A Simple Proof of the Brouwer Fixed Point Theorem

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Abstract: We give a new proof of the Brouwer fixed point theorem which is more elementary than all known ones. The only tool we use is the Tietze (continuous) extension theorem. The idea of the proof suggests some successful computation of a fixed point.

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1 Introduction

Brouwer Fixed Point Theorem: For the unit cube \([0,1]^d\) of the Euclidean space \(\mathbb{R}^d\), let \(T : [0,1]^d \to [0,1]^d\) be a continuous function. Then \(T\) has a fixed point, i.e., a point \(x \in [0,1]^d\) with \(T(x) = x\).

The proof is by induction on the dimension \(d\) and its idea of the proof can be extended from the one of \(d = 2\).

For a complete survey on the development of the Brouwer fixed point theorem, we refer the reader to Sehie Park [1].

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2 Preliminaries

The only tool we use in the proof in section 3 is the following theorem:

**Tietze Extension Theorem**: For a closed subset $K$ of the Euclidean space $\mathbb{R}^d$, let $T : K \to \mathbb{R}$ be a continuous function. Then $T$ has a continuous extension over $\mathbb{R}^d$, i.e., a continuous function $\overline{T} : \mathbb{R}^d \to \mathbb{R}$ such that the restriction of $\overline{T}$ over $K$ is $T$.

3 Proof of Main Result

Let $K$ be a nonempty compact convex subset of the Euclidean space $\mathbb{R}^d$. Any continuous function $T : K \to K$ has a fixed point. We prove the Theorem for $K$ of the form $K = [0, 1]^d$. Let us recall some familiar notations. For $j = 1, \ldots, d$, write $e_j = (\delta_{ji})_{i=1}^d$ where the Kronecker delta $\delta_{ji}$ is defined to be 1 or 0 according to $j = i$ or $j \neq i$. Thus $\{e_1, \ldots, e_d\}$ is the standard basis for $\mathbb{R}^d$. For $j = 1, \ldots, d$, let $\mathbb{O}^j = \{\sum_{i=1, i \neq j}^d x_i e_i : 0 \leq x_i \leq 1, i = 1, \ldots, d, i \neq j\}$ and $\mathbb{O}^j = \mathbb{O}^j + e_j$. Set $\hat{O} = (0, \ldots, 0), \hat{1} = (1, \ldots, 1) \in \mathbb{R}^d$, and for $0 \leq u \leq \sqrt{d}$, let $H_u$ be the hyperplane passing through $(u, \ldots, u)$ and having $\hat{1}$ as its normal vector and put $\Delta u = K \cap H_u$.

Note that $\Delta u$, for $u \leq 1/\sqrt{d}$, is a simplex $co(ae_i : i = 1, \ldots, d)$ for some $a \in [0, 1]$. Every point in $[0, 1]^d$ lies in a simplex $co(ae_i : i = 1, \ldots, d)$ for some $a \in [0, d]$.

We will consider the projection along $e_j$ defined by

$$\pi_j : (x_1, \ldots, x_d) \mapsto \sum_{i=1, i \neq j}^d x_i e_i,$$  \text{ for } (x_1, \ldots, x_d) \text{ in } K.

Set $\mathbb{O}_{ij} = \pi_j(\Delta u), \mathbb{S}_{ij} = \mathbb{O}_{ij} \setminus \mathbb{O}_{ij}$, and $\mathbb{S}_{ij} = \mathbb{S}_{ij}^+ = \mathbb{S}_{ij} + e_j$. Above the face $\mathbb{O}_{ij}$, let $S_{ij}$ be the continuous surface consisting of $\Delta u$ together with $\mathbb{S}_{ij}$ or $\mathbb{S}_{ij}$. For example, Figure 1 shows $S_{u1}$ and $S_{v1}$ in $\mathbb{R}^2$.

![Figure 1](image_url)

Note that $\pi_j : S_{ij} \to \mathbb{O}_{ij}$ is a bijection.

Write $T = (f_1, \ldots, f_d)$ where $f_j : K \to [0, 1]$ is continuous for each $j$. For each
u, draw the graph of $f_j$ restricted to $S_{u_j}$ via the formula

$$g_{u_j} : (x_1, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_d) \mapsto (x_1, \ldots, x_{j-1}, f_j(x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_d), x_{j+1}, \ldots, x_d)$$

for each $(x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_d)$ in $S_{u_j}$.

Observe that $(x_1, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_d) = \pi_j(x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_d)$.

Thus the graph of $f_j$ at $u$ means the set of points

$$(x_1, \ldots, x_{j-1}, f_j(x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_d), x_{j+1}, \ldots, x_d)$$

for $(x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_d)$ in $S_{u_j}$.

Figure 2 demonstrates the graphs of $f_1$ and $f_2$ for a given $u$.

![Figure 2](image_url)

The first result is fundamental whose proof based on the Brouwer fixed point theorem for $[0, 1]^{d-1}$.

**Lemma 3.1.** For each $u$, the graphs of $f_1, \ldots, f_d$ intersect at a point.

**Proof.** Consider the function

$$S := \pi_1 g_{u_d} \pi_d \cdots g_{u_3} \pi_3 g_{u_2} \pi_2 g_{u_1} : \square_1 \rightarrow \square_1.$$

By the Brouwer fixed point theorem, there is a point $v \in \square_1$ such that $v = S(v)$. Clearly, the point $g_1(v)$ is a desired point of intersection. □

A point of intersection in the proof can be found by successive projections on the graph of $f_1, \ldots, f_d$. It is the limit of each convergent subsequence.

In the sequence, we will refer to “a point of intersection of the graphs of” $f_1, \ldots, f_d$ shortly as “a point of intersection of” $f_1, \ldots, f_d$.

**Remark 3.2.** Note from Lemma 3.1 that

1. a point $w = (w_1, \ldots, w_d)$ is a point of intersection if and only if there are points $w^{(j)} = (w_1^{(j)}, \ldots, w_d^{(j)})$ in $S_{u_j}$ for $j = 1, \ldots, d$ such that $w_i^{(j)} = w_i$ for $i \neq j$ and $f_j(w^{(j)}) = w_j$. Thus,

2. if $w$ lies on $\Delta u$, then $T(w) = w$, i.e., $w$ is a fixed point of $T$. 
In the proof of the main result to follow, we will consider a negative part and a positive part of the function \( f \) over \( S_{u1} \), for each \( u \). They are defined respectively as

\[
N^0(f_1, u) = \{(x_1, \ldots, x_d) \in S_{u1} : f_1((x_1, \ldots, x_d)) < x_1\},
\]

\[
P^0(f_1, u) = \{(x_1, \ldots, x_d) \in S_{u1} : f_1((x_1, \ldots, x_d)) > x_1\}.
\]

As subsets of the \( d-1 \) dimensional Euclidean space \( \mathbb{R}^{d-1} \), let partition \( N^0(f_1, u) \) and \( P^0(f_1, u) \) into (open) components, say,

\[
N^0(f_1, u) = \bigcup_{\alpha} N^0_\alpha(f_1, u), \quad P^0(f_1, u) = \bigcup_{\beta} P^0_\beta(f_1, u).
\]

Let \( N(f_1, u) \) and \( P(f_1, u) \) be the closure of \( N^0(f_1, u) \) and \( P^0(f_1, u) \) respectively. Analogously, let \( N_\alpha(f_1, u) \) and \( P_\beta(f_1, u) \) be the closure of \( N^0_\alpha(f_1, u) \) and \( P^0_\beta(f_1, u) \) respectively. Put \( Z_u = S_{u1} \setminus (N(f_1, u) \cup P(f_1, u)) \). Thus, \( Z_u \subset \{(x_1, \ldots, x_d) \in S_{u1} : f_1((x_1, \ldots, x_d)) = x_1\} \). For \( 0 \leq u \leq \sqrt{d} \) and \( \lambda \in \mathbb{R} \), let \( \lambda_{u1}(w_0) = \lambda_{u1}(w_0) = \lambda_{u1}(w_0) + (1 - \lambda)w \) for \( w = (x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_d) \in S_{u1} \) where \( w_0 = (x_1, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_d) \). One may need to truncate the function \( \lambda_{u1} \) to lie within \( K \). For each \( \alpha \) and \( \beta \), put \( N^\alpha_{u} = \lambda_{u1}\cdot \chi_{N_{u}(f_1, u)} \) and \( P^\beta_{u} = \lambda_{u1}\cdot \chi_{P_{u}(f_1, u)} \), and \( O_{u} = \chi_{Z_{u}} \).

By Lemma 3.1 for a given closed interval \([a, b]\) in \( \mathbb{R} \), the functions

\[
N^\alpha_{u}, f_2, \ldots, f_d, \quad \text{for some } \alpha,
\]

\[
or \quad P^\beta_{u}, f_2, \ldots, f_d, \quad \text{for some } \beta,
\]

\[
O_{u}, f_2, \ldots, f_d,
\]

intersect at a point, for all \( \lambda \) in \([a, b]\).

To see why (3.1) holds, we argue on the contrary that the functions \( O_u, f_2, \ldots, f_d \), do not intersect and for each \( \alpha \) and \( \beta \), there exists respectively \( \lambda_\alpha \) and \( \lambda_\beta \) such that neither \( N^\alpha_{u}, f_2, \ldots, f_d \), nor \( P^\beta_{u}, f_2, \ldots, f_d \), intersect. We claim that the union, called \( h \), of the functions \( O_u, N^\alpha_{u}, P^\beta_{u} \), for all \( \alpha \) and \( \beta \) is continuous, and clearly it does not have a common point with the functions \( f_2, \ldots, f_d \) which contradicts Lemma 3.1. To verify the claim, let \( \{w_n\} \) be a sequence in \( S_{u1} \) converging to \( w \). Suppose that \( w \) belongs to \( P^\beta_{u}(f_1, u) \) for some \( \beta \) (the proof
for the case \( w \) belongs to \( N_\alpha(f_1, u) \) for some \( \alpha \) follows the same lines). Write \( w = (x_1, \ldots, x_d) \) and \( w_n = (x_{n1}, \ldots, x_{nd}) \). If \( f_1(w) > x_1 \), then \( f_1(w_n) > x_{n1} \) for all large \( n \), i.e. \( w_n \in P^\beta(f_1, u) \) for those \( n \). Now for such \( n \), \( h(w_n) = f_{\lambda n}(w_n) = \lambda_2 g_{11}((w_{n1})_0) + (1 - \lambda_2)w_n \to \lambda_2 P^\beta_{\lambda n}(w) + (1 - \lambda_2)w = h(w) \) as desired. On the other hand, if \( f_1(w) = x_1 \), then \( f_1(w_n) \to x_1 \). From the estimate

\[
\|h(w_n) - w_n\| \leq M\|f_1(w_n) - w_n\| \to \|x_1 - x_1\| = 0,
\]

and the fact that \( w_{n1} \to x_1 \), we get \( h(w_n) \to x_1 = h(w) \) as desired. Here \( M \) is a bound for \( \lambda \)'s. For the remaining case when \( w \in Z \), \( w_n \) belongs to a \((d-1)\)-ball \( B(w,n) \) for all large \( n \) which in turn \( h(w_n) = w_{n1} \to x_1 = h(w) \).

We are now ready to prove the Brouwer Fixed Point Theorem:

**Proof.** If there is a sequence \( \{w_n\} \) of points of intersection lying strictly below \( S_0 \), i.e. in the direction of the first component, for all \( u \), a convergent subsequence of \( \{w_n\} \), by Remark 3.2(3), must converge to a point of intersection which must lie in \( S_{01} = \{0\} \) and thus \( O \) is a fixed point of \( T \).

So we suppose that for some \( u \) there holds for each given bounded interval \([a, b]\) (which we will assume in the sequent that it properly contains \([0, 1]\) ), there exists a \( \beta \) such that (3.1) holds for \( P^\beta_{\lambda u}, f_2, \ldots, f_d \) for all \( \lambda \in [a, b] \). Now let

\[
u_0 = \sup \left\{ u \in [0, \sqrt{d}] : \text{for each bounded interval } [a, b], \text{there exists } \beta \text{ such that (3.1) holds for } P^\beta_{\lambda u}, f_2, \ldots, f_d \text{ for all } \lambda \in [a, b] \right\}.
\]

\[
:= \sup A
\]

We show that a point of intersection of \( f_1, f_2, \ldots, f_d \) lies in \( \Delta_{u_0} \) and we are done. We suppose that

all points of intersection of functions \( f_1, f_2, \ldots, f_d \) do not lie in \( \Delta_{u_0} \). \hspace{1cm} (3.2)

First, we claim that \( u_0 \in A \). To achieve this, we are given any bounded interval \([a, b]\) in \( \mathbb{R} \) and any \( \lambda \in [a, b] \), and choose a sequence \( \{u_n\} \) in \( A \) converging increasingly to \( u_0 \). Take a sequence \( \{\beta_n\} \) described in \( A \) corresponding to \( \{u_n\} \). Thus there exists, for each \( n \), a point of intersection \( w_n \) of functions \( P^\beta_{\lambda u_n}, f_2, \ldots, f_d \). Assume without loss of generality that \( w_n \to c_\lambda \).

By Remark 3.2(3), \( c_\lambda \) is a point of intersection of \( f_{\lambda_1}, f_2, \ldots, f_d \). In particular, when \( \lambda = 1 \), there exists a point \( c_1 \) of intersection of \( f_1, f_2, \ldots, f_d \). We note from the uniform continuity of \( T \) that \( P^\beta_{\lambda u}(f_1, u) \) converges to \( P^\beta_{\lambda u}(f_1, u) \), for some \( \beta_0 \), under the Hausdorff distance. Moreover, under our assumption (3.2), \( c_\lambda \) is a point of intersection of \( P^\beta_{\lambda u}, f_2, \ldots, f_d \). This proof holds for all \( \lambda \in [a, b] \), and it ends the proof of the claim that \( u_0 \in A \).
If \( u_0 = \sqrt{d} \), then \( I \) is a fixed point of \( T \) as we reasoned for \( O \). So we consider the case \( u_0 < \sqrt{d} \).

Case I [All \( c_1 \in \mathfrak{H}_0 \): The proof in this case is simpler than the following case, so we omit the proof.

Case II [Some \( c_1 \notin \mathfrak{H}_0 \)]. Choose \( \varepsilon_0 > 0 \) such that, under the subspace topology, \( B(w, \varepsilon_0) \cap \Delta_{u_0} \subset P_{\beta_0} (f_1, u_0) \) for all points of intersection \( w \) of \( P_{\beta_0}^1, f_2, \ldots , f_d \). Otherwise, a sequence of such points \( w \) would converge to a point of intersection of \( f_1, f_2, \ldots , f_d \) which lies in \( \Delta_{u_0} \) contradicting to our assumption. Thus the following sets are nonempty for all small \( \delta > 0 \). For \( \delta > 0 \) to be chosen appropriately later, let \( u_1 \in (u_0, u_0 + \delta) \).

Put

\[ P_0 = \{(0, x_2, \ldots , x_d) : (x_1, \ldots , x_d) \in S_{u_1}, P_{\lambda_{u_1}}^1 (x_1, \ldots , x_d) > x_1 + \delta \}, \]

for \( \delta > 0 \), and let \( c(P_0) \) be its closure in \([0, 1]^{d-1} \). Note that \( P_0 \) is open in \([0, 1]^{d-1} \). For each \( j = 2, \ldots , d \), let

\[ Q_\delta^j = (\pi_j f_j)^{-1}(c(P_2)) \quad \text{and} \quad R_\delta^j = [0, 1]^{d-1} \setminus (\pi_j f_j)^{-1}(P_0). \]

Thus both \( Q_\delta^j \) and \( R_\delta^j \) are nonempty for small \( \delta > 0 \) and they are disjoint compact sets in \([0, 1]^{d-1} \). Hence the minimum distance between elements of the two sets is positive. Thus the union of two continuous real valued functions on \( Q_\delta^j \) and \( R_\delta^j \) is always continuous, and in turn it is extendable continuously on \([0, 1]^{d-1} \). We use this fact to redefine \( f_1, \ldots , f_d \). First put

\[ h_j = f_j \chi_{Q_\delta^j}. \]

If \( O \notin Q_\delta^j \), we then redefine only \( h_2 \) as

\[ h_2 = f_2 \chi_{\bar{Q}_\delta^2} + a_1 \chi_{\bar{O}} \]

where \( \bar{Q}_\delta^2 = Q_\delta^2 \cap ([0, 1] \times [\eta, 1] \times [0, 1]^{d-2}) \) for some small \( \eta \) and \( a_1 \) is any fixed element chosen from \( Q_\delta^2 \). Extend each \( h_j \) over \([0, 1]^{d-1} \) continuously by Tietze extension theorem to obtain new set of \( f_1, \ldots , f_n \). Note that new functions \( f_1, \ldots , f_n \) remain unchanged on \( Q_\delta^1, \ldots , Q_\delta^j \) and they do not intersect on \( R_\delta^1, \ldots , R_\delta^j \).

Consequently, by Lemma 3.3 they intersect only on \( P_{\beta_1} (f_1, u_1) \). Now if there are sequences \( \{v_n\}, \{\delta_n\} \) and \( \{\lambda_n\} \) with \( \{v_n\} \) is strictly decreasing to \( u_0 \), \( \{\delta_n\} \) is strictly decreasing to \( 0 \) and \( \{\lambda_n\} \subset [a, b] \) such that \( f_{\lambda_{u_1}}, f_2, \ldots , f_d \) do not intersect when \( f_1 \) is restricted to \( Q_\delta^1 \). This means that intersection occurs for \( f_1 \) restricted to \( R_{\delta_{u_1}}^1 \setminus Q_{\delta_{u_1}}^1 \). But then under (3.2) and by uniform continuity of \( T \), we obtain a contradiction. At this point we can find small \( \delta > 0 \) and \( u_1 \in (u_0, u_0 + \delta) \) as planned so that (3.3) holds for \( P_{\lambda_{u_1}}^1, f_2, \ldots , f_d \) for all \( \lambda \in [a, b] \).

The above argument shows that there is a number \( u_1 \) in \( A \) which is bigger than \( u_0 \) and this is not possible. Hence our claim that some point of intersection of \( f_1, f_2, \ldots , f_d \) lies in \( \Delta_{u_0} \) is justified, and therefore the proof is complete. \( \square \)
Remark 3.3. It is clear from the proof that, in practice, we only need to consider the graphs of functions $f_1, \ldots, f_d$ restricted to each $\Delta_u$. To find a fixed point, we move the vector $\bar{u} = (u, \ldots, u) \in \mathbb{R}^d$ along the vector $\bar{1}$ until $\Delta_u$ meets a point of intersection. That point is a fixed point of $T$. This method of finding a fixed point can be described as “catching a fish by a fishing net”.

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References


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