \( n(a) = n(b) \). Since \( \beta \gamma c \in n(b \gamma c) \), we have \( b, c \in n(b \gamma c) \). Thus \( n(a) = n(b) \subseteq n(b \gamma c) \), so \( a, c \in n(b \gamma c) \). Hence \( a \gamma c \in n(b \gamma c) \), so \( n(a \gamma c) \subseteq n(b \gamma c) \). Similarly, \( n(b \gamma c) \subseteq n(a \gamma c) \).

Therefore \( n(a \gamma c) = n(b \gamma c) \), so \((a \gamma c, b \gamma c) \in n \). Similarly, \((c \gamma a, c \gamma b) \in n \). This proves that \( n \) is a congruence on \( M \). Next, let \( a, b, c \in \Gamma \) and \( \gamma \in \Gamma \). Then \( a \in n(a \gamma a) \) because \( a \gamma a \in n(a \gamma a) \), so \( n(a) \subseteq n(a \gamma a) \). Since \( a \in n(a) \), \( a \gamma a \in n(a) \). Hence \( n(a \gamma a) \subseteq n(a) \), so \( n(a \gamma a) = n(a) \). Therefore \((a \gamma a, a) \in n \). Similarly, \( n(a \gamma b) \subseteq n(b \gamma c) \).

(b) By (a) and Lemma 2.6 (b), \( f(x)_n = t \) where \( t \) is the filter of \( M \) generated by \( \bigcup_{y \in (x)_n} n(y) \). We note here that

\[ \bigcup_{y \in (x)_n} n(y) = n(x). \]

Hence \( t = n(x) \), so \( f(x)_n = n(x) \).

(c) By (a) and Lemma 2.6 (a),

\[ n(x) = f(x)_n = \{a \in M : a \in (x)_n \) or \( u \gamma a \in (x)_n \) for some \( u \in f(x)_n \) and \( \gamma \in \Gamma \} = \{a \in M : a \in (x)_n \) or \( u \gamma a \in (x)_n \) for some \( u \in n(x) \) and \( \gamma \in \Gamma \}. \]
Hence the proof is completed.

\[\text{Lemma 2.9. If } x \in M \text{ and } \sigma \in \text{OSC}(M), \text{ then the following statements hold.}\]

(a) \(F(x)_\sigma = \{ a \in M : a \in [(x)_\sigma] \text{ or } u \gamma a \in [(x)_\sigma] \text{ for some } u \in F(x)_\sigma \text{ and } \gamma \in \Gamma \}.\]

(b) \(F(x)_\sigma = T.\)

(c) If \(b \in F(x)_\sigma, \text{ then } F(b)_\sigma \subseteq F(x)_\sigma.\)

(d) \(\sigma = \{(x, y) \in M \times M : F(x)_\sigma = F(y)_\sigma \}.\)

**Proof.** (a) Let

\[N := \{ a \in M : u \gamma a \in [(x)_\sigma] \text{ or } a \in [(x)_\sigma] \text{ for some } u \in F(x)_\sigma \text{ and } \gamma \in \Gamma \}.\]

It is clear that \((x)_\sigma \subseteq N \subseteq F(x)_\sigma.\) Conversely, to show that \(N\) is an ordered filter of \(M,\) let \(a, b \in N\) and \(\gamma \in \Gamma.\) If \(u_1 \gamma_1 a, u_2 \gamma_2 b \in [(x)_\sigma]\) for some \(u_1, u_2 \in F(x)_\sigma\) and \(\gamma_1, \gamma_2 \in \Gamma,\) then \(y_1 \leq u_1 \gamma_1 a\) and \(y_2 \leq u_2 \gamma_2 b\) for some \(y_1, y_2 \in (x)_\sigma.\) Thus \(y_1 \gamma_2 \leq u_1 \gamma_1 a \gamma_2\) and \(y_1 \gamma_2 \leq y_2 \gamma_2 \in (x)_\sigma\) by Lemma 2.2 (a). Hence \((y_1 \gamma_2, y_1 \gamma_2 \gamma_1 \gamma_2 \gamma_2\) \(\in \sigma\) which implies that \((x, x \gamma u_1 \gamma_1 a \gamma_2 \gamma_2 b) \in \sigma.\) It follows from Lemma 2.2 (b) that

\[(x)_\sigma = (x \gamma u_1 \gamma_1 a \gamma_2 \gamma_2 b)_\sigma = (x \gamma u_1 \gamma_1 a \gamma_2 \gamma_2 a \gamma_2 b)_\sigma .\]

Thus \(a \gamma b \in N\) because \(x \gamma u_1 \gamma_1 u_2 \in F(x)_\sigma.\) Similarly, it is easy to verify in the remain cases that \(a \gamma b \in N.\) Hence \(N\) is a sub-\(\Gamma\)-semigroup of \(M.\) We note here that for any \(a, b \in M\) and \(\gamma \in \Gamma, a \gamma b \in N\) implies \(b \gamma a \in N.\) Let \(a, b \in M\) be such that \(a \gamma b \in N\) and \(\gamma \in \Gamma.\) Since \(N \subseteq F(x)_\sigma,\) we have \(a, b \in F(x)_\sigma.\) Since \(a \gamma b \in N,\)

\[a \gamma b \in [(x)_\sigma] \text{ or } u a a \gamma b \in [(x)_\sigma] \text{ for some } u \in F(x)_\sigma \text{ and } \alpha \in \Gamma.\]

Thus \(b \in N.\) Since \(b \gamma a \in N, a \in N.\) Hence \(N\) is a filter of \(M.\) Next, let \(b \in M\) and \(a \in N\) be such that \(a \leq b.\) Then \(a \in [x)_\sigma] \text{ or } u a a \in [(x)_\sigma] \text{ for some } u \in F(x)_\sigma \text{ and } \alpha \in \Gamma \text{ which implies that } b \in [(x)_\sigma] \text{ or } u a b \in [(x)_\sigma].\)

(b) It is similar to the proof of Lemma 2.6 (b).

(c) Let \(b \in F(x)_\sigma\) and \(\gamma \in \Gamma.\) By (a), we have \(b \in [(x)_\sigma] \text{ or } u a b \in [(x)_\sigma] \text{ for some } u \in F(x)_\sigma \text{ and } \alpha \in \Gamma.\) Thus \((x)_\sigma = (x \gamma b)_\sigma \text{ or } (x)_\sigma = (x \gamma u a b)_\sigma \text{ which implies that } (b)_\sigma \subseteq F(x)_\sigma.\) Therefore \(F(b)_\sigma \subseteq F(x)_\sigma.\)

(d) Let

\[\tau := \{(x, y) \in M \times M : F(x)_\sigma = F(y)_\sigma \}.\]

It is clear that \(\sigma \subseteq \tau.\) Conversely, let \(x, y \in M\) be such that \((x, y) \in \tau \text{ and } \gamma \in \Gamma.\) Then \(F(x)_\sigma = F(y)_\sigma, \text{ so } x \in F(y)_\sigma \text{ and } y \in F(x)_\sigma.\) By (a), it suffices to show that the following case is satisfied. If \(u_1 \gamma_1 x \in [(y)_\sigma] \text{ and } u_2 \gamma_2 y \in [(x)_\sigma]\) for some \(u_1, u_2 \in F(x)_\sigma, y \in F(y)_\sigma\) and \(\gamma_1, \gamma_2 \in \Gamma,\) then \((y, y \gamma u_1 \gamma_1 x) \in \sigma \text{ and } (x, x \gamma u_2 \gamma_2 y) \in \sigma.\) It follows from Lemma 2.2 (b) that
On the Least (Ordered) Semilattice Congruences in Ordered $\Gamma$-Semigroups

$$(u_2\gamma_2 y)\sigma = (u_2\gamma_2 yu_1\gamma_1 x)\sigma = (u_1\gamma_1 xu_2\gamma_2 y)\sigma = (u_1\gamma_1 x)\sigma.$$ 

Hence $(x)_\sigma = (xu_2\gamma_2 y)\sigma = (u_2\gamma_2 yx)\sigma = (u_1\gamma_1 xu_2\gamma_2 y)\sigma = (u_2\gamma_2 y)\sigma = (xu_2\gamma_2 y)\sigma = (y\gamma_1 xu_2\gamma_2 y)\sigma = (y)\sigma$, so $(x, y) \in \sigma$. Similarly, it is easy to verify in the remain cases that $(x, y) \in \sigma$. Therefore $\sigma = \tau$. □

Immediately from Lemma 2.8, we have

**Corollary 2.10.** If $x \in M$ and $\sigma \in \text{OSC}(M)$, then $F(x)_\sigma \subseteq N(x)$.

**Proof.** (a) By the similar way of proof of Corollary 2.8 (a), we have $N \in \text{SC}(M)$. Now, let $a, b \in M$ be such that $a \leq b$ and $\gamma, \mu \in \Gamma$. Then $a, b \in N(a\gamma b)$ because $a\gamma b \in N(a\gamma b)$, so $N(a) \subseteq N(a\gamma b)$. Since $a \in N(a), b \in N(a)$. Thus $a\gamma b \in N(a), N(a\gamma b) \subseteq N(a)$. Hence $N(a) = N(a\gamma b)$, so $a, a\gamma b \in N$. Therefore $N \in \text{OSC}(M)$.

(b) It is similar to the proof of Corollary 2.8 (b).

(c) It is similar to the proof of Corollary 2.8 (c).

Hence the proof is completed. □

**3 Main Results**

In last section, we characterize the least semilattice congruences and ordered semilattice congruences on ordered $\Gamma$-semigroups and show that $N$ is not the least semilattice congruence on ordered $\Gamma$-semigroups in general.

**Theorem 3.1.**

(a) $N = \bigcap_{I \in \text{SP}(M)} \sigma_I$.

(b) $N = \bigcap_{I \in \text{OSP}(M)} \sigma_I$.

(b) $N \subseteq N$.

**Proof.** (a) Let
Hence the theorem is proved. □

Let \( x, y \in M \) be such that \((x, y) \in n\). Then \( n(x) = n(y) \). Suppose that there exists \( I \in SP(M) \) such that \((x, y) \notin \sigma_I\). By Corollary 1.2, \( M \setminus I \) is a filter of \( M \). Without loss of generality, we may assume that \( x \in I \) and \( y \in M \setminus I \). Then \( x \in n(x) = n(y) \subseteq M \setminus I \), which is impossible. Hence \((x, y) \in \sigma_I\) for all \( I \in SP(M) \), so \((x, y) \in \tau\). Conversely, let \( x, y \in M \) be such that \((x, y) \in \tau\). Then \((x, y) \in \sigma_I\) for all \( I \in SP(M) \). Suppose that \((x, y) \notin n\). Then \( n(x) \neq n(y) \). By Corollary 2.8 \( (b) f(x)_n = n(x) \neq n(y) = f(y)_n \). Without loss of generality, we may assume that \( f(x)_n \not\subseteq f(y)_n \). By Lemma 2.6 \( (c) x \notin f(y)_n \). Then \((x, y) \notin \sigma_{M \setminus f(y)_n}\). Since \( M \setminus f(y)_n \neq \emptyset \), it follows from Corollary 1.2 that \( M \setminus f(y)_n \in SP(M) \). This implies that \((x, y) \notin \sigma_{M \setminus f(y)_n}\), which is impossible. Hence \((x, y) \in n\), this proves that \( n = \bigcap \{ \sigma_I : I \in SP(M) \}\).

(b) It is similar to the proof of \((a)\).

(c) Since \( OSP(M) \subseteq SP(M) \), it follows from \((a)\) and \((b)\) that \( n \subseteq N\).

Hence the theorem is proved.

**Theorem 3.2.** If \( \sigma \in SC(M) \), then the following statements hold.

\[
(a) \quad \sigma = \bigcap_{x \in M} \sigma_{M \setminus f(x)_{\sigma}}.
\]

\[
(b) \quad n \subseteq \sigma, \text{i.e., } n \text{ is the least element of } SC(M).
\]

**Proof.** \((a)\) Let

\[
\tau := \bigcap_{x \in M} \sigma_{M \setminus f(x)_{\sigma}}.
\]

Let \(x, y \in M\) be such that \((x, y) \in \sigma\). Then \( f(x)_\sigma = f(y)_\sigma \) by Lemma 2.6 \((d)\). Suppose that \((x, y) \notin \sigma_{M \setminus f(a)_\sigma}\) for some \( a \in M \). Without loss of generality, we may assume that \( x \in M \setminus f(a)_\sigma \) and \( y \notin M \setminus f(a)_\sigma \). Then \( y \in f(a)_\sigma\), it follows from Lemma 2.6 \((c)\) that \( x \in f(x)_\sigma = f(y)_\sigma \subseteq f(a)_\sigma\). It is impossible, so \((x, y) \in \sigma_{M \setminus f(a)_\sigma}\) for all \( a \in M \). Conversely, let \( x, y \in M \) be such that \((x, y) \in \tau\). Then \((x, y) \in \sigma_{M \setminus f(a)_\sigma}\) for all \( a \in M \). Suppose that \((x, y) \notin \sigma\). By Lemma 2.6 \((d)\), \( f(x)_\sigma \neq f(y)_\sigma \). Without loss of generality, we may assume that \( f(x)_\sigma \not\subseteq f(y)_\sigma\). By Lemma 2.6 \((c)\), \( x \notin f(y)_\sigma\). Then \((x, y) \notin \sigma_{M \setminus f(y)_\sigma}\), which is impossible. Hence \((x, y) \in \sigma\), this proves that

\[
\sigma = \bigcap_{x \in M} \sigma_{M \setminus f(x)_{\sigma}}.
\]

\((b)\) By Corollary 1.2, \( M \setminus f(x)_\sigma = \emptyset \) or \( M \setminus f(x)_\sigma \in SP(M) \) for all \( x \in M \). Thus

\[
\{ \sigma_{M \setminus f(x)_\sigma} : x \in M \} \subseteq \{ \sigma_I : I \in SP(M) \}.
\]
On the Least (Ordered) Semilattice Congruences in Ordered Γ-Semigroups

By (a) and Theorem 3.1 (a), \( n \subseteq \sigma \). Therefore \( n \) is the least semilattice congruence on \( M \).

By the similarity of the proof of Theorem 3.2, we obtain

**Theorem 3.3.** If \( \sigma \in \text{OSC}(M) \), then the following statements hold.

(a) \( \sigma = \bigcap_{x \in M} \sigma_{M \setminus F(x), x} \).

(b) \( N \subseteq \sigma \), i.e., \( N \) is the least element of \( \text{OSC}(M) \).

Immediately from Theorem 3.2 and Theorem 3.3, we have

**Corollary 3.4.**

(a) \( n = \bigcap_{x \in M} \sigma_{M \setminus n(x)} \).

(b) \( N = \bigcap_{x \in M} \sigma_{M \setminus N(x)} \).

We shall give an example of an ordered Γ-semigroup \( M \) with \( N \) is not the least semilattice congruence on \( M \).

**Example 3.5.** Let \( M = \{a, b, c, d\} \) and \( \Gamma = \{\gamma\} \) with the multiplication defined by

\[
x \gamma y = \begin{cases} 
  b & \text{if } x, y \in \{a, b\}, \\
  c & \text{otherwise.}
\end{cases}
\]

First to show that \( M \) is a Γ-semigroup, suppose not. Then there exist \( x, y, z \in M \) such that \( (x \gamma y) \gamma z \neq x \gamma (y \gamma z) \). If \( (x \gamma y) \gamma z = b \), then \( x, y, z \in \{a, b\} \). Thus \( x \gamma (y \gamma z) = b \), which is impossible. If \( x \gamma (y \gamma z) = b \), then \( x, y, z \in \{a, b\} \). Thus \( (x \gamma y) \gamma z = b \), which is impossible. Hence \( (x \gamma y) \gamma z = x \gamma (y \gamma z) \) for all \( x, y, z \in M \). Obviously, \( x \gamma y = y \gamma x \) for all \( x, y \in M \). Therefore \( M \) is a commutative Γ-semigroup.

Define a relation \( \leq \) on \( M \) as follows:

\[
\leq := \{(a, a), (b, b), (c, c), (d, d), (a, b), (b, a), (c, d), (d, c)\}.
\]

Then \( (M, \leq) \) is a partially ordered set. Let \( x, y \in M \) be such that \( x \leq y \). Since \( x \gamma c = c \gamma x \) and \( x \gamma d = c \gamma d \) for all \( x, y \in M \) and \( b \leq c \), \( x \gamma z \leq y \gamma z \) and \( z \gamma x \leq z \gamma y \) for all \( z \in M \). Hence \( M \) is an ordered Γ-semigroup. We shall show that \( SC(M) = \{n, N\} \) and \( n \subset N \). Let

\[
\begin{align*}
\sigma_1 &= M \times M, \\
\sigma_2 &= \{(a, a), (b, b), (c, c), (d, d), (a, b), (b, a), (c, d), (d, c)\}.
\end{align*}
\]
It is easy to see that $\sigma_1, \sigma_2 \in SC(M)$. Since $(a\gamma a, a) = (b, a)$ and $(d\gamma d, d) = (c, d), \sigma_2 \subseteq \sigma$ for all $\sigma \in SC(M)$. Let $\sigma \in SC(M)$. Then we have the following two cases:

**Case 1:** $(b, c) \in \sigma$. Since $(a, b) \in \sigma$, $(a, c) \in \sigma$. Thus $(a, d), (b, d) \in \sigma$ because $(c, d) \in \sigma$. Hence $\sigma = \sigma_1$.

**Case 2:** $(b, c) \notin \sigma$. If $(a, c) \in \sigma$, then $(b, c) \in \sigma$ because $(b, a) \in \sigma$, which is impossible. If $(a, d) \in \sigma$, then $(a, c) \in \sigma$ because $(d, c) \in \sigma$, which is impossible. Hence $\sigma = \sigma_2$.

This proves that $SC(M) = \{\sigma_1, \sigma_2\}$. We shall show that $\sigma_1 = N$ and $\sigma_2 = n$.

We can easily get all ideals of $M$ as follows:

$$P_1 = M, P_2 = \{c, d\}, P_3 = \{b, c\}, P_4 = \{c\}, P_5 = \{a, b, c\}, P_6 = \{b, c, d\}.$$ 

It is easy to see that $SP(M) = \{P_1, P_2\}$ and $OSP(M) = \{P_1\}$. By Theorem 3.1, we obtain that

$$N = \bigcap_{I \in OSP(M)} \sigma_I = \sigma_{P_1} = M \times M = \sigma_1$$

and

$$n = \bigcap_{I \in SP(M)} \sigma_I = \sigma_{P_1} \cap \sigma_{P_2} = \sigma_{P_2}.$$ 

We note here that

$$\sigma_{P_2} = \{(x, y) \in M \times M : x, y \in P_2 \text{ or } x, y \notin P_2\} \setminus \{(a, a), (b, b), (c, c), (d, d), (a, b), (b, a), (c, d), (d, c)\} = \sigma_2.$$ 

Hence $n = \sigma_2$, so $n \subset N$.

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**References**


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