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Extendability of the Complementary Prism of 2-Regular Graphs[1]

P. Janseana†, S. Rueangthampisan† and N. Ananchuen†,‡,2

†Department of Mathematics, Faculty of Science
Silpakorn University, Nakorn Pathom 73000, Thailand
e-mail : jpongthep@yahoo.com (P. Janseana)
pang_sriphan@hotmail.com (S. Rueangthampisan)
ananchuen_n@su.ac.th (N. Ananchuen)

‡Centre of Excellence in Mathematics
CHE, Si Ayutthaya Rd., Bangkok 10400, Thailand

Abstract : Let G be a simple graph. The complementary prism of G, denoted by $G\overline{G}$, is the graph formed from the disjoint union of G and $\overline{G}$, the complement of G, by adding the edges of a perfect matching between the corresponding vertices of G and $\overline{G}$. A connected graph G of order at least $2k+2$ is k-extendable if for every matching M of size k in G, there is a perfect matching in G containing all edges of M. The problem that arises is that of investigating the extendability of $G\overline{G}$. In this paper, we investigate the extendability of $G\overline{G}$ where G contains $G_1, \ldots, G_l$ as its components and the extendability of $G_i\overline{G}_i$ is known for $1 \leq i \leq l$. We then apply this result to establish the extendability of $G\overline{G}$ when G is 2-regular.

Keywords : extendable; complementary prism; regular graph.
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1 Introduction

Let G denote a finite simple undirected graph with vertex set $V(G)$ and edge set $E(G)$. The complement of G is denoted by $\overline{G}$. For a vertex v of G, $deg_G(v)$

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[2]Corresponding author.

and $N_G(v)$ denote the degree and the neighbour set of v, respectively. Further, the closed neighbour set of v, denoted by $N_G[v]$, is $N_G(v) \cup \{v\}$. For disjoint graphs G_1 and G_2, the join of G_1 and G_2 is denoted by $G_1 \vee G_2$. For positive integers m and $n \geq 3$, K_m and C_n denote a complete graph of order m and a cycle of order n, respectively. For $S \subseteq V(G)$, the induced subgraph of S in G is denoted by $G[S]$. A graph G is said to be H-free if G does not contain H as an induced subgraph. A subset M of $E(G)$ is called a matching in G if no two edges of M have a common end vertex. M is a maximum matching in G if there is no matching M' in G such that $|M'| > |M|$. A vertex v of G is said to be M-saturated if v is an end vertex of some edge in a matching M; otherwise, v is M-unsaturated. If each vertex of G is M-saturated, then M is called a perfect matching. Note that if M is a perfect matching of G, then $|M| = \frac{|V(G)|}{2}$.

In 1980, Plummer [1] introduced a concept of k-**extendable**. For a positive integer k, a connected graph G of order at least $2k+2$ is said to be k-extendable if for every matching $M$of size k in G, there is a perfect matching in G containing all edges of M. It is easy to see that K_{2n} is k-extendable for $1 \leq k \leq n-1$ and a cycle of even order is 1-extendable but not 2-extendable. Since 1980 the concept of k-extendable graphs was investigated by several researchers. For excellence surveys in this topic, a reader is directed to ([2],[3] and [4]). A closely concept to k-extendable graphs is k-**factor-critical** graphs introduced by Favaron [5]. A graph G is said to be k-factor-critical if for every subset $S \subseteq V(G)$ with $|S| = k$, $G - S$ has a perfect matching. Favaron also pointed out some relationship between extendable non-bipartite graphs and factor-critical graphs as we shall see in Theorem 2.4, Section 2.

Haynes et al. [6] introduced the concept of **complementary prism** of a graph. For a simple graph G, the complementary prism of G, denoted by $G\overline{G}$, is the graph formed from the disjoint union of G and $\overline{G}$ by adding the edges of a perfect matching between the corresponding vertices of G and $\overline{G}$. Examples of the complementary prism of graphs are shown in Figures 1 and 2. Note that the graph $C_5\overline{C}_5$ in Figure 1 is isomorphic to the Petersen graph. One might ask what property that a graph G should have so that $G\overline{G}$ is k-extendable for some k. A problem that arises is that of investigating the extendability of $G\overline{G}$. In this paper, we first consider the extendability of $G\overline{G}$ where G contains $G_1, \ldots, G_l$ as its components and the extendability of $G_i\overline{G}_i$ is known for $1 \leq i \leq l$. In fact, we prove the following theorem:

Theorem 1.1. *For positive integers i and l where $1 \leq i \leq l$, let $G_1, \ldots, G_l$ be components of G. If $G_i\overline{G}_i$ is k-extendable of order $p_i \geq 2k+2$ for some positive integer k, then $G\overline{G}$ is k-extendable.*

We then apply Theorem 1.1 to establish the extendability of 2-regular graphs. We show that:

Theorem 1.2. *Let G be a 2-regular H-free graph where $H \in \{C_3, C_4, C_5\}$. Then $G\overline{G}$ is 2-extendable.*

The condition of H-free and the extendability of $G\overline{G}$ stated in Theorem 1.2 are all best possible. For positive integers $n \geq 8$ and $3 \leq i \leq 5$, let $H_i = C_i \cup C_{n-i}$. Then the graph $H_i\overline{H_i}$, shown in Figure 2, is not 2-extendable since there is no perfect matching containing the edge x_1x_2 and y_1y_2. Note that "a double line" in our diagram denotes the join between corresponding graphs. Hence, the hypothesis H-free where $H \in \{C_3, C_4, C_5\}$ in Theorem 1.2 cannot be dropped. Finally, the extendability of $G\overline{G}$ in Theorem 1.2 is best possible by Theorem 2.2(2), stated in Section 2, and the fact that the minimum degree of $G\overline{G}$ is 3. The proof of Theorems 1.1 and 1.2 are in Sections 3 and 4, respectively.

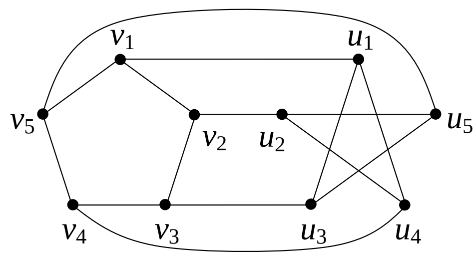

Figure 1: The graph $C_5\overline{C}_5$

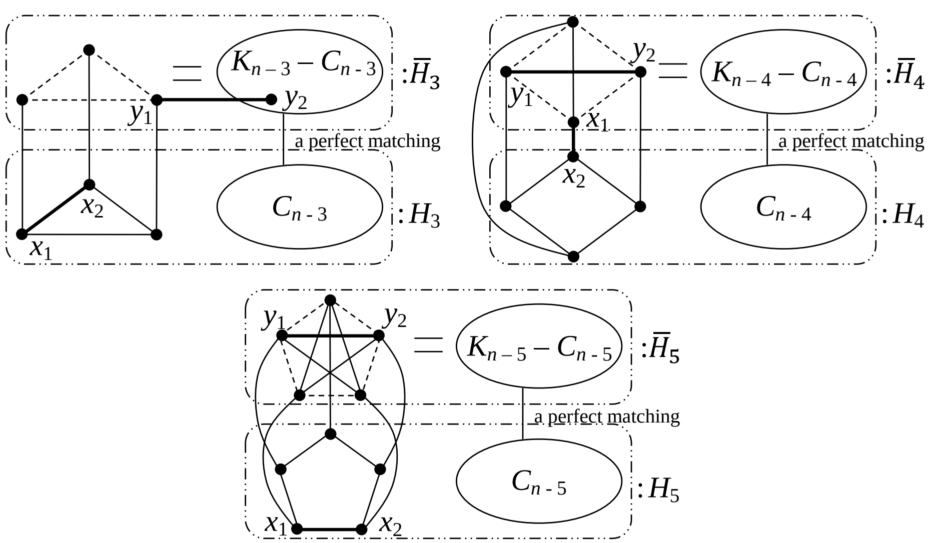

Figure 2: The graph $H_i\overline{H_i}$, $i \in \{3, 4, 5\}$

2 Preliminaries

In this section, we provide results that we make use of in establishing our results in the next two sections. We begin with a result on an existence of a perfect matching in a graph.

Theorem 2.1 ([7]). *(Tutte's Theorem) A graph G has a perfect matching if and only if for a subset S of $V(G)$, the number of odd components of $G-S$ is at most $|S|$.*

The next two theorems proved by Plummer concern some properties of extendable graphs.

Theorem 2.2 ([1]). *For positive integers k and p, let G be a graph of order $p \geq 2k+2$. If G is k-extendable, then*

1. *G is $(k-1)$-extendable, and*
2. *G is $(k+1)$-connected.*

Theorem 2.3 ([8]). *Let $k \geq 1$ be an integer and let G be a (2k+1)-connected $K_{1,3}$-free graph with an even number of vertices. Then G is k-extendable.*

Our next result provides a relationship between extendable non-bipartite graphs and factor-critical graphs proved by Favaron.

Theorem 2.4 ([5]). *If G is a $2k$-extendable non–bipartite graph for $2k \geq 2$, then G is a $2k$-factor-critical graph.*

We conclude this section with our results proved in [9].

Lemma 2.5 ([9]). *Let G be a k-extendable non-bipartite graph and M a matching of G with $|M| \leq k-1$. Then $G-V(M)$ is a $(k-|M|)$-extendable non-bipartite graph. Further, if $k-|M|$ is even, then $G-V(M)$ is $(k-|M|)$-factor critical.*

Lemma 2.6 ([9]). *Let G be a k-extendable graph for some integer $k \geq 2$ and let $S \subseteq V(G)$ be a cutset of G. If $G[S]$ contains $t \leq k-1$ independent edges, then $|S| \geq k+t+1$.*

3 Fundamental results

In this section, we provide the proof of Theorem 1.1. We first establish a useful lemma. For a matching M, we simply denote the set of end vertices of edges in M by $V(M)$.

Lemma 3.1. *Let $G\overline{G}$ be a k-extendable graph for some positive integer k. Suppose M is a matching in $G\overline{G}$ and $S \subseteq V(\overline{G})$ where $V(M) \cap S = \emptyset$ and $|M|+|S| \leq k$. Then*

1. *If $|S|$ is even, then there is a perfect matching in $G\overline{G} - (V(M) \cup S)$.*
2. *If $|S|$ is odd, then there is a vertex $\overline{y} \in V(\overline{G}) - (V(M) \cup S)$ such that $G\overline{G} - (V(M) \cup S \cup \{\overline{y}\})$ contains a perfect matching.*

Proof. Observe that $G\overline{G}$ is non-bipartite.

(1) It is easy to see that if $S = \emptyset$, then, by Theorem 2.2(1), $G\overline{G}-(V(M)\cup S) = G\overline{G}-V(M)$ contains a perfect matching since $G\overline{G}$ is k-extendable. So we may now assume that $S \neq \emptyset$. Then $2 \leq |S| \leq |M| + |S| \leq k$. Thus $|M| \leq k-2$. By Lemma 2.5 and the fact that $G\overline{G}$ is non–bipartite, $G\overline{G} - V(M)$ is $(k - |M|)$-extendable non–bipartite. Since $|S| \leq k - |M|$, $G\overline{G} - V(M)$ is $|S|$-extendable non–bipartite by Theorem 2.2(1). Hence, $G\overline{G} - V(M)$ is $|S|$-factor-critical by Theorem 2.4 and the fact that $|S|$ is even. Therefore, $G\overline{G}-(V(M)\cup S)$ contains a perfect matching. This proves (1).

(2) Since $|S|$ is odd, $|S| \geq 1$ and thus $|M| \leq k-|S| \leq k-1$. We first show that there are a vertex $\bar{u} \in S$ and a vertex $\bar{v} \in V(\overline{G})-(V(M)\cup S)$ such that $\bar{u}\bar{v} \in E(\overline{G})$. Suppose this is not the case. Let $\bar{u}_0 \in S$. Then $N_{G\overline{G}}[\bar{u}_0] \subseteq S\cup V(M)\cup\{u_0\}$ where u_0 is the only vertex in G which is adjacent to $\bar{u}_0$. Put $S' = (S - \{\bar{u}_0\}) \cup \{u_0\}$. Clearly, $\bar{u}_0$ becomes an isolated vertex in $G\overline{G} - (V(M) \cup S')$ and $|V(M) \cup S'| = 2|M| + |S'| = 2|M| + |S| \leq k + |M|$. So $V(M) \cup S'$ is a cutset of $G\overline{G}$. But this contradicts Lemma 2.6 since $G\overline{G}[V(M) \cup S']$ contains a matching of size at least $|M|$ and at most $|M| + \frac{1}{2}|S'| < |M| + |S| \leq k$ and $|V(M) \cup S'| \leq k + |M|$. Hence, there are a vertex $\bar{u} \in S$ and a vertex $\bar{v} \in V(\overline{G}) - (V(M) \cup S)$ such that $\bar{u}\bar{v} \in E(\overline{G})$ as required.

Now let $\bar{x} \in S$ and a vertex $\bar{y} \in V(\overline{G}) - (V(M) \cup S)$ such that $\bar{x}\bar{y} \in E(\overline{G})$. Consider $M\cup\{\bar{x}\bar{y}\}$. Clearly, $|M\cup\{\bar{x}\bar{y}\}| \leq k$. We first suppose that $|M\cup\{\bar{x}\bar{y}\}| = k$. Because $|M| \leq k - |S|$, $|S| = 1$ and thus $S = \{\bar{x}\}$. Since $G\overline{G}$ is k-extendable, $G\overline{G} - (V(M) \cup \{\bar{x}\bar{y}\}) = G\overline{G} - (V(M) \cup S \cup \{\bar{y}\})$ contains a perfect matching as required. So we now suppose that $|M \cup \{\bar{x}\bar{y}\}| \leq k - 1$. By Lemma 2.5 and the fact that $G\overline{G}$ is non–bipartite, $G\overline{G} - (V(M) \cup \{\bar{x}\bar{y}\})$ is $(k - (|M| + 1))$-extendable non–bipartite. Since $k - |M| - 1 \geq |S| - 1$ and $|S| - 1$ is even, it then follows by Theorems 2.2(1) and 2.4 that $G\overline{G} - (V(M) \cup \{\bar{x}\bar{y}\})$ is $(|S| - 1)$-factor-critical. Hence, $G\overline{G} - (V(M) \cup S \cup \{\bar{y}\})$ contains a perfect matching as required. This proves (2) and completes the proof of our lemma. □

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1

Proof. Clearly, our result holds for $l = 1$. So we now suppose $l \geq 2$. For simplicity, the induced subgraphs $G\overline{G}[V(G_i)]$, $G\overline{G}[V(\overline{G}_i)]$ and $G\overline{G}[V(G_i\overline{G}_i)]$ are denoted by $G_i, \overline{G}_i$ and $G_i\overline{G}_i$, respectively.

Let M be a matching of size k in $G\overline{G}$. For $1 \le i \le l$, let $M_i = M \cap E(G_i\overline{G}_i)$ and $S_i = \{x \in V(G_i\overline{G}_i) | xy \in M \text{ and } y \notin V(G_i\overline{G}_i)\}$. Observe that $S_i \subseteq V(\overline{G}_i)$ and $E(G\overline{G}[\bigcup_{i=1}^{l} S_i]) = M - \bigcup_{i=1}^{l} M_i$. We first suppose that $|S_i|$ is even for $1 \le i \le l$. Then, by Lemma 3.1(1), there is a perfect matching F_i in $G_i\overline{G}_i - (V(M_i) \cup S_i)$ for $1 \le i \le l$. Hence, $(\bigcup_{i=1}^{l} F_i) \cup M$ is a perfect matching in $G\overline{G}$ containing M as required.

We now suppose that $|S_i|$ is odd for some i. Let l_o be the number of components G_i of G in which $|S_i|$ is odd. We may now renumber the components of G in such a way that for the first l_0 components of G, $|S_i|$ is odd for $1 \le i \le l_o$ and for the last $l - l_0$ components of G, $|S_i|$ is even. Since $\sum_{i=1}^{l_o} |S_i| = 2(|M - \bigcup_{i=1}^{l} M_i|) - \sum_{i>l_0} |S_i|$ is even, l_o is even. By Lemma 3.1(2), there is $\bar{y}_i \in V(\overline{G}_i) - (V(M_i) \cup S_i)$ such that $G_i\overline{G}_i - (V(M_i) \cup S_i \cup \{\bar{y}_i\})$ contains a perfect matching, say F_i', for $1 \le i \le l_o$. Clearly, $G\overline{G}[\{\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_{l_0}\}]$ is a complete graph of even order. So there is a perfect matching in $G\overline{G}[\{\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_{l_0}\}]$, say F'. By Lemma 3.1(1), if $l_0 < l$, then there is a perfect matching F_i' in $G_i\overline{G}_i - (V(M_i) \cup S_i)$ for $l_o + 1 \le i \le l$. Therefore, $\bigcup_{i=1}^{l} F_i' \cup F' \cup M$ is a perfect matching in $G\overline{G}$ containing M as required. Hence, $G\overline{G}$ is k-extendable. This completes the proof of our theorem. □

Our next result follows immediately from Theorems 1.1 and 2.2(1).

Corollary 3.1. *For positive integers i and l where $1 \le i \le l$, let $G_1, \ldots, G_l$ be components of G. If $G_i\overline{G}_i$ is k_i-extendable of order $p_i \ge 2k_i + 2$ for some positive integer k_i, then $G\overline{G}$ is k_0-extendable where $k_0 = min\{k_1, k_2, \ldots, k_l\}$.*

4 The extendability of 2-regular graphs

To establish the proof of Theorem 1.2, we need to set up some lemmas. Observe that if x is a vertex of C_n for $n \ge 3$, then $C_n - x$ is a path of order $n - 1$. Our first lemma follows immediately by this fact.

Lemma 4.1. *Let $G \cong C_n$ for $n \ge 3$ with $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G) = \{v_iv_{i+1} | 1 \le i \le n\}$ where the subscript is read modulo n. Then*

1. *If n is even and e is an edge of G, then there is a perfect matching in G containing the edge e.*
2. *If n is odd, then, for each $1 \le k \le n$, $G - v_k$ contains a maximum matching of size $\frac{n-1}{2}$. In fact, a maximum matching of size $\frac{n-1}{2}$ is $\{v_{k+1}v_{k+2}, v_{k+3}v_{k+4}, \ldots, v_{k+n-2}v_{k+n-1}\}$ which is also a perfect matching in $G - v_k$.*

Lemma 4.2. *Let $G \cong C_n$ for $n \ge 5$. Then $\overline{G}$ is $(n-3)$-connected.*

Proof. Observe that $\overline{G}$ is $(n-3)$-regular. Let S be a minimum cutset of $\overline{G}$. For a positive integer $k \ge 2$, let $H_1, \ldots, H_k$ be components of $\overline{G} - S$. Since $\overline{G}$ is $(n-3)$-regular, $|V(H_i)| \ge n - 2 - |S|$. Then $n = |V(\overline{G})| = \sum_{i=1}^{k} |V(H_i)| + |S| \ge 2(n - 2 - |S|) + |S| = 2n - 4 - |S|$ and thus $|S| \ge n - 4$. Suppose $|S| = n - 4$. It is easy to see that $|V(H_i)| = 2$ and $k = 2$. Thus $n \geq 7$ since $\overline{G}$ is $(n-3)$-regular. It follows that $\overline{G} \cong 2K_2 \vee H$ where H is $(n-7)$-regular of order $n-4$. Thus G contains C_4 as an induced subgraph. But this contradicts the fact that $G \cong C_n$ where $n \geq 5$. Hence, $|S| \geq n-3$ and then $\overline{G}$ is $(n-3)$-connected. This completes the proof of our lemma. □

Lemma 4.3. *Let $G \cong C_n$ for $n \geq 6$. If n is even, then $\overline{G}$ is $(\frac{n-4}{2})$-extendable and if n is odd, then, for $1 \leq k \leq n$, $\overline{G} - v_k$ is $(\frac{n-5}{2})$-extendable.*

Proof. Observe that $\overline{G}$ is $K_{1,3}$-free otherwise G contains C_3 as an induced subgraph which contradicts the fact that $G \cong C_n$ and $n \geq 6$. By Theorem 2.3 and Lemma 4.2, $\overline{G}$ is $(\frac{n-4}{2})$-extendable if n is even. We now suppose that n is odd. Then $n \geq 7$ and $\overline{G} - v_k$ is $(n-4)$-connected by Lemma 4.2. Hence, by Theorem 2.3, $\overline{G} - v_k$ is $(\frac{n-5}{2})$-extendable. This proves our lemma. □

As a consequence of Theorem 2.2(1) and Lemma 4.3, we have the following corollaries.

Corollary 4.1. *Let $G \cong C_n$ for $n \geq 8$. If n is even, then $\overline{G}$ is 2-extendable and if n is odd, then, for $1 \leq k \leq n$, $\overline{G} - v_k$ is 2-extendable.*

Corollary 4.2. *Let $G \cong C_n$ for $n \geq 6$. If n is even, then $\overline{G}$ is 1-extendable and if n is odd, then, for $1 \leq k \leq n$, $\overline{G} - v_k$ is 1-extendable.*

Corollary 4.3. *Let $G \cong C_n$ for $n \geq 6$. Further, let v_i, v_j, v_k be three distinct vertices of $\overline{G}$ where $1 \leq i, j, k \leq n$, then $\overline{G} - \{v_i, v_j\}$ has a perfect matching if n is even and $\overline{G} - \{v_i, v_j, v_k\}$ has a perfect matching if n is odd.*

Proof. Our result follows from Theorems 2.2(1) and 2.4 together with Corollary 4.1 if $n \geq 8$. For $6 \leq n \leq 7$, our result follows from Theorem 2.1, Lemma 4.2 and the fact that $\overline{G}$ is $K_{1,3}$-free. □

Theorem 4.4. *Let G be a connected 2-regular graph of order $n \geq 6$. Then $G\overline{G}$ is 2-extendable.*

Proof. Clearly, $G \cong C_n$. Put $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G) = \{v_i v_{i+1} | 1 \leq i \leq n\}$ where the subscript is read modulo n. For simplicity, put $V(\overline{G}) = \{u_1, \ldots, u_n\}$ where $u_i \in V(\overline{G})$ corresponds to $v_i \in V(G)$. Then $V(G\overline{G}) = \{v_1, \ldots, v_n\} \cup \{u_1, \ldots, u_n\}$ and $E(G\overline{G}) = E(G) \cup E(\overline{G}) \cup \{v_i u_i | 1 \leq i \leq n\}$.

Let $T = \{e_1, e_2\}$ be a matching of size 2 in $G\overline{G}$. It is easy to see that if $\{e_1, e_2\} \subseteq \{v_i u_i | 1 \leq i \leq n\}$, then $\{v_i u_i | 1 \leq i \leq n\}$ is a perfect matching in $G\overline{G}$ containing the edges e_1 and e_2. So we may now assume without loss of generality that $e_1 \notin \{v_i u_i | 1 \leq i \leq n\}$. For simplicity, the set of end vertices of the edge e_i is denoted by $V(e_i)$ for $1 \leq i \leq 2$. To show that there is a perfect matching in $G\overline{G}$ containing the edges e_1 and e_2, we distinguish five cases according to the edges e_1

and e_2.

Case 1: $\{e_1, e_2\} \subseteq E(\overline{G})$.

By Corollary 4.1 and the fact that $G \cong C_n$, it is easy to see that there is a perfect matching in $G\overline{G}$ containing the edges e_1 and e_2 if $n \geq 8$ is even. For $n = 6$, it is not difficult to show that there is a perfect matching in $G\overline{G}$ containing the edges e_1 and e_2 as well.

So we now suppose that $n \geq 9$ is odd. Choose a vertex $u_j \in V(\overline{G}) - (V(e_1) \cup V(e_2))$. Then, by Corollary 4.1, there is a perfect matching $\overline{M}_1$, in $\overline{G} - u_j$, containing the edges e_1 and e_2. By Lemma 4.1(2), there is a perfect matching M_1 in $G - v_j$. Hence, $M_1 \cup \overline{M}_1 \cup \{v_j u_j\}$ is a perfect matching in $G\overline{G}$ containing the edges e_1 and e_2.

We now consider $n = 7$. Observe that $V(\overline{G})$ $-(V(e_1) \cup V(e_2))$ contains an edge, say e_3, otherwise G contains C_3 as an induced subgraph. Put $\{u_{j'}\} = V(\overline{G}) - \bigcup_{i=1}^{3} V(e_i)$. By Lemma 4.1(2), there is a perfect matching M_2 in $G - v_{j'}$. Thus $M_2 \cup \{e_1, e_2, e_3, v_{j'}u_{j'}\}$ is a perfect matching in $G\overline{G}$ containing the edges e_1 and e_2. This proves Case 1.

Case 2: $e_1 \in E(G), e_2 \in E(\overline{G})$.

Suppose $e_1 = v_j v_{j+1}$ and $e_2 = u_k u_{k'}$ where $1 \leq j, k, k' \leq n$ and $k \neq k'$. By Lemma 4.1(1) and Corollary 4.2, it is easy to see that there is a perfect matching containing the edges e_1 and e_2 if n is even. So we now suppose that n is odd.

We first suppose that $j + 2 \notin \{k, k'\}$. Then a maximum matching M_1, in $G - v_{j+2}$, containing the edge $e_1 = v_j v_{j+1}$ is a matching of size $\frac{n-1}{2}$. Thus M_1 is a perfect matching in $G - v_{j+2}$ by Lemma 4.1(2). By Corollary 4.2, $\overline{G} - u_{j+2}$ has a perfect matching $\overline{M}_1$ containing the edge e_2. Then $M_1 \cup \overline{M}_1 \cup \{v_{j+2}u_{j+2}\}$ is a perfect matching in $G\overline{G}$ containing the edges e_1 and e_2.

By similar arguments, if $j - 1 \notin \{k, k'\}$, then there is a perfect matching in $G\overline{G}$ containing the edges e_1 and e_2. We may now assume that $\{j - 1, j + 2\} = \{k, k'\}$. Then $e_2 = u_k u_{k'} = u_{j-1}u_{j+2}$. Now consider $G - v_{j+4}$. Since $n \geq 7$, $j + 4 \notin \{j - 1, j + 2\}$. Then a maximum matching M_2, in $G - v_{j+4}$, of size $\frac{n-1}{2}$ must contain the edge $e_1 = v_j v_{j+1}$. By Lemma 4.1(2), M_2 is a perfect matching in $G - v_{j+4}$. By Corollary 4.2, $\overline{G} - u_{j+4}$ has a perfect matching $\overline{M}_2$ containing the edge e_2. Then $M_2 \cup \overline{M}_2 \cup \{v_{j+4}u_{j+4}\}$ is a perfect matching in $G\overline{G}$ containing the edges e_1 and e_2.

Case 3: $e_1 \in E(G), e_2 \in \{v_i u_i | 1 \leq i \leq n\}$.

Let $e_2 = v_k u_k$ for some $1 \leq k \leq n$. Consider $G - v_k$. Observe that $G - v_k$ is a path of order $n-1$. Let M_1 and M_2 be matchings in $G - v_k$ where $E(G - v_k) = M_1 \cup M_2$ and $M_1 \cap M_2 = \emptyset$. We may assume that $|M_1| \geq |M_2|$. We first suppose that n is odd. Then $|M_1| = \frac{n-1}{2}$ and $|M_2| = \frac{n-3}{2}$. Further, v_{k-1} and v_{k+1} are M_2-unsaturated. By Lemma 4.1(2), M_1 is a perfect matching in $G - v_k$. If $e_1 \in M_1$, then $M_1 \cup \overline{M}_1 \cup \{v_k u_k\}$ is a perfect matching in $G\overline{G}$ containing the edges e_1 and e_2 where $\overline{M}_1$ is a perfect matching, in $\overline{G} - u_k$. Note that $\overline{M}_1$ exists by Corollary

4.2. We now suppose that $e_1 \in M_2$. By Corollary 4.3, there is a perfect matching $\overline{M}_2$, in $\overline{G} - \{u_{k-1}, u_k, u_{k+1}\}$. Hence, $M_2 \cup \overline{M}_2 \cup \{v_{k-1}u_{k-1}, v_k u_k, v_{k+1}u_{k+1}\}$ is a perfect matching in $G\overline{G}$ containing the edges e_1 and e_2.

We now suppose that n is even. Then $|M_1| = |M_2| = \frac{n-2}{2}$. Then either v_{k-1} or v_{k+1} is M'-unsaturated where $M' \in \{M_1, M_2\}$. Suppose without loss of generality that $e_1 \in M_1$ and v_{k-1} is M_1-unsaturated. By Corollary 4.3, there is a perfect matching $\overline{M}_3$, in $\overline{G} - \{u_{k-1}, u_k\}$. Hence, $M_1 \cup \overline{M}_3 \cup \{v_{k-1}u_{k-1}, v_k u_k\}$ is a perfect matching in $G\overline{G}$ containing the edges e_1 and e_2. This proves Case 3.

Case 4: $e_1 \in E(\overline{G}), e_2 \in \{v_i u_i | 1 \leq i \leq n\}$.

Let $e_1 = u_j u_{j'}$ and $e_2 = v_k u_k$ for some $1 \leq j, j', k \leq n$. Clearly, $k \notin \{j, j'\}$. We first suppose that n is odd. By Lemma 4.1(2), $G - v_k$ contains M_1 as a perfect matching. By Corollary 4.2, $\overline{G} - u_k$ has a perfect matching containing the edge e_1, say $\overline{M}_1$. Thus $M_1 \cup \overline{M}_1 \cup \{v_k u_k\}$ is a perfect matching in $G\overline{G}$ containing the edges e_1 and e_2.

We now suppose that $n \geq 8$ is even. Let M_2 and M_3 be perfect matchings in G containing the edges $v_k v_{k+1}$ and $v_{k-1}v_k$, respectively. Observe that if $S \subseteq V(\overline{G})$ with $|S| = 4$, then $\overline{G}[S]$ contains a matching of size two since $\overline{G}$ is $(n-3)$-regular and G does not contain C_3 as an induced subgraph. We first suppose that $k+1 \notin \{j, j'\}$. By Corollary 4.1, $\overline{G} - \{u_j, u_{j'}, u_k, u_{k+1}\}$ contains $\overline{M}_2$ as a perfect matching. Then $(M_2 - \{v_k v_{k+1}\}) \cup \overline{M}_2 \cup \{u_j u_{j'}, v_k u_k, v_{k+1}u_{k+1}\}$ is a perfect matching in $G\overline{G}$ containing the edges e_1 and e_2. By similar arguments, if $k-1 \notin \{j, j'\}$, then $(M_3 - \{v_{k-1}v_k\}) \cup \overline{M}_3 \cup \{u_j u_{j'}, v_{k-1}u_{k-1}, v_k u_k\}$ is a perfect matching in $G\overline{G}$ containing the edges e_1 and e_2 where $\overline{M}_3$ is a perfect matching in $\overline{G} - \{u_j, u_{j'}, u_{k-1}, u_k\}$. Finally, we suppose that $\{j, j'\} = \{k-1, k+1\}$. By Corollary 4.1 and the observation that $\overline{G}[S]$ contains a matching of size two if $S \subseteq V(\overline{G})$ with $|S| = 4$, $\overline{G} - \{u_{k-1}, u_k, u_{k+1}, u_{k+3}\}$ contains $\overline{M}_4$ as a perfect matching. Then $(M_2 - \{v_k v_{k+1}, v_{k+2}v_{k+3}\}) \cup \overline{M}_4 \cup \{u_j u_{j'}, v_k u_k, v_{k+1}v_{k+2}, v_{k+3}u_{k+3}\}$ is a perfect matching in $G\overline{G}$ containing the edges e_1 and e_2. For $n = 6$, it is routine to show that there is a perfect matching in $G\overline{G}$ containing the edges e_1 and e_2. This proves Case 4.

Case 5: $\{e_1, e_2\} \subseteq E(G)$.

Let M be a maximum matching in G containing the edge e_1. Clearly, M is a perfect matching if n is even and if n is odd, then there is exactly one M-unsaturated vertex, say v_j, for some $1 \leq j \leq n$. We first suppose that $e_2 \in M$. Then there is a perfect matching F containing the edges e_1 and e_2 where $F = M \cup \overline{M}$ if n is even and $F = M \cup \overline{M}_1 \cup \{v_j u_j\}$ if n is odd where $\overline{M}$ and $\overline{M}_1$ are perfect matchings in $\overline{G}$ and $\overline{G} - u_j$, respectively. Such $\overline{M}$ and $\overline{M}_1$ exist by Lemma 4.3.

We now suppose that $e_2 \notin M$. Put $e_2 = v_k v_{k+1}$ where $1 \leq k \leq n$. We first assume that n is even. Then $\{v_{k-1}v_k, v_{k+1}v_{k+2}\} \subseteq M - \{e_1\}$ since $\{e_1, e_2\}$ is a matching, M is a perfect matching and $G \cong C_n$. Clearly, $\{v_{k-1}, v_{k+2}\} \cap V(e_1) = \emptyset$. By Corollary 4.3, there exists a perfect matching in $\overline{M}_2$ in $\overline{G} - \{u_{k-1}, u_{k+2}\}$. Then $(M - \{v_{k-1}v_k, v_{k+1}v_{k+2}\}) \cup \overline{M}_2 \cup \{v_k v_{k+1}, v_{k-1}u_{k-1}, v_{k+2}u_{k+2}\}$ is a perfect matching in $G\overline{G}$ containing the edges e_1 and e_2.

We now suppose that n is odd. Recall that v_j is the only M-unsaturated of G. If $\{v_k, v_{k+1}\} \cap \{v_j\} = \{v_k\}$, then $\{v_{k+1}v_{k+2}\} \subseteq M - \{e_1\}$ and thus $(M - \{v_{k+1}v_{k+2}\}) \cup \overline{M}_3 \cup \{v_kv_{k+1}, v_{k+2}u_{k+2}\}$ is a perfect matching in $G\overline{G}$ containing the edges e_1 and e_2 where $\overline{M}_3$ is a perfect matching in $\overline{G} - u_{k+2}$. Note that $\overline{M}_3$ exists by Corollary 4.2. Similarly, if $\{v_k, v_{k+1}\} \cap \{v_j\} = \{v_{k+1}\}$, then $M - \{v_{k-1}v_k\} \cup \overline{M}_4 \cup \{v_kv_{k+1}, v_{k-1}u_{k-1}\}$ is a perfect matching in $G\overline{G}$ containing the edges e_1 and e_2 where $\overline{M}_4$ is a perfect matching in $\overline{G} - u_{k-1}$. We now consider the case that $\{v_k, v_{k+1}\} \cap \{v_j\} = \emptyset$. Observe that $j \notin \{k-1, k+2\}$ since $e_2 \notin M$ and v_j is M-unsaturated. Then $\{v_{k-1}v_k, v_{k+1}v_{k+2}\} \subseteq M - \{e_1\}$. By Corollary 4.3, there exists a perfect matching $\overline{M}_5$ in $\overline{G} - \{u_j, u_{k-1}, u_{k+2}\}$. Then $(M - \{v_{k-1}v_k, v_{k+1}v_{k+2}\}) \cup \overline{M}_5 \cup \{v_kv_{k+1}, v_{k-1}u_{k-1}, v_{k+2}u_{k+2}, v_ju_j\}$ is a perfect matching in $G\overline{G}$ containing the edges e_1 and e_2. This proves Case 5 and completes the proof of our theorem □

Note that the bound on n in Theorem 4.4 is sharp since the graph $C_5\overline{C}_5$ in Figure 1 is not 2-extendable because there is no perfect matching containing the edges v_1u_1 and v_3v_4.

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2

It is easy to see that our theorem follows immediately from Theorems 1.1 and 4.4. □

Corollary 4.4. *Let G be a connected 2-regular graph of order $n \geq 4$. Then $G\overline{G}$ is 1-extendable.*

Proof. Our result follows from Theorems 2.2(1) and 4.4 if $n \geq 6$. It is not difficult to show that the result is true for $4 \leq n \leq 5$. □

The next corollary follows immediately from Theorem 1.1 and Corollary 4.4.

Corollary 4.5. *Let G be a 2-regular C_3-free graph. Then $G\overline{G}$ is 1-extendable.*

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