A Graphical Proof of the Brouwer Fixed Point Theorem

N. Chuensupantharat†, P. Kumam‡ and S. Dhompongsa¶,†

†KMUTT Fixed Point Research Laboratory, Department of Mathematics
Room SCL 802 Fixed Point Laboratory, Science Laboratory Building
Faculty of Science, King Mongkuts University of Technology Thonburi
(KMUTT), 126 Pracha-Uthit Road, Bang Mod, Thrung Khru
Bangkok 10140, Thailand
e-mail : nantaporn.joy@mail.kmutt.ac.th (N. Chuensupantharat)

‡KMUTT-Fixed Point Theory and Applications Research Group
Theoretical and Computational Science Center (TaCS)
Science Laboratory Building, Faculty of Science
King Mongkuts University of Technology Thonburi (KMUTT)
126 Pracha-Uthit Road, Bang Mod, Thrung Khru
Bangkok 10140, Thailand
e-mail : poom.kum@kmutt.ac.th (P. Kumam)

¶Department of Mathematics, Faculty of Science
Chiang Mai University, Chiang Mai 50200, Thailand
e-mail : sompong.d@cmu.ac.th (S. Dhompongsa)
sompong.dho@kmutt.ac.th (S. Dhompongsa)

Abstract : By simplifying the proof in [1], we give a new proof of the Brouwer
fixed point theorem without using the Tietze (continuous) extension theorem.

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1 Introduction

The Brouwer fixed point theorem states that: For the unit cube \([0, 1]^d\) of the Euclidean space \(\mathbb{R}^d\), any continuous mapping \(T : [0, 1]^d \to [0, 1]^d\) has a fixed point, i.e., a point \(x \in [0, 1]^d\) with \(T(x) = x\).

As in [1], we prove the theorem by induction on the dimension \(d\).

2 Preliminaries

We recall notations introduced in [1]. Put \(K = [0, 1]^d\) and let \(\{e_1, \ldots, e_d\}\) be the standard basis for \(\mathbb{R}^d\), that is, based on the Kronecker delta \(\delta_{ji}\), \(e_j = (\delta_{ji})_{i=1}^d\).

For \(j = 1, \ldots, d\), write \(j = \{\sum_{i=1, i \neq j}^d x_i e_i : 0 \leq x_i \leq 1, i = 1, \ldots, d, i \neq j\}\) and \(j' = j + e_j\). Let \(H_u\) for \(0 \leq u \leq \sqrt{d}\) be the hyperplane passing through \((u, \ldots, u) \in \mathbb{R}^d\) having \(\bar{1} = (1, 1, \ldots, 1) \in \mathbb{R}^d\) as its normal vector and put \(\Delta_u = K \cap H_u\).

The mapping

\[\pi_p : (x_1, \ldots, x_d) \mapsto \sum_{i=1, i \neq p}^d x_i e_i, \quad \text{for} \quad (x_1, \ldots, x_d) \in K\]

is the projection onto \(\Box_p\) for \(p = j, j'\). Set \(S_{uj}\) to be the component of a subset of \(\Box_j \setminus \pi_j(\Delta_u) \cup \Box_j' \setminus \pi_j'(\Delta_u) \cup \Delta_u\) containing \(\Delta_u\). Above the face \(\Box_j\), let \(S_{uj}\) be the continuous surface consisting of \(\Delta_u\) together with \(\Box_{uj}\).

![Figure 1](image_url)

Write the given continuous function \(T = (f_1, \ldots, f_d)\) where \(f_j : K \to [0, 1]\) is continuous for each \(j\). For each \(u\), draw the graph of \(f_j\) restricted to \(S_{uj}\) via the formula

\[g_{uj} : (x_1, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_d) \mapsto (x_1, \ldots, x_{j-1}, f_j(x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_d), x_{j+1}, \ldots, x_d)\]

for each \((x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_d) \in S_{uj}\).

Observe that \((x_1, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_d) = \pi_j(x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_d)\). Thus the graph of \(f_j\) at \(u\) means the set of points

\[(x_1, \ldots, x_{j-1}, f_j(x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_d), x_{j+1}, \ldots, x_d)\]
for \((x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_d) \in S_{u_j}\). Write \(f_{u_j}\) for \(f_j\big|_{S_{u_j}}\).

Our proof relies on this result:

**Lemma 2.1.** [1, Lemma 3.1] For each \(u\), the graphs of \(f_1, \ldots, f_d\) intersect at a point.

In the sequel, we will refer to “a point of intersection of the graphs of \(f_1, \ldots, f_d\)” shortly as “a point of intersection of \(f_1, \ldots, f_d\).” Define \(F_u(f_1, \ldots, f_k)\) to be the set of the points of intersection of \(f_1, \ldots, f_k\). For example, if \(k_1\) is the identity mapping on \(S_{u_1}\), \(F_u(s_1, f_2)\) is the intersection of the graph of \(f_2\) and \(S_{u_1}\). The negative part \(N^0(f, u)\) and the positive part \(P^0(f, u)\) of \(f\) over \(S_{u_1}\) are defined as

\[
N^0(f_1, u) = \{(x_1, \ldots, x_d) \in S_{u_1} : f_1(x_1, \ldots, x_d) < x_1\},
\]

\[
P^0(f_1, u) = \{(x_1, \ldots, x_d) \in S_{u_1} : f_1(x_1, \ldots, x_d) > x_1\}.
\]

The nonpositive \(N(f, u)\) and the nonnegative \(P(f, u)\) are defined by replacing < and > by \(\leq\) and \(\geq\) respectively. Clearly, \(N^0(f_1, u)\) and \(P^0(f_1, u)\) can be partitioned into relatively open components, say,

\[
N^0(f_1, u) = \bigcup_\alpha N^0_\alpha(f_1, u), \quad P^0(f_1, u) = \bigcup_\beta P^0_\beta(f_1, u).
\]

For each pair \((u, u_0)\) for which \(0 < u < u_0 < \sqrt{d}\) and for a continuous mapping \(h_{u_j} : S_{u_j} \to [0, 1]\), we write \(H^u_{u_j}\) for a copy of \(h_{u_j}\) by translating \(h_{u_j}\) along the vector \(\frac{u_0 - u}{\sqrt{d}} e_j\). We may need to project \(H^u_{u_j}\) back to \(K\) if necessary. Thus, \(H^u_{u_j}\) can be considered as a continuous mapping defined on \(S_{u_0}\).

In the course of the proof, we need the following construction:

\[\text{(2.1)}\]

For a given nonempty closed subset \(A\) of \(\Delta_u\) formed by a finite union of closed \((d-1)\)-dimensional boxes and for a pair of continuous mappings \(g, h : S_{u_j} \to [0, 1]\), we draw the segment joining \(g(x)\) and \(h(x)\) for \(x \in A\). By slight shrinking the graph of \(g\) over \(A\) and call the new mapping as \(\hat{g}\), we obtain a continuous surface \(\hat{h}\) so that \(\hat{h} = h\) over \(S_{u_j} \setminus A\) and \(\hat{h} = \hat{g}\) over \(A\).

We will apply the construction \(\text{(2.1)}\) to \((g, h) = (f_{u_0}, H)\) where \(u_0\) and \(H\) are to be specified later.

## 3 Proof

Assume that \(\bar{0}\) is not a fixed point of \(T\) and suppose that \(f_1(\bar{0}) > 0\). We shall consider \(N^0(f_1, u)\) when \(u\) moves from \(0\) toward \(\sqrt{d}\). Obviously, under the above assumption, \(N^0(f_1, u) = \emptyset\) for all small \(u\). It is also clear that each point in the intersection

\[
F_u := F_u(s_1, f_2, f_3, \ldots, f_d) \cap \{(x_1, \ldots, x_d) \in \Delta_u : f(x_1, \ldots, x_d) = x_1\}
\]
is a fixed point of $T$.

For each $u$, we say that $f_2$ and $f_3$ are removable from $S_{u_1}$ if there are mappings $h_2$ and $h_3$ such that $h_j = f_j$ for $j = 2, 3$ on $P(f_1, u)$ and $F_u(s_1, h_2, h_3, f_4, \ldots, f_d) \cap N^0(f_1, u) = \emptyset$. The term “removable” describes the removal of points in $F_u(s_1, h_2, h_3, f_4, \ldots, f_d) \cap N^0(f_1, u)$. Let

$$\mathcal{U} = \{u > 0 : \text{for each } v \leq u, F_v = \emptyset \text{ and } f_2, f_3 \text{ are removable from } S_v \}.$$ 

Clearly $\mathcal{U} \neq \emptyset$, let $u_0 = \sup \mathcal{U}$. If $u_0 = \sqrt{d}$, then $\bar{1}$ is a fixed point. This follows from Lemma 2.1 and the fact that under new definition of $f_2$ and $f_3$, $F_u(s_1, f_2, f_3, \ldots, f_d) \cap N^0(f_1, u) = \emptyset$ for all $u < \sqrt{d}$. Now suppose $u_0 < \sqrt{d}$. If $F_{u_0} \neq \emptyset$, we are done. If $F_{u_0} = \emptyset$, we will find a contradiction. First construct a subset $A_\alpha$ of $N^0_\alpha(f_1, u_0)$ formed by a finite union of $(d-1)$ - dimensional boxes lie in each $N^0_\alpha(f_1, u_0)$ for which $F_{u_0}(s_1, f_2, f_3, \ldots, f_d) \cap N^0_\alpha(f_1, u_0) \neq \emptyset$. The set $A_\alpha$ can be constructed so that

$$[F_u(s_1, f_2, f_3, \ldots, f_d) + \frac{u_0 - u}{\sqrt{d}} e_j] \cap N^0_\alpha(f_1, u_0) \subset A_\alpha$$

for all $u < u_0$ with $u_0 - u$ sufficiently small. For some such $u$, we apply construction (2.1) to $(g, h) = (f_{u_0 j}, H_{u_0 j})$ for $j = 2, 3$. It is observed by continuity that, for some $u$ with $u_0 - u$ sufficiently small, $F_u(s_1, H_{u_0 2}, H_{u_0 3}, f_4, \ldots, f_d) \cap N^0(f_1, u_0) = \emptyset$. This shows that $u_0 \in \mathcal{U}$. With the similar argument, we can show that $u \in \mathcal{U}$ for some (and actually for all) $u > u_0$ with $u - u_0$ sufficiently small. We do this by letting $(u_0, u)$ take the role of $(u, u_0)$ in the previous case, and this leads to a contradiction as claimed.

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