Convergence Criteria of Viscosity Common Fixed Point Iterative Process for Asymptotically Nonexpansive Nonself Mappings in Banach Spaces

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Abstract: Let $X$ be a real arbitrary Banach space and let $C$ be a nonempty closed convex nonexpansive retract of $X$ with $P$ as a nonexpansive retraction. For $i = 1, 2$, let $T_i : C \to X$ be an asymptotically nonexpansive nonself mapping such that $F(T_1) \cap F(T_2) \neq \emptyset$ in $C$. Let $f : C \to C$ be a contractive mapping and let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0, 1]$. Define $\{x_n\}$ and $\{y_n\}$ to be the iterative sequences

$$y_n = P(\alpha_nf(x_n) + (1 - \alpha_n)(\beta_nx_n + (1 - \beta_n)T_2(PT_2)^{n-1}x_n)),$$
$$x_{n+1} = P(\gamma_nf(y_n) + (1 - \gamma_n)(\delta_ny_n + (1 - \delta_n)T_1(PT_1)^{n-1}y_n)), \quad n \geq 1.$$

Some strong convergence theorems of the sequence $\{x_n\}$ to a common fixed point of $T_1$ and $T_2$ are established under appropriate conditions.

Keywords: Asymptotically nonexpansive mapping; Nonexpansive retraction; Banach space; Common fixed point.

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1 Introduction

The concept of asymptotically nonexpansive self-mappings which is a generalization of the class of nonexpansive self-mappings was first introduced in 1972 by Goebel and Kirk [1]. They proved that any asymptotically nonexpansive self-mapping of a nonempty closed convex bounded subset of a uniformly convex Banach space possesses a fixed point. Since then, the weak and strong convergence problems of iterative sequences (with errors) for asymptotically nonexpansive self-mappings have been studied by many authors (see, for example, [2–7]). In 2003, Chidume et al. [8] introduced the concept of asymptotically nonexpansive nonself-mappings. Such a nonself mapping is defined as follows. Let $X$ be a real normed space, $C$ a nonempty subset of $X$ and $P : X \to C$ the nonexpansive retraction of $X$ onto $C$. A nonself mapping $T : C \to X$ is called asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\|T(P(T)x) - T(P(T)y)\| \leq k_n \|x - y\|,$$

for all $x, y \in C$ and $n \geq 1$. They proved the following.

**Theorem 1.1.** Let $E$ be a real uniformly convex Banach space, $K$ closed convex nonempty subset of $E$. Let $T : K \to E$ be completely continuous and asymptotically nonexpansive mapping with sequence $\{k_n\} \subset [1, \infty)$ such that $\sum_{n \geq 1} (k_n^2 - 1) < \infty$ and $F(T) \neq \emptyset$. Let $\alpha_n \in (0, 1)$ be such that $\epsilon \leq 1 - \alpha_n \leq 1 - \epsilon$, $\forall n \geq 1$ and some $\epsilon > 0$. From arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ by

$$x_n = P((1 - \alpha_n)x_n + \alpha_n T(P(T)x_{n-1})),$$

$n \geq 1$.

Then $\{x_n\}$ converges strongly to some fixed point of $T$.

They also proved the following theorem which was about the weak convergence of $\{x_n\}$ to some fixed point of $T$.

**Theorem 1.2.** Let $E$ be a real uniformly convex Banach space which has a Fréchet differentiable norm, $K$ closed convex nonempty subset of $E$. Let $T : K \to E$ be asymptotically nonexpansive mapping with sequence $\{k_n\} \subset [1, \infty)$ such that $\sum_{n \geq 1} (k_n^2 - 1) < \infty$ and $F(T) \neq \emptyset$. Let $\alpha_n \in (0, 1)$ be such that $\epsilon \leq 1 - \alpha_n \leq 1 - \epsilon$, $\forall n \geq 1$ and some $\epsilon > 0$. From arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ by

$$x_n = P((1 - \alpha_n)x_n + \alpha_n T(P(T)x_{n-1})),$$

$n \geq 1$.

Then $\{x_n\}$ converges weakly to some fixed point of $T$.

Recently, in 2008, Lou et al. [6] studied the viscosity approximation fixed point for asymptotically nonexpansive self-mappings in Banach spaces. They proved the following theorems.
Theorem 1.3. Let $K$ be a nonempty closed convex subset of a Banach space $X$ which has a uniformly Gâteaux differentiable norm and $T : K \to K$ an asymptotically nonexpansive mapping with $F(T) \neq \emptyset$ and $f$ a contraction on $C$. Let $\{\alpha_n\}, \{\beta_n\}$ be sequences in $(0, 1)$ satisfying
\[
C1 : \lim_{n \to \infty} \alpha_n = 0; \quad C2 : \lim_{n \to \infty} \frac{k_n - 1}{\alpha_n} = 0.
\]
Then the sequence $\{z_n\}$ defined by
\[
z_{n+1} = \alpha_n f(z_n) + (1 - \alpha_n) T^n z_n,
\]
converges strongly to the unique solution of the variational inequality:
\[
p \in F(T) \text{ such that } \langle (I - f)p, j(p - x^*) \rangle \leq 0 \quad \forall x^* \in F(T).
\]

Theorem 1.4. Let $K$ be a nonempty closed convex subset of a uniformly convex Banach space $X$ which has a uniformly Gâteaux differentiable norm and $T : K \to K$ an asymptotically nonexpansive mapping with $F(T) \neq \emptyset$ and $f$ a contraction on $C$. Let $\{\alpha_n\}, \{\beta_n\}$ be sequences in $(0, 1)$ satisfying
\[
C1 : \lim_{n \to \infty} \alpha_n = 0; \quad C2 : \sum_{n=1}^{\infty} \alpha_n = \infty \quad C3 : \lim_{n \to \infty} \frac{k_n - 1}{\alpha_n} = 0.
\]
For arbitrary $x_0 \in K$, let the sequence $\{x_n\}$ be defined iteratively by
\[
x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T^n x_n.
\]
Assume
(i) $\alpha_n, \beta_n, \gamma_n \in [0, 1], \alpha_n + \beta_n + \gamma_n = 1$;
(ii) $0 < \inf_{n \to \infty} \beta_n \leq \sup_{n \to \infty} \beta_n < 1$;
(iii) $T$ satisfies the asymptotically regularity; $\lim_{n \to \infty} \|T^{n+1} x_n - T^n x_n\| = 0$.
Then the sequence $\{x_n\}$ converges strongly to the unique solution of the variational inequality:
\[
p \in F(T) \text{ such that } \langle (I - f)p, j(p - x^*) \rangle \leq 0 \quad \forall x^* \in F(T).
\]

2 Preliminaries

In this paper, we study a viscosity approximation for some common fixed point of asymptotically nonexpansive nonself mappings in Banach spaces as follows.

Let $X$ be a real arbitrary Banach space and let $C$ be a nonempty closed convex nonexpansive retract of $X$ with $P$ as a nonexpansive retraction. A mapping $f : C \to C$ is called a contractive mapping if there exists a constant $\alpha \in (0, 1)$ such that
\[
\|f(x) - f(y)\| \leq \alpha \|x - y\|,
\]
for all \( x, y \in C \). We use \( d(x, F) \) for the distance from the point \( x \) to the set \( F \) and \( F(T) \) for the set of all fixed points of the mapping \( T \). For \( i = 1, 2 \), let \( T_i : C \to X \) be an asymptotically nonexpansive nonself mapping such that \( F(T_1) \cap F(T_2) \neq \emptyset \).

Let \( f : C \to C \) be a contractive mapping and let \( \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \) and \( \{\delta_n\} \) be real sequences in \([0, 1]\). For arbitrary \( x_1 \in C \), let \( \{x_n\} \) and \( \{y_n\} \) be the iterative sequences defined by

\[
\begin{align*}
y_n & = P(\alpha_n f(x_n) + (1 - \alpha_n)(\beta_n x_n + (1 - \beta_n)T_2(PT_2)^n-1 x_n)), \\
x_{n+1} & = P(\gamma_n f(y_n) + (1 - \gamma_n)(\delta_n y_n + (1 - \delta_n)T_1(PT_1)^n-1 y_n)), \quad n \geq 1. \quad (2.1)
\end{align*}
\]

Here, for convenience, we use the following definition of asymptotically nonexpansive nonself mapping. A nonself mapping \( T : C \to X \) is called asymptotically nonexpansive if there exists a sequence \( \{r_n\} \subset [0, 1) \) with \( r_n \to 0 \) as \( n \to \infty \) such that

\[
\|T(PT)^n-1 x - T(PT)^n-1 y\| \leq (1 + r_n)\|x - y\|,
\]

for all \( x, y \in C \) and \( n \geq 1 \).

We need the following lemmas for the main results in this paper.

**Lemma 2.1** ([9, Lemma 2.1]). Let \( \{a_n\}, \{b_n\} \) and \( \{\delta_n\} \) be sequences of nonnegative real numbers satisfying the inequality

\[
a_{n+1} \leq (1 + \delta_n)a_n + b_n \quad \text{for all } n.
\]

If \( \sum_{n=1}^{\infty} \delta_n < \infty \) and \( \sum_{n=1}^{\infty} b_n < \infty \), then

1. \( \lim_{n \to \infty} a_n \) exists.
2. \( \lim_{n \to \infty} a_n = 0 \) if \( \{a_n\} \) has a subsequence converging to zero.

**Lemma 2.2.** Let \( C \) be a nonempty closed subset of a Banach space \( X \) and \( T : C \to X \) be an asymptotically nonexpansive nonself mapping with the fixed point set \( F(T) \neq \emptyset \). Then \( F(T) \) is a closed subset in \( C \).

**Proof.** Assume that \( T : C \to X \) is an asymptotically nonexpansive nonself mapping with respect to \( \{r_n\} \). Let \( \{p_n\} \) be a sequence in \( F(T) \) such that \( p_n \to p \) as \( n \to \infty \). Since \( C \) is closed and \( \{p_n\} \) is a sequence in \( C \), we must have \( p \in C \). Since \( T : C \to X \) is asymptotically nonexpansive, we obtain

\[
\|Tp - p_n\| = \|Tp - Tp_n\| \leq (1 + r_1)\|p - p_n\|.
\]

Taking limit as \( n \to \infty \) and using the continuity of the norm, we obtain \( \|Tp - p\| \leq 0 \), which implies that \( Tp = p \). The proof is complete. \( \square \)
3 Main Results

In this section, we present our main results. The first theorem gives the necessary and sufficient condition for the convergence of the sequence \( \{x_n\} \) defined by (2.1).

**Theorem 3.1.** Let \( X \) be a real arbitrary Banach space and let \( C \) be a nonempty closed convex nonexpansive retract of \( X \) with \( P \) as a nonexpansive retraction. For \( i = 1, 2 \), let \( T_i : C \to X \) be an asymptotically nonexpansive nonself mapping with respect to \( \{r_i^{(n)}\} \) such that \( F(T_1) \cap F(T_2) \neq \emptyset \) and \( \sum_{n=1}^{\infty} r_n < \infty \), where \( r_n = \max\{r_1^{(n)}, r_2^{(n)}\} \). Let \( f : C \to C \) be a contractive mapping and let \( \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \) and \( \{\delta_n\} \) be real sequences in \([0, 1]\) such that \( \sum_{n=1}^{\infty} \alpha_n < \infty \) and \( \sum_{n=1}^{\infty} \gamma_n < \infty \). Then, the iterative sequence \( \{x_n\} \) defined by (2.1) converges to a common fixed point of \( T_1 \) and \( T_2 \) if and only if \( \lim \inf_{n \to \infty} d(x_n, F(T_1) \cap F(T_2)) = 0 \).

**Proof.** The necessity is obvious, so it is omitted. We now prove the sufficiency. Assume that \( T_i : C \to X \) is an asymptotically nonexpansive nonself mapping with respect to \( \{r_i^{(n)}\} \). Let \( p \in F(T_1) \cap F(T_2) \). Note that \( T_i(PT_i)^{-1}p = p \). By assumption, we have

\[
\|y_n - p\| = \|P(\alpha_n f(x_n) + (1 - \alpha_n)(\beta_n x_n + (1 - \beta_n)T_2(PT_2)^{n-1}x_n)) - Pp\|
\]

Using the contraction property of \( f \), we have

\[
\|y_n - p\| \leq \alpha_n \|f(x_n) - p\| + (1 - \alpha_n)\beta_n \|x_n - p\| + (1 - \alpha_n)\beta_n \|x_n - p\| + (1 - \alpha_n)\|x_n - p\| + (1 - \alpha_n)\|x_n - p\| + (1 - \alpha_n)\|x_n - p\| + (1 - \alpha_n)\|x_n - p\| + (1 - \alpha_n)\|x_n - p\|
\]

Similarly, we have that

\[
\|x_{n+1} - p\| \leq (1 + r_n)\|y_n - p\| + \gamma_n \|f(p) - p\|.
\]

From this and (3.1), we have

\[
\|x_{n+1} - p\| \leq (1 + r_n)((1 + r_n)\|x_n - p\| + \alpha_n\|f(p) - p\|) + \gamma_n\|f(p) - p\|
\]

\[
\leq (1 + r_n)(1 + r_n)\|x_n - p\| + [(1 + r_n)\alpha_n + \gamma_n]\|f(p) - p\|
\]

\[
\leq (1 + r_n(2 + r_n))\|x_n - p\| + [(1 + r_n)\alpha_n + \gamma_n]\|f(p) - p\|
\]

\[
= (1 + c_n)\|x_n - p\| + b_n,
\]

where \( c_n = r_n(2 + r_n) \) and \( b_n = [(1 + r_n)\alpha_n + \gamma_n]\|f(p) - p\| \). Since \( \sum_{n=1}^{\infty} r_n < \infty \), we have that \( \{2 + r_n\} \) and \( \{1 + r_n\} \) are bounded. Thus \( \sum_{n=1}^{\infty} c_n < \infty \) and \( \sum_{n=1}^{\infty} b_n < \infty \).
Also from (3.6), we obtain
\[
\lim_{n \to \infty} \|x_n - p\| = 0.
\]
Hence Lemma 2.1 implies that \(\lim_{n \to \infty} \|x_n - p\| \) exists. Thus \(\{x_n\}\) is bounded and so are \(\{T_1(PT_2)^n x_n\}\) and \(\{f(x_n)\}\) because \(T_1\) is asymptotically nonexpansive and \(f\) is contractive. Now since \(\{x_n\}\) is bounded and from (3.1), we conclude that \(\{y_n\}\) is bounded and so are \(\{T_1(PT_1)^n y_n\}\) and \(\{f(y_n)\}\).

We next turn to another calculation for \(\|y_n - p\|\) and \(\|x_{n+1} - p\|\) as follows.

\[
\|y_n - p\| = \|P(\alpha_nf(x_n) + (1 - \alpha_n)(\beta_n x_n + (1 - \beta_n)T_2(PT_2)^n x_n)) - Pp\|
\leq \|\alpha_nf(x_n) + (1 - \alpha_n)(\beta_n x_n + (1 - \beta_n)T_2(PT_2)^n x_n) - p\|
\leq \alpha_n \|f(x_n) - T_2(PT_2)^n x_n + (1 - \alpha_n)\beta_n x_n - p\|
+ (1 - \beta_n + \alpha_n \beta_n) \|T_2(PT_2)^n x_n - p\|
\leq \alpha_n \|f(x_n) - T_2(PT_2)^n x_n + (1 - \alpha_n)\beta_n x_n - p\|
+ (1 - \beta_n + \alpha_n \beta_n) \|x_n - p\| + \alpha_n \|f(x_n) - T_2(PT_2)^n x_n - p\|
\leq (1 + r_n) \|x_n - p\| + \alpha_n \|f(x_n) - T_2(PT_2)^n x_n\|.
\] (3.3)

Similarly, we have that
\[
\|x_{n+1} - p\| \leq (1 + 2r_n) \|y_n - p\| + \gamma_n \|f(y_n) - T_1(PT_1)^n y_n\|.
\] (3.4)

Putting (3.3) in (3.4), we obtain that
\[
\|x_{n+1} - p\| \leq (1 + 2r_n) \|x_n - p\| + (1 + 2r_n) \alpha_n \|f(x_n) - T_2(PT_2)^n x_n\|
+ \gamma_n \|f(y_n) - T_1(PT_1)^n y_n\|
= (1 + d_n) \|x_n - p\| + e_n,
\] (3.5)
where \(d_n = 4r_n(1 + r_n)\) and \(e_n = (1 + 2r_n) \alpha_n \|f(x_n) - T_2(PT_2)^n x_n\| + \gamma_n \|f(y_n) - T_1(PT_1)^n y_n\|\). By the assumption that \(\sum_{n=1}^\infty r_n < \infty\), \(\sum_{n=1}^\infty \alpha_n < \infty\), \(\sum_{n=1}^\infty \gamma_n < \infty\), and \(\{T_2(PT_2)^n x_n\}, \{T_1(PT_1)^n y_n\}, \{f(x_n)\}\) and \(\{f(y_n)\}\) are bounded, we have that \(\sum_{n=1}^\infty d_n < \infty\) and \(\sum_{n=1}^\infty e_n < \infty\). Hence Lemma 2.1 tells us that \(\lim_{n \to \infty} \|x_n - p\|\) exists. Thus \(\{|x_n - p|\}\) is bounded. Let \(L = \sup_n \|x_n - p\|\). We can rewrite (3.5) as
\[
\|x_{n+1} - p\| \leq \|x_n - p\| + Ld_n + e_n \quad \text{for} \quad n \geq 1.
\] (3.6)

From this and by induction, we obtain, for \(m, n \geq 1\) and \(p \in F(T_1) \cap F(T_2)\), that
\[
\|x_{n+m} - p\| \leq \|x_n - p\| + L \sum_{i=n}^{n+m-1} d_i + \sum_{i=n}^{n+m-1} e_i.
\] (3.7)

Also from (3.6), we obtain
\[
d(x_{n+1}, F(T_1) \cap F(T_2)) \leq d(x_n, F(T_1) \cap F(T_2)) + Ld_n + e_n.
\]
But, the assumption \( \liminf_{n \to \infty} d(x_n, F(T_1) \cap F(T_2)) = 0 \) implies that there exists a subsequence of \( \{d(x_n, F(T_1) \cap F(T_2))\} \) converging to zero. From this and because \( \sum_{n=1}^{\infty} (Ld_n + e_n) < \infty \), Lemma 2.1 tells us that

\[
\lim_{n \to \infty} d(x_n, F(T_1) \cap F(T_2)) = 0.
\] (3.8)

We now show that \( \{x_n\} \) is a Cauchy sequence in \( X \). Let \( \epsilon > 0 \). From (3.8), \( \sum_{n=1}^{\infty} d_n < \infty \) and \( \sum_{n=1}^{\infty} e_n < \infty \), there exists \( n_0 \) such that, for \( n \geq n_0 \), we have

\[
d(x_n, F(T_1) \cap F(T_2)) < \epsilon/6, \sum_{i=n}^{\infty} d_i < \epsilon/(3L) \text{ and } \sum_{i=n}^{\infty} e_i < \epsilon/3.
\] (3.9)

By the first inequality in (3.9) and the definition of infimum, there exists \( p_0 \in F(T_1) \cap F(T_2) \) such that

\[
||x_{n_0} - p_0|| < \epsilon/6.
\] (3.10)

Combining (3.6), (3.9) and (3.10), we obtain

\[
||x_{n_0+m} - x_{n_0}|| \leq ||x_{n_0+m} - p_0|| + ||x_{n_0} - p_0||
\]

\[
\leq 2||x_{n_0} - p_0|| + L \sum_{i=n_0}^{n_0+m-1} d_i + \sum_{i=n_0}^{n_0+m-1} e_i
\]

\[
< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon,
\]

which implies that \( \{x_n\} \) is a Cauchy sequence in \( X \). But \( X \) is a Banach space, so there must be some \( q \in X \) such that \( x_n \to q \). Since \( C \) is closed and \( \{x_n\} \) is a sequence in \( C \), we have that \( q \in C \). Now \( d(x_n, F(T_1) \cap F(T_2)) \to 0 \) and \( x_n \to q \) as \( n \to \infty \), the continuity of \( d(\cdot, F(T_1) \cap F(T_2)) \) implies that \( d(q, F(T_1) \cap F(T_2)) = 0 \). Thus \( q \in F(T_1) \cap F(T_2) \) because \( F(T_1) \cap F(T_2) \) is closed, by Lemma 2.2. Therefore \( \{x_n\} \) converges to a common fixed point of \( T_1 \) and \( T_2 \), as desired. \( \Box \)

If \( T_1 = T_2 = T \), then the iterative sequences (2.1) become

\[
y_n = P(\alpha_n f(x_n) + (1 - \alpha_n) \beta_n x_n + (1 - \beta_n) T(PT)^{n-1} x_n),
\]

\[
x_{n+1} = P(\gamma_n f(y_n) + (1 - \gamma_n) \delta_n y_n + (1 - \delta_n) T(PT)^{n-1} y_n), \quad n \geq 1.
\] (3.11)

We then have the following result for a fixed point of a single asymptotically nonexpansive nonself mapping.

**Corollary 3.2.** Let \( X \) be a real Banach space and let \( C \) be a nonempty closed convex nonexpansive retract of \( X \) with \( P \) as a nonexpansive retraction. Let \( T : X \to X \) be an asymptotically nonexpansive nonself mapping with respect to \( \{r_n\} \) such that \( F(T) \neq \emptyset \) in \( C \) and \( \sum_{n=1}^{\infty} r_n < \infty \). Let \( f : C \to C \) be a contractive mapping and let \( \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \) and \( \{\delta_n\} \) be real sequences in \([0,1]\) such that \( \sum_{n=1}^{\infty} \alpha_n < \infty \) and \( \sum_{n=1}^{\infty} \gamma_n < \infty \). Then, the sequence \( \{x_n\} \), defined by (3.11), converges to a fixed point of \( T \) if and only if \( \liminf_{n \to \infty} d(x_n, F(T)) = 0 \).
We also have the following results involving asymptotic regularity as in Lou et al. [6] and an auxiliary strictly increasing nonnegative function as in Ayaragnaranchakanul [10].

**Corollary 3.3.** Let $X, C, T_i$ $(i = 1, 2)$ and the iterative sequence $\{x_n\}$ be as in Theorem 3.1. Suppose that the conditions in Theorem 3.1 hold and

1. the mapping $T_i$ $(i = 1, 2)$ is asymptotically regular in $x_n$, i.e.,
$$\liminf_{n \to \infty} \|x_n - T_ix_n\| = 0, \quad i = 1, 2;$$
2. $\liminf_{n \to \infty} \|x_n - T_ix_n\| = 0$ implies that
$$\liminf_{n \to \infty} d(x_n, F(T_1) \cap F(T_2)) = 0.$$

Then the sequences $\{x_n\}$ converges to a common fixed point of $T_1$ and $T_2$.

**Theorem 3.4.** Let $X, C, T_i$ $(i = 1, 2)$ and the iterative sequence $\{x_n\}$ be as in Theorem 3.1. Suppose that the conditions in Theorem 3.1 hold, the mapping $T_i$ is asymptotically regular in $x_n$, and there exists an increasing function $g : \mathbb{R}^+ \to \mathbb{R}^+$ with $g(r) > 0$ for all $r > 0$ such that for $i = 1, 2$,
$$\|x_n - T_ix_n\| \geq g(d(x_n, F(T_1) \cap F(T_2))), \quad \forall n \geq 1.$$ 

Then the sequence $\{x_n\}$ converges to a common fixed point of $T_1$ and $T_2$.

**Proof.** To apply Theorem 3.1, we prove that $\liminf_{n \to \infty} d(x_n, F(T_1) \cap F(T_2)) = 0$. From the assumption that $\|x_n - T_ix_n\| \geq g(d(x_n, F(T_1) \cap F(T_2)))$ for $i = 1, 2$ and for all $n \geq 1$, we have
$$\frac{1}{2} \sum_{i=1}^{2} \|x_n - T_ix_n\| \geq g(d(x_n, F(T_1) \cap F(T_2))),$$
for all $n \geq 1$. Since $T_i$ is asymptotically regular in $x_n$, this implies that
$$\liminf_{n \to \infty} g(d(x_n, F(T_1) \cap F(T_2))) = 0. \quad (3.12)$$

Suppose that $\liminf_{n \to \infty} d(x_n, F(T_1) \cap F(T_2)) = L > 0$. By definition of infimum, there exists an $N$ such that
$$\inf_{n \geq m} d(x_n, F(T_1) \cap F(T_2)) - L < \frac{L}{2}, \quad \forall m \geq N.$$ 

Equivalently,
$$d(x_n, F(T_1) \cap F(T_2)) > \frac{L}{2}, \quad \forall n \geq m \geq N.$$
Since $g$ is increasing, we have that
\[ g(d(x_n, F(T_1) \cap F(T_2))) \geq g \left( \frac{L}{2} \right), \quad \text{for all } n \geq m \geq N. \]

This implies that
\[ \liminf_{n \to \infty} g(d(x_n, F(T_1) \cap F(T_2))) \geq g \left( \frac{L}{2} \right) > 0, \]
which contradicts (3.12). Hence $\liminf_{n \to \infty} d(x_n, F(T_1) \cap F(T_2)) = 0$, as desired.

If $T_i$ is a self-mapping, then the iterative sequences (2.1) become
\[
\begin{align*}
y_n &= \alpha_n f(x_n) + (1 - \alpha_n)(\beta_n x_n + (1 - \beta_n)T_2^n x_n), \\
x_{n+1} &= \gamma_n f(y_n) + (1 - \gamma_n)(\delta_n y_n + (1 - \delta_n)T_1^n y_n), \quad n \geq 1. \tag{3.13}
\end{align*}
\]

We have the following theorem for common fixed point of two asymptotically nonexpansive self-mappings.

**Corollary 3.5.** Let $X$ be a real Banach space and let $C$ be a nonempty closed convex subset of $X$. For $i = 1, 2$, let $T_i : C \to C$ be an asymptotically nonexpansive self-mapping with respect to $\{r_i^{(n)}\}$ such that $F(T_1) \cap F(T_2) \neq \emptyset$ and $\sum_{n=1}^{\infty} r_n < \infty$, where $r_n = \max\{r_1^{(n)}, r_2^{(n)}\}$. Let $f : C \to C$ be a contractive mapping and let $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\delta_n\}$ be real sequences in $[0, 1]$ such that $\sum_{n=1}^{\infty} \alpha_n < \infty$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$. Then, the iterative sequence $\{x_n\}$ defined by (3.13) converges to a common fixed point of $T_1$ and $T_2$ if and only if $\liminf_{n \to \infty} d(x_n, F(T_1) \cap F(T_2)) = 0$.

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**References**


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