Semigroups of Full Transformations with Restriction on the Fixed Set is Bijective

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Abstract: Let $T(X)$ be the full transformation semigroup of the set $X$ and let $S(X,Y) = \{ \alpha \in T(X) : Y\alpha \subseteq Y \}$ where $Y$ is a nonempty subset of $X$. Then $S(X,Y)$ is a subsemigroup of $T(X)$. In this paper, for a fixed nonempty subset $Y$ of $X$, let

$$PG_Y(X) = \{ \alpha \in T(X) : \alpha|_Y \in G(Y) \}$$

where $G(Y)$ is the permutation group on $Y$. Then $PG_Y(X)$ is a subsemigroup of $S(X,Y)$. Some relationships between $PG_Y(X)$ it’s subsemigroup and $S(X,Y)$ are considered. Moreover, it is shown that $PG_Y(X)$ is regular and characterizations of left regularity, right regularity, and completely regularity of elements of $PG_Y(X)$ are also described.

Keywords: transformation semigroup; regularity; left regularity; right regularity; completely regularity.

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1 Introduction

Let $X$ be a nonempty set and let $T(X)$ denote the semigroup of the full transformations from $X$ into itself under composition of mappings. This semigroup is an important object in semigroup theory, combinatorics, many-valued logic, etc. It is known that $T(X)$ is a regular semigroup, that is, for every $\alpha \in T(X)$, $\alpha = \alpha \beta \alpha$
for some \( \beta \in T(X) \). For a fixed nonempty subset \( Y \) of \( X \), we denote

\[
S(X, Y) = \{ \alpha \in T(X) : Y \alpha \subseteq Y \}.
\]

Then \( S(X, Y) \) is a semigroup of full transformations on \( X \) which leave \( Y \) invariant. In 1966, Magill [1] introduced and studied this semigroup. Later, many classical notions of this semigroup have been investigated, see [2], [3] and [4].

In [3], Nenthein, Youngkhong and Kemprasit showed that \( S(X, Y) \) is regular if and only if \( X = Y \) or \( Y \) contains exactly one element.

In 1994, Umar [5] constructed a subsemigroup of \( T(X) \) as follows:

\[
F_Y(X) = \{ \alpha \in T(X) : C(\alpha) \subseteq Y = Y \alpha \text{ and } \alpha|_Y \text{ is injective} \}
\]

where

\[
C(\alpha) = \bigcup \{ ya^{-1} : y \in X \alpha \text{ and } |ya^{-1}| \geq 2 \}.
\]

\( F_Y(X) \) is called an *Umar semigroup*. He proved that \( F_Y(X) \) is a regular semigroup and considered the Green’s relations on this semigroup. It is clear that \( F_Y(X) \) is a subsemigroup of \( S(X, Y) \).

Later, Sanwong and Sommanee [6] investigated regularity and Green’s relations on a subsemigroup of \( S(X, Y) \) which defined by

\[
T(X, Y) = \{ \alpha \in T(X) : X \alpha \subseteq Y \}.
\]

Recently, a subsemigroup of \( S(X, Y) \) defined by \( F(X, Y) = \{ \alpha \in T(X, Y) : X \alpha \subseteq Y \alpha \} \) was studied by Sanwong [7].

It is the aim of the paper to introduce a new subsemigroup of \( S(X, Y) \) which is defined as follows:

\[
PG_Y(X) = \{ \alpha \in T(X) : \alpha|_Y \in G(Y) \}
\]

where \( G(Y) \) is the permutation group on a nonempty subset \( Y \) of \( X \). Some algebraic properties of \( PG_Y(X) \) are studied. For examples, \( PG_Y(X) \) is a regular semigroup, \( T(X) \) can be embeddable into \( PG_Y(Z) \) for some set \( Z \) and relationships between \( PG_Y(X) \), it’s subsemigroup and \( S(X, Y) \) are given. In the last section, left regularity, right regularity, and completely regularity of elements of \( PG_Y(X) \) are determined.

Throughout of the paper, the symbol \( \pi(\alpha) \) will denote the partition of \( X \) induced by \( \alpha \in T(X) \) namely,

\[
\pi(\alpha) = \{ ya^{-1} : y \in X \alpha \}.
\]

The set \( X \) can be finite or infinite. The cardinality of a set \( A \) is denoted by \( |A| \).
2 Preliminaries

Let $X$ be an arbitrary set and $Y$ a nonempty subset of $X$. Define a subset of $T(X)$ as follows:

$$PG_Y(X) = \{ \alpha \in T(X) : \alpha|_Y \in G(Y) \}$$

where $G(Y)$ is the permutation group on $Y$. Note that $id_X$, the identity mapping on $X$, belongs to $PG_Y(X)$.

**Remark 2.1.** We note that $PG_Y(X) = G(X)$ if $Y = X$. For arbitrary singleton subset $Y$ of $X$, we obtain that $S(X, Y) = PG_Y(X)$. Moreover, if $|X| = 2$, then we have $S(X, Y) = PG_Y(X) = F_Y(X)$.

**Theorem 2.2.** $PG_Y(X)$ is a regular semigroup.

**Proof.** To prove that $PG_Y(X)$ is a subsemigroup of $T(X)$, let $\alpha, \beta \in PG_Y(X)$. Then we have $\alpha|_Y, \beta|_Y \in G(Y)$ whence $\alpha\beta|_Y \in T(Y)$. It is easy to verify that $\alpha\beta|_Y \in G(Y)$. To show $PG_Y(X)$ is regular, let $\alpha \in PG_Y(X)$. We obtain via $Y\alpha = Y$ that $X\alpha = Y \cup (X\alpha\setminus Y)$. For each $x \in Y$, there exists a unique $x' \in Y$ such that $x'\alpha = x$ since $\alpha|_Y \in G(Y)$. For $x \in X\alpha\setminus Y$, we choose $x' \in x\alpha^{-1}$. Define $\beta : X \to X$ by

$$x\beta = \begin{cases} x', & \text{if } x \in X\alpha, \\ x, & \text{otherwise}. \end{cases}$$

Obviously, $\beta|_Y : Y \to Y$ is bijective, that is $\beta \in PG_Y(X)$. Let $x \in X$. Then $x\alpha\beta\alpha = (x\alpha)\alpha = x\alpha$. This means that $\alpha = \alpha\beta\alpha$ whence $PG_Y(X)$ is a regular semigroup.

From the definition of $PG_Y(X)$ and Theorem 2.2, we conclude that $F_Y(X)$ is a subsemigroup of $PG_Y(X)$ and $PG_Y(X)$ is a subsemigroup of $S(X, Y)$. Next, the conditions under which the semigroups coincide are given.

**Theorem 2.3.** $F_Y(X) = PG_Y(X)$ if and only if $|X\setminus Y| \leq 1$.

**Proof.** Assume that $|X\setminus Y| \geq 2$. There exist $a, b \in X\setminus Y$ such that $a \neq b$. We define $\alpha : X \to X$ by

$$x\alpha = \begin{cases} a, & \text{if } x = b, \\ x, & \text{otherwise}. \end{cases}$$

We obtain that $\alpha|_Y$ is the identity mapping on $Y$, that is $\alpha \in PG_Y(X)$. Since $aa^{-1} = \{a, b\}$, we have $a \in C(\alpha)\alpha$ whence $C(\alpha)\alpha \not\subseteq Y$. This implies that $\alpha \not\in F_Y(X)$ and then $F_Y(X) \neq PG_Y(X)$.

Conversely, assume that $|X\setminus Y| \leq 1$. It is enough to show that $PG_Y(X) \subseteq F_Y(X)$. Let $\alpha \in PG_Y(X)$ and $x \in C(\alpha)$. Then we get $\alpha|_Y \in G(Y)$ and $x \in y\alpha^{-1}$ for some $y \in X$ and $|y\alpha^{-1}| \geq 2$. To verify that $y = x\alpha \in Y$, suppose that $y \not\in Y$. Since $Y\alpha = Y$, we have $x \not\in Y$. By the assumption, we conclude that $x = y$. This means that $y\alpha^{-1} = \{y\}$ which is a contradiction. Hence $y \in Y$ and so $C(\alpha)\alpha \subseteq Y$. It is clear that $\alpha|_Y$ is injective. Therefore $\alpha \in F_Y(X)$ whence $F_Y(X) = PG_Y(X)$.
Theorem 2.4. \( PG_Y(X) = S(X,Y) \) if and only if \( |Y| = 1 \).

Proof. Assume that \( |Y| > 1 \). Let \( a, b \in Y \) be such that \( a \neq b \) and define \( \alpha : X \to X \) by

\[
x\alpha = \begin{cases} a, & \text{if } x = b, \\ x, & \text{otherwise}. \end{cases}
\]

It is easy to verify that \( Y\alpha \subseteq Y \) and \( \alpha|_Y \) is not injective. Hence \( \alpha \in S(X,Y) \) and \( \alpha \notin PG_Y(X) \).

The converse follows from Remark 2.1.

Theorem 2.5. If \( PG_Y(X) \) is an inverse semigroup, then \( |X| \leq 1 \).

Proof. Suppose that \( |X \setminus Y| \geq 2 \). Then there exist \( a, b \in X \setminus Y \) such that \( a \neq b \). Choose \( c \in Y \) and define \( \alpha, \beta : X \to X \) by

\[
x\alpha = \begin{cases} c, & \text{if } x = b, \\ x, & \text{otherwise}, \end{cases}
\]

and

\[
x\beta = \begin{cases} a, & \text{if } x = b, \\ x, & \text{otherwise}. \end{cases}
\]

We note that \( \alpha|_Y \) and \( \beta|_Y \) are the identity mapping. Thus \( \alpha, \beta \in PG_Y(X) \).

Moreover, we obtain that \( \alpha = \alpha\beta\alpha, \beta = \beta\alpha\beta \) and \( \alpha^2 = \alpha \). Hence \( \alpha, \beta \in V(\alpha) \).

Consequently, \( PG_Y(X) \) is not an inverse semigroup.

Corollary 2.6. \( PG_Y(X) \) is an inverse semigroup if and only if \( |X| \leq 2 \) or \( Y = X \).

Proof. Assume that \( |X| > 2 \) and \( Y \neq X \). If \( |X \setminus Y| > 1 \), then we have \( PG_Y(X) \) is not an inverse semigroup by Theorem 2.5.

Suppose that \( |X \setminus Y| = 1 \). Let \( a, b \in Y \) be such that \( a \neq b \) and \( X \setminus Y = \{c\} \). Define \( \alpha, \beta : X \to X \) by

\[
x\alpha = \begin{cases} a, & \text{if } x = c, \\ x, & \text{otherwise}, \end{cases}
\]

and

\[
x\beta = \begin{cases} b, & \text{if } x = c, \\ x, & \text{otherwise}. \end{cases}
\]

Since \( \alpha|_Y, \beta|_Y \) are the identity mappings, we deduce \( \alpha, \beta \in PG_Y(X) \). Note that \( \alpha = \alpha\beta\alpha, \beta = \beta\alpha\beta \) and \( \alpha^2 = \alpha \) whence \( \alpha, \beta \in V(\alpha) \). Therefore \( PG_Y(X) \) is not an inverse semigroup.

From Remark 2.1 we obtain the converse.

Let \( P(X) \) be the partial transformation semigroup on \( X \). We note that \( T(X) \) is a subsemigroup of \( P(X) \). The following theorem shows that each full transformations semigroup can be embedded into \( PG_Y(X) \) for some set \( X \).
Theorem 2.7. Let $0 \notin X$. Then $P(X)$ is isomorphic to $PG_{(0)}(X \cup \{0\})$.

Proof. Let $\alpha \in P(X)$. Then $\text{dom}(\alpha) \subseteq X$. We let $\pi : X \cup \{0\} \to X \cup \{0\}$ be defined by

$$x\pi = \begin{cases} x\alpha, & \text{if } x \in \text{dom}(\alpha), \\ 0, & \text{otherwise.} \end{cases}$$

Obviously, $\pi_{(0)} \in G(\{0\})$ whence $\pi \in PG_{(0)}(X \cup \{0\})$. We claim that $\pi\beta = \pi\beta$ for each $\alpha, \beta \in P(X)$. Let $\alpha, \beta \in P(X)$ and $x \in X \cup \{0\}$.

Case 1. $x\pi \notin \text{dom}(\beta)$. If $x \in \text{dom}(\alpha)$ then $x\alpha \notin \text{dom}(\beta)$ which implies $x \notin \text{dom}(\alpha\beta)$. If $x \notin \text{dom}(\alpha)$ then $x \notin \text{dom}(\alpha\beta)$ since $\text{dom}(\alpha\beta) \subseteq \text{dom}(\alpha)$. Consequently, $x\pi\beta = 0 = x\alpha\beta$.

Case 2. $x\pi \in \text{dom}(\beta)$. Then $x\alpha = x\pi$ and we conclude that $x \in \text{dom}(\alpha\beta)$. Hence $x\pi\beta = x\alpha\beta = x\alpha\beta$.

These imply that $\pi\beta = \pi\beta$ for all $\alpha, \beta \in P(X)$. It follows that the mapping $\varphi : P(X) \to PG_{(0)}(X \cup \{0\})$ defined by $\alpha\varphi = \pi$ is a homomorphism.

To verify injectivity of $\varphi$, let $\alpha, \beta \in P(X)$ be such that $\pi = \beta$. Let $D = \{x \in X : x\pi \neq 0\}$. Obviously, $\text{dom}(\alpha) = D = \text{dom}(\beta)$. Moreover, we obtain that $x\alpha = x\beta$ for all $x \in D$ which implies $\alpha = \beta$. Finally, let $\beta \in PG_{(0)}(X \cup \{0\})$. Then define $\alpha \in P(X)$ by $x\alpha = x\beta$ for all $x \in \{x \in X : x\beta \neq 0\}$. Clearly, $\pi = \beta$.

Hence $P(X)$ is isomorphic to $PG_{(0)}(X \cup \{0\})$. □

Immediately, we obtain the following corollary.

Corollary 2.8. Let $0 \notin X$. Then $T(X)$ can be embedded into $PG_{(0)}(X \cup \{0\})$.

3 Regularity

Recall that an element $x$ in a semigroup $S$ is called left [right] regular if $x = yx^2$ $[x = x^2y]$ for some $y \in S$ and $x$ is completely regular if $x = xyx$ and $xy = yx$ for some $y \in S$. In this section, the left regularity, right regularity, and completely regularity of elements in $PG_Y(X)$ are studied.

Theorem 3.1. Let $\alpha \in PG_Y(X)$. Then $\alpha$ is a right regular element if and only if $\alpha|_{X\alpha}$ is injective.

Proof. Assume that $\alpha = \alpha^2\beta$ for some $\beta \in PG_Y(X)$. Let $x, y \in X\alpha$ be such that $x\alpha = y\alpha$. Thus $x = x'\alpha$ and $y = y'\alpha$ for some $x', y' \in X$. It follows that

$$x = x'\alpha = x'\alpha^2\beta = x\alpha\beta = y\alpha\beta = y'\alpha^2\beta = y'\alpha = y.$$

Hence $\alpha|_{X\alpha}$ is injective.

Suppose that $\alpha|_{X\alpha}$ is injective. We will construct $\beta \in PG_Y(X)$ satisfying $\alpha = \alpha^2\beta$. Let $x \in X\alpha^2$. Then by the assumption, we have $x'\alpha = x$ for a unique $x' \in X\alpha$. Define $\beta : X \to X$ by

$$x\beta = \begin{cases} x', & \text{if } x \in X\alpha^2, \\ x, & \text{otherwise.} \end{cases}$$
We conclude via $\alpha|_Y \in G(Y)$ that $Y = Y\alpha = Y\alpha^2$. To verify $Y\beta = Y$, let $x \in Y$. Since $x \in Y = Y\alpha$ and by the uniqueness of $x'$, we have $x' = y\alpha$ for some $y \in Y$ whence $x\beta = x$. On the other hand, let $y \in Y$. Since $Y = Y\alpha$, we conclude that $y\alpha \in Y\alpha^2$ and hence $(y\alpha)\beta = (y\alpha)' = y$. Thus $Y = Y\beta$. Let $x,y \in Y$ be such that $x\beta = y\beta$. Then $x,y \in Y = Y\alpha = Y\alpha^2$. By the uniqueness of $x'$ and $y'$, we obtain $\beta|_Y$ is injective. Hence $\beta \in PG_Y(X)$. Finally, let $x \in X$. Since $(x\alpha)\alpha = x\alpha^2$, we have $(x\alpha^2)' = x\alpha$. That is $x\alpha^2\beta = (x\alpha^2)' = x\alpha$.

**Theorem 3.2.** Let $\alpha \in PG_Y(X)$. Then $\alpha$ is a left regular element if and only if $X\alpha = X\alpha^2$.

**Proof.** Assume that $\alpha = \beta\alpha^2$ for some $\beta \in PG_Y(X)$. Clearly, $X\alpha^2 \subseteq X\alpha$. Let $x \in X\alpha$. Then $x = x'\alpha$ for some $x' \in X$. Hence $x = x'\alpha = x'\beta\alpha^2 \in X\alpha^2$ which implies that $X\alpha = X\alpha^2$.

Suppose that $X\alpha = X\alpha^2$. We note from $\alpha|_Y \in G(Y)$ that for each $x \in Y$, there exists a unique $x' \in Y$ such that $x'\alpha = x$ whence $x'\alpha^2 = x\alpha$. Let $x \in X\setminus Y$. Then by the assumption, we choose $x' \in X$ such that $x'\alpha^2 = x\alpha$. We construct $\beta \in PG_Y(X)$ as follows: $x\beta = x'$ for each $x \in X$. To verify $Y = Y\beta$, let $y \in Y$. By the definition of $x'$, we deduce $x\beta = x' \in Y$. Let $y \in Y$, then $y\alpha = x$ for some $x \in Y$ since $Y\alpha = Y$. By the uniqueness of $x'$, we conclude that $x\beta = x' = y$ which implies that $Y = Y\beta$. Assume that $x\beta = y\beta$ for some $x,y \in Y$. Then $x' = y'$ which implies $x = x'\alpha = y'\alpha = y$. Therefore $\beta|_Y \in G(Y)$. Let $x \in X$. We conclude from the definition of $x'$ that $x\beta\alpha^2 = x'\alpha^2 = x\alpha$.

**Theorem 3.3.** Let $\alpha \in PG_Y(X)$. Then $\alpha$ is a completely regular element if and only if $|\pi(x)\cap X\alpha| = 1$ for all $x \in \pi(\alpha)$.

**Proof.** Assume that $\alpha = \alpha\beta\alpha$ and $\alpha\beta = \beta\alpha$ for some $\beta \in PG_Y(X)$. Let $P \in \pi(\alpha)$. Then $P = x\alpha^{-1}$ for some $x \in X\alpha$. Choose $x' \in P$, we conclude that $x = x'\alpha = x'\alpha\beta\alpha = x\beta\alpha$ which implies $x\beta \in P$. Since $x\beta = x'\alpha\beta = x'\beta\alpha \in X\alpha$, we obtain $P\cap X\alpha \neq \emptyset$. To verify $|P\cap X\alpha| = 1$, suppose that $a,b \in P\cap X\alpha$. Then $a = a'\alpha, b = b'\alpha$ for some $a', b' \in X$ and $aa = ba$. It follows that $a = a'\alpha = a'\alpha\beta\alpha = a\beta\alpha = ba\beta = b\beta\alpha = b'\alpha\beta\alpha = b'\alpha = b$.

Assume that for each $P \in \pi(\alpha), |P\cap X\alpha| = 1$. Let $P \in \pi(\alpha)$. By assumption, we denote $x_P \in P\cap X\alpha$. Let $P' = x_P\alpha^{-1}$. Then $y\alpha = x_P$ for all $y \in P'$. In particular, $x_P\alpha = x_P$. Define $\beta : X \rightarrow X$ by

$$x\beta = x_P \text{ if } x \in P \text{ for some } P \in \pi(\alpha).$$

Since $\pi(\alpha)$ is a partition of $X$, $\beta$ is well-defined. To show that $Y\beta = Y$, let $x \in Y$. Then $x \in P$ for some $P \in \pi(\alpha)$. We note from $Y = Y\alpha$ that $x \in P\cap X\alpha$ whence $x = x_P$. Since $Y = Y\alpha$, we have $x_P = y\alpha$ for some $y \in Y$. This means that $y\alpha \in x_P\alpha^{-1} = P'$. Thus $y\alpha \in P'\cap X\alpha$ which implies $y\alpha = x_P'$. It follows that $x\beta = x_P' = x_P \in Y\alpha = Y$. Hence $Y\beta \subseteq Y$. Let $y \in Y$. Then $y = y'\alpha$ for some $y' \in Y$ since $Y = Y\alpha$. From $\pi(\alpha)$ is a partition of $X$, we obtain that
$y'\alpha^2 \in P$ for some $P \in \pi(\alpha)$. Since $y'\alpha^2 \in P \cap X\alpha$, we have $y'\alpha^2 = x_P$. This implies that $y'\alpha \in x_P\alpha^{-1} = P'$ whence $y'\alpha \in P' \cap X\alpha$. Thus $y'\alpha = x_{P'}$. We conclude that $y'\alpha^2\beta = x_{P'} = y'\alpha = y$ then we get $Y\beta = Y$. Next, let $x, y \in Y$ be such that $x\beta = y\beta$. By the definition of $\beta$, we have that $x \in P$, $y \in Q$ for some $P, Q \in \pi(\alpha)$ and $x\beta = x_{P'}, y\beta = x_Q$ where $x_P\alpha = x_P$ and $x_Q\alpha = x_Q$. Thus $x_P = x_{P'}\alpha = x\beta\alpha = y\beta\alpha = x_Q\alpha = x_Q$ whence $P \cap Q \neq \emptyset$. Since $\pi(\alpha)$ is a partition of $X$, we have $P = Q$. We note that $x, y \in Y = Y\alpha$ which implies $x = x_P = y$. Hence $\beta|_Y$ is injective and so $\beta \in PG_Y(X)$.

To verify that $\alpha = \alpha\beta\alpha$ and $\alpha\beta = \beta\alpha$, let $x \in X$. We note that $x\alpha \in P$ for a unique $P \in \pi(\alpha)$. Then we obtain that $x\alpha = x_P$. Hence $x\alpha\beta\alpha = x_P\alpha = x_P = x\alpha$. Since $x \in x_P\alpha^{-1} = P'$, we conclude that $x\beta\alpha = x_{P'}\alpha = x_{P'} = x\alpha\beta$. Therefore, $\alpha$ is a completely regular element.

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