Common Fixed Point Theorems for the Finite Family of Uniformly Quasi-sup(f_n) Lipschitzian Mappings and g_n—Expansive Mappings in Convex Metric Spaces

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Abstract: In this paper we introduce a new n-steps iterative process, which converges to a common fixed point of finite families of uniformly quasi-sup (f_n) Lipschitzian mappings and g_n—expansive mappings in convex metric spaces.

Keywords: convex metric space; common fixed point theorems; uniformly quasi-sup (f_n) Lipschitzian mapping; g_n—expansive mapping.

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1 Introduction

In 2011, Byung—Soo Lee defined new family of mappings: infinite family \( \{T_i\}_{i=1}^{\infty} \) of uniformly quasi-sup (f_n) Lipschitzian mappings and infinite family \( \{S_i\}_{i=1}^{\infty} \) of g_n—expansive mappings for approximating a common fixed point in convex metric spaces using a Noor—type iterative. The iterative is defined as follows: let \( C \) be a nonempty convex subset of \( (X,d,W) \), \( \{T_i\}_{i=1}^{\infty} \) an infinite family of uniformly quasi-sup (f_n) Lipschitzian mappings and \( \{S_i\}_{i=1}^{\infty} \) a g_n—expansive
mappings of \( C \). Suppose that \( \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}, \{e_n\}, \{l_n\} \) are sequences in \([0, 1]\) for which \( \alpha_n + \beta_n + \gamma_n = a_n + b_n + c_n = d_n + e_n + l_n = 1 \) for all \( n \in \mathbb{N} \). For \( x_1 \in X \), let \( \{x_n\} \) be a sequence defined by

\[
\begin{align*}
x_{n+1} &= W(S_n x_n, T_n^a y_n, u_n; \alpha_n, \beta_n, \gamma_n) \\
y_n &= W(S_n x_n, T_n^a z_n, v_n; \alpha_n, b_n, c_n) \\
z_n &= W(S_n x_n, T_n^a w_n; \alpha_n, e_n, l_n)
\end{align*}
\]

(1.1)

where \( \{u_n\}, \{v_n\}, \{w_n\} \) are any sequences in \( X \).

In 2013, Phueangratana and Suantai [2] introduced a new iterative process for approximating a common fixed point of a finite family \( \{T_i\}_{i=1}^N \) of generalized asymptotically quasi–nonexpansive mappings in a convex metric space. The following is the iterative process: let \( C \) be a convex subset of a convex metric space \((X, d, W)\) and \( \{T_i\}_{i=1}^N \) a finite family of generalized asymptotically quasi–nonexpansive mappings. Suppose that \( \{\alpha_n^{(i)}\}, \{\beta_n^{(i)}\}, \{\gamma_n^{(i)}\} \) are sequences in \([0, 1]\) for which \( \alpha_n^{(i)} + \beta_n^{(i)} + \gamma_n^{(i)} = 1 \) for each \( n \in \mathbb{N} \) and \( i = 1, 2, \ldots, N \). For \( x_1 \in C \), let \( \{x_n\} \) be a sequence defined by

\[
\begin{align*}
y_n^{(0)} &= x_n \\
y_n^{(1)} &= W(T_n y_n^{(0)}, y_n^{(0)}; \alpha_n^{(1)}) \\
y_n^{(2)} &= W(T_n^a y_n^{(1)}, y_n^{(1)}; \alpha_n^{(2)}) \\
y_n^{(3)} &= W(T_n^a y_n^{(2)}, y_n^{(2)}; \alpha_n^{(3)}) \\
&\vdots \\
y_n^{(N-1)} &= W(T_{N-1} y_n^{(N-2)}, y_n^{(N-2)}; \alpha_n^{(N-1)}) \\
x_{n+1} &= W(T_N y_n^{(N-1)}, y_n^{(N-1)}; \alpha_n^{(N)})
\end{align*}
\]

(1.2)

for all \( n \in \mathbb{N} \).

Motivated by [1] and [2], we define a new \( n \)-steps iterative process to approximate a common fixed point of a finite family \( \{T_i\}_{i=1}^N \) of uniformly quasi–sup (\( f_n \)) Lipschitzian mappings and a finite family \( \{S_i\}_{i=1}^N \) of \( g_n \)-expansive mappings in convex metric spaces. Our new iterative process is explained as follows: let \( C \) be a nonempty convex subset of a convex metric space \((X, d, W)\). Let \( \{T_i\}_{i=1}^N \) be a finite family of uniformly quasi–sup (\( f_n \)) Lipschitzian mappings and \( \{S_i\}_{i=1}^N \) a finite family of \( g_n \)-expansive mappings of \( C \). Suppose that \( \{\alpha_n^{(i)}\}, \{\beta_n^{(i)}\}, \{\gamma_n^{(i)}\} \) are sequences in \([0, 1]\) such that \( \alpha_n^{(i)} + \beta_n^{(i)} + \gamma_n^{(i)} = 1 \) for each \( n \in \mathbb{N} \) and \( i = 1, 2, \ldots, N \). For \( x_1 \in C \), let \( \{x_n\} \) be a sequence defined by

\[
\begin{align*}
y_n^{(0)} &= x_n \\
y_n^{(1)} &= W(S_n y_n^{(0)}, T_n^a y_n^{(0)}, u_n^{(0)}; \alpha_n^{(1)}, \beta_n^{(1)}, \gamma_n^{(1)}) \\
y_n^{(2)} &= W(S_n^a y_n^{(1)}, T_n^a y_n^{(1)}, u_n^{(1)}; \alpha_n^{(2)}, \beta_n^{(2)}, \gamma_n^{(2)}) \\
y_n^{(3)} &= W(S_n^a y_n^{(2)}, T_n^a y_n^{(2)}, u_n^{(2)}; \alpha_n^{(3)}, \beta_n^{(3)}, \gamma_n^{(3)}) \\
&\vdots \\
y_n^{(N-1)} &= W(S_n^{N-1} y_n^{(N-2)}, T_n^{N-1} y_n^{(N-2)}, u_n^{(N-2)}; \alpha_n^{(N-1)}, \beta_n^{(N-1)}, \gamma_n^{(N-1)}) \\
x_{n+1} &= y_n^{(N)} = W(S_n^N y_n^{(N-1)}, T_n^{N} y_n^{(N-1)}, u_n^{(N-1)}; \alpha_n^{(N)}, \beta_n^{(N)}, \gamma_n^{(N)})
\end{align*}
\]

(1.3)
for all $n \in \mathbb{N}$ and $\{u_n^{(i)}\}$ are any bounded sequences in $C$.

The purpose of this paper is to extend and improve some results of [1] and [2].

## 2 Preliminaries

In this section, the definition of mapping which will be used in the paper is presented as follow.

**Definition 2.1** ([1] Definition 1.1), [2]). Let $C$ be a nonempty subset of a metric space $(X, d)$, $T$ a self-mapping on $C$ and $f : C \to (0, \infty)$ a function which is bounded above. The set of fixed point of $T$ is denote by $F(T)$, i.e., $F(T) = \{x \in C : Tx = x\}$.

(i) $T$ is $f$–expansive if

$$d(Tx, Ty) \leq \sup_{z \in C} f(z) \cdot d(x, y)$$

for all $x, y \in C$.

(ii) $T$ is asymptotically $f$–expansive if there exists a sequence $\{x_n\}$ in $C$ such that $\lim_{n \to \infty} f(x_n) = 1$ satisfying

$$d(T^n x, T^n y) \leq f(x_n) \cdot d(x, y)$$

for all $x, y \in C$ and $n \in \mathbb{N}$.

(iii) $T$ is asymptotically quasi–$f$–expansive if there exists a sequence $\{x_n\}$ in $C$ such that $\lim_{n \to \infty} f(x_n) = 1$ satisfying

$$d(T^n x, p) \leq f(x_n) \cdot d(x, p)$$

for all $x \in C, p \in F(T)$ and $n \in \mathbb{N}$.

(iv) $T$ is uniformly quasi–sup $(f)$ Lipschitzian if

$$d(T^n x, p) \leq \sup_{z \in C} f(z) \cdot d(x, p)$$

for all $x \in C, p \in F(T)$ and $n \in \mathbb{N}$.

(v) $T$ is uniformly $L$–Lipschitzian if there exists constant $L > 0$ such that

$$d(T^n x, T^n y) \leq L \cdot d(x, y)$$

for all $x, y \in C$ and $n \in \mathbb{N}$.

Some definitions and useful results related to convex structure and convex metric space are recalled next.
Definition 2.2 ([1] Definition 1.2). Let \((X, d)\) be a metric space. A mapping \(W : X^3 \times I^3 \to X\) is said to be a *convex structure* on \(X\) if for each \(x, y, z \in X\) and \(\alpha, \beta, \gamma \in I\) with \(\alpha + \beta + \gamma = 1\) satisfy
\[
d(W(x, y, z; \alpha, \beta, \gamma), u) \leq \alpha d(x, u) + \beta d(y, u) + \gamma d(z, u)
\]
for all \(u \in X\). Moreover, a metric space \((X, d)\) with a convex structure \(W\) is called a convex metric space which will be denoted by \((X,d,W)\). A nonempty subset \(C\) of a convex metric space \((X,d,W)\) is said to be a convex subset of \((X,d)\) if \(W(x,y,z;\alpha,\beta,\gamma) \in C\) for \((x,y,z) \in C^3\) and \((\alpha,\beta,\gamma) \in I^3\) with \(\alpha + \beta + \gamma = 1\).

Definition 2.3 ([2]). Let \(C\) be a subset of a metric space \((X,d)\). A finite family of self mappings \(\{T_i\}_{i=1}^N\) and \(\{S_i\}_{i=1}^N\) of \(C\) are said to have *Condition A* if there exists a nondecreasing function \(f : [0, \infty) \to [0, \infty)\) with \(f(0) = 0\) and \(f(r) > 0\) and function \(g : [0, \infty) \to [0, \infty)\) with \(g(0) = 0\) and \(g(r) > 0\), respectively for all \(r > 0\) such that
\[
d(x,T_i(x)) \geq f(d(x,D)) \quad \text{and} \quad d(x,S_i(x)) \geq g(d(x,D))
\]
for some \(i, 1 \leq i \leq N\) and for all \(x \in C\), where \(d(x,D) = \inf \left\{ d(x,p) : p \in D = \left( \bigcap_{i=1}^N F(T_i) \right) \cap \left( \bigcap_{i=1}^N F(S_i) \right) \right\}\).

Definition 2.4 ([2]). Let \(C\) be a subset of a metric space \((X,d)\). A mapping \(T\) is *semi-compact* if for a sequence \(\{x_n\}\) in \(C\) with \(\lim_{n \to \infty} d(x_n,Tx_n) = 0\), there exists a subsequence \(\{x_{n_i}\}\) of \(\{x_n\}\) such that \(x_{n_i} \to p \in C\).

Lemma 2.5 ([3] Lemma 1.1, Remark 1.3, [4] Lemma 2). Let \(\{a_n\}, \{b_n\}, \{c_n\}\) be sequences of nonnegative real numbers such that \(\sum_{n=1}^\infty b_n < \infty\) and \(\sum_{n=1}^\infty c_n < \infty\) and for all \(n \in \mathbb{N}\),
\[
a_{n+1} \leq (1 + b_n) a_n + c_n.
\]
Then,
(i) \(\lim_{n \to \infty} a_n\) exists,
(ii) If \(\lim_{n \to \infty} a_n = 0\) then \(\lim_{n \to \infty} a_n = 0\),
(iii) If either \(\lim_{n \to \infty} a_n = 0\) or \(\limsup_{n \to \infty} a_n = 0\) then \(\lim_{n \to \infty} a_n = 0\).

Definition 2.6 ([2] Definition 2.3). Let \(\{r_n\}\) be a sequence in a metric space \((X,d)\) and \(D\) a subset of \(X\). We say that \(\{r_n\}\) is of *monotone type 1* with respect to \(D\) if there exist sequences \(\{r_n\}\) and \(\{s_n\}\) of nonnegative real numbers such that
\[
\sum_{n=1}^\infty r_n < \infty, \sum_{n=1}^\infty s_n < \infty \quad \text{and} \quad d(x_{n+1},p) \leq (1 + r_n) d(x_n,p) + s_n
\]
for all \( n \in \mathbb{N} \) and \( p \in D \). A sequence \( \{x_n\} \) is of monotone type II with respect to \( D \) if for each \( p \in D \) there exist sequences \( \{r_n\} \) and \( \{s_n\} \) of nonnegative real numbers such that
\[
\sum_{n=1}^{\infty} r_n < \infty, \sum_{n=1}^{\infty} s_n < \infty \quad \text{and} \quad \d(x_{n+1}, p) \leq (1 + r_n)d(x_n, p) + s_n
\]
for all \( n \in \mathbb{N} \).

**Lemma 2.7** ([2 Theorem 2.4]). Let \((X, d)\) be a complete metric space, \( D \) a subset of \( X \) and \( \{x_n\} \) a sequence in \( X \). Then one has the following assertions:

(i) If \( \{x_n\} \) is of monotone type I with respect to \( D \) then \( \lim_{n \to \infty} d(x_n, D) \) exists.

(ii) If \( \{x_n\} \) is of monotone type I with respect to \( D \) and \( \lim \inf_{n \to \infty} d(x_n, D) = 0 \) then \( x_n \to p \) for some \( p \in X \) satisfying \( d(p, D) = 0 \). In particular, if \( D \) is closed then \( p \in D \).

(iii) If \( \{x_n\} \) is of monotone type II with respect to \( D \) then \( \lim_{n \to \infty} d(x_n, p) \) exists for all \( p \in D \).

### 3 Main Results

In this section, we let \( C \) be a nonempty convex subset of a convex metric space \((X, d, W)\). Let \( \{T_i\}_{i=1}^{N} \) be a finite family of uniformly quasi-sup \( \{f_n\} \) Lipschitzian self-mappings of \( C \) and \( \{S_i\}_{i=1}^{N} \) a finite family of \( g_n \)-expansive self-mappings of \( C \). Let \( f_n, g_n \) be functions which is bounded above such that
\[
U_n = \sup_{x \in C} f_n(x) \quad \text{and} \quad E_n = \sup_{x \in C} g_n(x). \quad \text{Suppose that} \quad U = \sup_{n \in \mathbb{N}} U_n \quad \text{and} \quad E = \sup_{n \in \mathbb{N}} E_n \quad \text{are finite and} \quad \{\alpha_n\}^{(i)}, \{\beta_n^{(i)}\}, \{\gamma_n^{(i)}\} \quad \text{are sequences in} \quad [0, 1] \quad \text{such that} \quad \alpha_n^{(i)} + \beta_n^{(i)} + \gamma_n^{(i)} = 1 \quad \text{for each} \quad n \in \mathbb{N} \quad \text{and} \quad \delta_n^{(i)} = \alpha_n^{(i)} + \beta_n^{(i)}, \delta_n = \max_{1 \leq i \leq N} \{\delta_n^{(i)}\}, \gamma_n = \max_{1 \leq i \leq N} \{\gamma_n^{(i)}\} \quad \text{and} \quad d(u_n, p) = \max_{1 \leq i \leq N} \{d(u_n^{(i-1)}, p)\}. \quad \text{Let} \quad A = \max(E, U) \quad \text{sequences} \quad \{\delta_n A\} \subset [1, \infty) \quad \text{and} \quad \{\gamma_n d(u_n, p)\} \subset [0, \infty) \quad \text{such that} \quad \sum_{n=1}^{\infty} (\delta_n A - 1) < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \gamma_n d(u_n, p) < \infty. \quad \text{Suppose} \quad D = \left( \bigcap_{i=1}^{N} F(T_i) \right) \bigcap \left( \bigcap_{i=1}^{N} F(S_i) \right) \quad \text{is nonempty. Let} \quad x_1 \in C \quad \text{and the sequence} \quad \{x_n\} \quad \text{be the iteration defined as \[ \text{Lemma 3.3}. \] We shall first begin by constructing the following useful inequalities.

**Lemma 3.1.** For each \( i = 1, 2, \ldots, N-1, n \in \mathbb{N} \) and \( p \in D \). The following results hold.

(i) \( d(g_n^{(i)}, p) \leq \delta_n^{(i)} \cdot A \cdot d(g_n^{(i-1)}, p) + \gamma_n d(u_n, p). \)

(ii) \( d(g_n^{(i)}, p) \leq \delta_n^{(i)} \cdot A \cdot d(x_n, p) + \left( \sum_{j=1}^{i} \delta_n^{j-1} A^{j-1} \right) \gamma_n d(u_n, p). \)
(iii) \( d(x_{n+1}, p) \leq \delta_n \cdot A^i \cdot d(y_n^{(N-i)}, p) + \sum_{j=1}^{i} \delta_n^{j-1} A^{j-1} \gamma_n d(u_n, p). \)

**Proof.** (i) We need to show that

\[
d(y_n^{(i)}, p) \leq \delta_n \cdot A \cdot d(y_n^{(i-1)}, p) + \gamma_n d(u_n, p)
\]

for any \( i = 1, 2, \ldots, N - 1, \ n \in \mathbb{N} \) and \( p \in D. \) By Definition 2.2, Definition 2.1(i), (iv), \( U_n = \sup_{x \in C} f_n(x), \ E_n = \sup_{x \in C} g_n(x), \ U = \sup_n U_n \) and \( E = \sup_n E_n, \) we have the following inequality,

\[
d(y_n^{(i)}, p) = d(W(S_n^{(i-1)}, T_n^{(i-1)}, u_n^{(i-1)}; \alpha_n^{(i)}, \beta_n^{(i)}, \gamma_n^{(i)}), p)
\]

\[
\leq \alpha_n^{(i)} d(S_n^{(i-1)}, y_n^{(i-1)}, p) + \beta_n^{(i)} d(T_n^{(i-1)}, y_n^{(i-1)}, p) + \gamma_n^{(i)} d(u_n^{(i-1)}, p)
\]

\[
\leq \alpha_n^{(i)} d(S_n^{(i-1)}, y_n^{(i-1)}, p) + \beta_n^{(i)} d(T_n^{(i-1)}, y_n^{(i-1)}, p) + \gamma_n^{(i)} d(u_n^{(i-1)}, p)
\]

\[
\leq \alpha_n^{(i)} E_n d(y_n^{(i-1)}, p) + \beta_n^{(i)} U_n d(y_n^{(i-1)}, p) + \gamma_n^{(i)} d(u_n^{(i-1)}, p)
\]

\[
= (\alpha_n^{(i)} E + \beta_n^{(i)} U) d(y_n^{(i-1)}, p) + \gamma_n^{(i)} d(u_n^{(i-1)}, p).
\]

Thus by \( \delta_n^{(i)} = \alpha_n^{(i)} + \beta_n^{(i)}, \ A = \max\{E, U\}, \ \delta_n = \max_{1 \leq i \leq N} \{\delta_n^{(i)}\}, \ \gamma_n = \max_{1 \leq i \leq N} \{\gamma_n^{(i)}\} \)

and \( d(u_n, p) = \max_{1 \leq i \leq N} \{d(u_n^{(i-1)}, p)\}, \) we can rewrite the above inequalities as

\[
d(y_n^{(i)}, p) \leq \delta_n^{(i)} A d(y_n^{(i-1)}, p) + \gamma_n^{(i)} d(u_n^{(i-1)}, p)
\]

\[
\leq \delta_n A d(y_n^{(i-1)}, p) + \gamma_n d(u_n, p). \tag{3.1}
\]

(ii) We are going to show that

\[
d(y_n^{(i)}, p) \leq \delta_n^i A^i d(x_n, p) + \sum_{j=1}^{i} \delta_n^{j-1} A^{j-1} \gamma_n d(u_n, p)
\]

for all \( i = 1, 2, \ldots, N - 1, \) by using mathematical induction. Recall inequality (3.1), for any \( i = 1, 2, \ldots, N - 1, \ n \in \mathbb{N} \) and \( p \in D, \) we know that

\[
d(y_n^{(i)}, p) \leq \delta_n A d(y_n^{(i-1)}, p) + \gamma_n d(u_n, p).
\]

For \( i = 1, \)

\[
d(y_n^{(1)}, p) \leq \delta_n A d(y_n^{(0)}, p) + \gamma_n d(u_n, p)
\]

\[
= \delta_n A d(x_n, p) + \gamma_n d(u_n, p) \sum_{j=1}^{1} \delta_n^{j-1} A^{j-1}.
\]

Assume that for some \( m, 1 \leq m \leq N - 2, \)

\[
d(y_n^{(m)}, p) \leq \delta_n^m A^m d(x_n, p) + \gamma_n d(u_n, p) \sum_{j=1}^{m} \delta_n^{j-1} A^{j-1}.
\]
Hence,
\[
d(y^{(m+1)}_n, p) \leq \delta_n Ad(y^{(m)}_n, p) + \gamma_n d(u_n, p)
\leq \delta_n A\left[\sum_{j=1}^{m} \delta_n^{j-1} A^{j-1} \gamma_n d(u_n, p)\right] + \gamma_n d(u_n, p)
= \delta_n^{m+1} A^{m+1} d(x_n, p) + \left[\sum_{j=1}^{m+1} \delta_n^{j-1} A^{j-1}\right] \gamma_n d(u_n, p).
\]

Therefore, by mathematical induction, we have,
\[
d(y^{(i)}_n, p) \leq \delta_n^i A^i d(x_n, p) + \left[\sum_{j=1}^{i} \delta_n^{j-1} A^{j-1}\right] \gamma_n d(u_n, p)
\]
for all \(i = 1, 2, \ldots, N-1\).

(iii) We prove the following inequality, for any \(i = 1, 2, \ldots, N-1\),
\[
d(x_{n+1}, p) \leq \delta_n^i \cdot A^i \cdot d(y^{N-i}_n, p) + \left[\sum_{j=1}^{i} \delta_n^{j-1} A^{j-1}\right] \gamma_n d(u_n, p).
\]

Now, for any \(i = 1, 2, \ldots, N-1, n \in \mathbb{N}\) and \(p \in D\),
\[
d(x_{n+1}, p) = d(W(S_N y^{(N-1)}_n, T_N y^{(N-1)}_n, u^{(N-1)}_n; \alpha^{(N)}_n, \beta^{(N)}_n, \gamma^{(N)}_n), p)
\leq \alpha^{(N)}_n d(S_N y^{(N-1)}_n, p) + \beta^{(N)}_n d(T_N y^{(N-1)}_n, p) + \gamma^{(N)}_n d(u^{(N-1)}_n, p)
\leq \alpha^{(N)}_n Ed(y^{(N-1)}_n, p) + \beta^{(N)}_n Ud(y^{(N-1)}_n, p) + \gamma^{(N)}_n d(u^{(N-1)}_n, p)
\leq \alpha^{(N)}_n Ad(y^{(N-1)}_n, p) + \beta^{(N)}_n Ad(y^{(N-1)}_n, p) + \gamma^{(N)}_n d(u^{(N-1)}_n, p).
\]

Together with (i), we have
\[
d(x_{n+1}, p) \leq \delta_n^{(N)} Ad(y^{(N-1)}_n, p) + \gamma^{(N)}_n d(u^{(N-1)}_n, p)
\leq \delta_n Ad(y^{(N-1)}_n, p) + \gamma_n d(u_n, p)
\leq \delta_n A\left[\delta_n Ad(y^{(N-2)}_n, p) + \gamma_n d(u_n, p)\right] + \gamma_n d(u_n, p)
= \delta_n^2 A^2 d(y^{(N-2)}_n, p) + \delta_n A\gamma_n d(u_n, p) + \gamma_n d(u_n, p)
\leq \delta_n^2 A^2 \left[\delta_n Ad(y^{(N-3)}_n, p) + \gamma_n d(u_n, p)\right] + (\delta_n A + 1) \gamma_n d(u_n, p)
= \delta_n^3 A^3 d(y^{(N-3)}_n, p) + \delta_n^2 A^2 \gamma_n d(u_n, p) + \delta_n A + 1) \gamma_n d(u_n, p)
= \delta_n^3 A^3 d(y^{(N-3)}_n, p) + (\delta_n^2 A^2 + \delta_n A + 1) \gamma_n d(u_n, p)
\vdots
\]
\[
\leq \delta_n^i A^i d(y^{(N-i)}_n, p) + \left[\sum_{j=1}^{i} \delta_n^{j-1} A^{j-1}\right] \gamma_n d(u_n, p)
\]
for all \(i = 1, 2, \ldots, N-1\).
Remark 3.2. Lemma 3.1 generalizes main results of Lemma 3.1 in [2].

Lemma 3.3. The following results hold.

(i) There exist two sequences \( \{\eta_n\} \) and \( \{\sigma_n\} \) such that \( \sum_{n=1}^{\infty} \eta_n < \infty \) and

\[
\sum_{n=1}^{\infty} \sigma_n < \infty \quad \text{and} \quad d(x_{n+1},p) \leq (1 + \eta_n)d(x_n,p) + \sigma_n
\]

for all \( p \in D \) and \( n \in \mathbb{N} \).

(ii) \( \lim_{n \to \infty} d(x_n,p) \) exists for all \( p \in D \).

Proof. (i) Let \( p \in D \) and \( n \in \mathbb{N} \). By Lemma 3.1 (ii), (iii), we get

\[
d(x_{n+1},p) \leq \delta_nAd(y^{(N-1)}_n,p) + \gamma_n d(u_n,p)
\]

\[
\leq \delta_nA^j[x^{(N-1)}A^{N-1}d(x_n,p) + \sum_{j=1}^{N-1} \delta^{j-1}_n A^{j-1} \gamma_n d(u_n,p)] + \gamma_n d(u_n,p)
\]

\[
= \delta^n_n A^N d(x_n,p) + \sum_{j=1}^{N} \delta^{j-1}_n A^{j-1} \gamma_n d(u_n,p)
\]

\[
= (1 + \eta_n)d(x_n,p) + \sigma_n
\]

where \( \eta_n = \sum_{j=1}^{N} \binom{N}{j} (\delta_n A - 1)^j \) and \( \sigma_n = \sum_{j=1}^{N} \delta^{j-1}_n A^{j-1} \gamma_n d(u_n,p) \).

Since \( \sum_{n=1}^{\infty} (\delta_n A - 1) \) and \( \sum_{n=1}^{\infty} \gamma_n d(u_n,p) \) are both finite, it follows that \( \sum_{n=1}^{\infty} \eta_n < \infty \)

and \( \sum_{n=1}^{\infty} \sigma_n < \infty \).

(ii) Let \( p \in D \). Then, by the above result, there exist two sequences \( \{\eta_n\} \) and \( \{\sigma_n\} \) such that \( \sum_{n=1}^{\infty} \eta_n < \infty \) and \( \sum_{n=1}^{\infty} \sigma_n < \infty \) satisfying

\[
d(x_{n+1},p) \leq (1 + \eta_n)d(x_n,p) + \sigma_n
\]

for all \( n \in \mathbb{N} \). Then, by Lemma 2.3 (i), \( \lim_{n \to \infty} d(x_n,p) \) exists for all \( p \in D \).

Remark 3.4. Lemma 3.3 generalizes main results of Lemma 3.2 in [2].

Theorem 3.5. The following results hold when \( D \) is closed.

(i) If \( \{x_n\} \) converges to a common fixed point in \( D \) then \( \lim_{n \to \infty} d(x_n,D) = \lim_{n \to \infty} d(x_n,D) = 0 \).

(ii) If either \( \lim_{n \to \infty} d(x_n,D) = 0 \) or \( \lim_{n \to \infty} d(x_n,D) = 0 \) then \( \{x_n\} \) converges to a common fixed point in \( D \).
Lipschitzian self−self \( U \), \( d, W \) space (\( X \)).

Then, by Lemma 2 result together with Lemma 3.

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Proof. (i) Suppose that \( \{ x_n \} \) converges to a common fixed point \( p \) in \( D \). Then for \( \epsilon > 0 \), there exists \( N \in \mathbb{N} \) such that if \( n \geq N \) then

\[
d(x_n, p) < \frac{\epsilon}{2}.
\]

Taking infimum over \( p \in D \), for \( n \geq N \) we have

\[
dx_n, D \leq \frac{\epsilon}{2} < \epsilon.
\]

We have \( \lim \limits_{n \to \infty} d(x_n, D) = 0 \), i.e., \( \lim \inf \limits_{n \to \infty} d(x_n, D) = \lim \sup \limits_{n \to \infty} d(x_n, D) = 0 \).

(ii) Assume that \( \lim \inf \limits_{n \to \infty} d(x_n, D) = 0 \) or \( \lim \sup \limits_{n \to \infty} d(x_n, D) = 0 \). Using this result together with Lemma 3(i) and Lemma 3(iii) we have \( \lim \limits_{n \to \infty} d(x_n, D) = 0 \).

Then, by Lemma 2(ii), \( \lim \limits_{n \to \infty} x_n \) exists that is there exists \( q \in X \) such that \( x_n \to q \) as \( n \to \infty \). Since \( D \) is closed, \( \{ x_n \} \) converges to a common fixed point in \( D \).

4 Applications

In this section, we let \( C \) be a nonempty convex subset of a convex metric space \( (X, d, W) \). Let \( \{ T_i \}_{i=1}^N \) be a finite family of uniformly quasi−sup \((f_n)\) Lipschitzian self−mappings of \( C \) and \( \{ S_i \}_{i=1}^N \) a finite family of \( g_n \)−expansive self−mappings of \( C \). Let \( f_n, g_n \) be functions which is bounded above such that \( U_n = \sup f_n(x) \) and \( E_n = \sup g_n(x) \). Suppose that \( U = \sup \limits_{n \to \infty} U_n \) and \( E = \sup \limits_{n \in \mathbb{N}} E_n \) are finite and \( \{ \alpha_n^{(i)} \}, \{ \beta_n^{(i)} \}, \{ \gamma_n^{(i)} \} \) are sequences in \([0, 1]\) such that \( \alpha_n^{(i)} + \beta_n^{(i)} + \gamma_n^{(i)} = 1 \) for each \( n \in \mathbb{N} \). Let \( \delta_n^{(i)} = \alpha_n^{(i)} + \beta_n^{(i)} \), \( \delta_n = \max \{ \delta_n^{(i)} \} \), \( \gamma_n = \max \{ \gamma_n^{(i)} \} \) and \( d(u_n, p) = \max \{ d(u_n^{(i)}, p) \} \). Let \( A = \max \{ E, U \} \), sequences \( \{ \delta_n A \} \subset [1, \infty) \) and \( \{ \gamma_n d(u_n, p) \} \subset [0, \infty) \) such that \( \sum \limits_{n=1}^\infty (\delta_n A − 1) < \infty \) and \( \sum \limits_{n=1}^\infty \gamma_n d(u_n, p) < \infty \). Suppose \( D = \left( \bigcap \limits_{i=1}^N F(T_i) \right) \bigcap \left( \bigcap \limits_{i=1}^N F(S_i) \right) \) is nonempty and closed. Let \( x_1 \in C \) and the sequence \( \{ x_n \} \) be the iteration defined by \( 1.3 \). First, we prove the following lemma.

Lemma 4.1. A sequence \( \{ x_n \} \) converges to common fixed point of the families \( \{ T_i \}_{i=1}^N \) and \( \{ S_i \}_{i=1}^N \) if and only if \( \lim \inf \limits_{n \to \infty} d(x_n, D) = 0 \), \( d(x_n, D) = \inf \{ d(x_n, p) : p \in D \} \).

Proof. The necessity condition is obvious. Thus we will only prove the sufficiency. Suppose that \( \lim \inf \limits_{n \to \infty} d(x_n, D) = 0 \). Then by Lemma 3(i), there exist two se-
quences \( \{ \eta_n \} \) and \( \{ \sigma_n \} \) such that \( \sum_{n=1}^{\infty} \eta_n < \infty \) and \( \sum_{n=1}^{\infty} \sigma_n < \infty \) satisfying

\[
d(x_{n+1},p) \leq (1 + \eta_n)d(x_n,p) + \sigma_n
\]

for all \( p \in D \) and all \( n \in \mathbb{N} \). By Definition 2.6 we have \( \{ x_n \} \) is of monotone type 1 with respect to \( D \). By Lemma 2.7(ii), we have desired.

**Remark 4.2.** Lemma 4.1 generalizes main results of Theorem 3.3 in [2].

The following theorem shows that the sequence \( \{ x_n \} \) converges to common fixed point of the families \( \{ T_i \}_{i=1}^{N} \) and \( \{ S_i \}_{i=1}^{N} \) with two added properties: Condition A (Definition 2.3) and semi-compact (Definition 2.4).

**Theorem 4.3.** Let \( C \) be a closed convex subset of a complete uniformly convex metric space \( (X,d,W) \) with continuous convex structure. Let \( \{ T_i \}_{i=1}^{N} \) and \( \{ S_i \}_{i=1}^{N} \) be finite families of uniformly \( L \)-Lipschitzian self-mappings of \( C \). Suppose that \( \lim_{n \to \infty} d(x_n,T_ix_n) = 0 = \lim_{n \to \infty} d(x_n,S_ix_n) \) for all \( i = 1,2,\ldots,N \). If one of the following is satisfied:

(i) \( \{ T_i \}_{i=1}^{N} \) and \( \{ S_i \}_{i=1}^{N} \) satisfy Condition A,

(ii) one member of the families \( \{ T_i \}_{i=1}^{N} \) and \( \{ S_i \}_{i=1}^{N} \) are semi-compact, then \( \{ x_n \} \) converges to common fixed point of two families \( \{ T_i \}_{i=1}^{N} \) and \( \{ S_i \}_{i=1}^{N} \).

**Proof.** (i) Suppose that \( \{ T_i \}_{i=1}^{N} \) and \( \{ S_i \}_{i=1}^{N} \) are satisfy Condition A. Then by Definition 2.3 we have there exists a nondecreasing functions such that \( f : [0, \infty) \to [0, \infty) \) with \( f(0) = 0 \) and \( f(r) > 0 \) for all \( r \in (0, \infty) \),

\[
g : [0, \infty) \to [0, \infty) \with g(0) = 0 \text{ and } g(r) > 0 \text{ for all } r \in (0, \infty)
\]

with \( d(x_n, T_ix_n) \geq f(d(x_n,D)) \) and \( d(x_n, S_ix_n) \geq g(d(x_n,D)) \). Hence,

\[
\lim_{n \to \infty} d(x_n, T_ix_n) \geq \lim_{n \to \infty} f(d(x_n,D)),
\]

\[
\lim_{n \to \infty} d(x_n, S_ix_n) \geq \lim_{n \to \infty} g(d(x_n,D)),
\]

for some \( 1 \leq i \leq N \). By the assumption we know that \( \lim_{n \to \infty} d(x_n, T_ix_n) = 0 = \lim_{n \to \infty} d(x_n, S_ix_n) \). It follows that

\[
\lim_{n \to \infty} d(x_n, D) = 0.
\]

Then by Lemma 4.1 the sequence \( \{ x_n \} \) converges to common fixed point of the families \( \{ T_i \}_{i=1}^{N} \) and \( \{ S_i \}_{i=1}^{N} \).

(ii) Suppose that \( \{ T_i \}_{i=1}^{N} \) is semi-compact. Then, by Definition 2.4 there exists a subsequence \( \{ x_{n_j} \} \) of \( \{ x_n \} \) such that \( x_{n_j} \to p \in C \). Hence, for each \( 1 \leq i \leq N \),

\[
d(p, T_ip) \leq d(p, x_{n_j}) + d(x_{n_j}, T_ix_{n_j}) + d(T_ix_{n_j}, T_ip)
\]

\[
\leq d(p, x_{n_j}) + d(x_{n_j}, T_ix_{n_j}) + Ld(x_{n_j}, p)
\]

\[
\leq (1 + L)d(p, x_{n_j}) + d(x_{n_j}, T_ix_{n_j})
\]

\[
\to 0.
\]
Thus, we have $T_i p \to p$ for $1 \leq i \leq N$. The proof in case of $\{S_i\}_{i=1}^{N}$ being semi–compact is similar to prove the above case. Thus, $p \in D = \left( \bigcap_{i=1}^{N} F(T_i) \right) \bigcap \left( \bigcap_{i=1}^{N} F(S_i) \right)$. By continuity of $x \mapsto d(x, D)$, we obtain

$$\lim_{j \to \infty} d(x_{n_j}, D) = d(p, D) = 0,$$

$$\lim_{k \to \infty} d(x_{n_k}, D) = d(p, D) = 0.$$

It follows by Lemma 3(iii) that $\lim_{n \to \infty} d(x_n, D) = 0$. Hence, by Lemma 4.1 we have $\{x_n\}$ converges to common fixed point of the family $\{T_i\}_{i=1}^{N}$ and $\{S_i\}_{i=1}^{N}$. 

**Remark 4.4.** Theorem 4.3 generalizes main results of Theorem 3.7 in [2].

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**References**


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