Regularity of Full Order-Preserving Transformation Semigroups on Some Dictionary Posets

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Abstract: For a poset $X$, let $OT(X)$ be the full order-preserving transformation semigroup on $X$. The following results are known. If $X$ is any nonempty subset of $\mathbb{Z}$, then $OT(X)$ is a regular semigroup, that is, for every $\alpha \in OT(X)$, $\alpha = \alpha \beta \alpha$ for some $\beta \in OT(X)$. If $X$ is an interval in $\mathbb{R}$, then $OT(X)$ is regular if and only if $X$ is closed and bounded. We deal with the regularity of $OT(A \times A, \leq_d)$ where $\phi \neq A \subseteq \mathbb{Z}$ and $\leq_d$ is the dictionary partial order on $A \times A$. We have that if $A$ is infinite, then the chain $(A \times A, \leq_d)$ is neither order-isomorphic to a subset of $\mathbb{Z}$ nor order-isomorphic to an interval in $\mathbb{R}$. Our purpose is to show that $OT(A \times A, \leq_d)$ is regular if and only if $A$ is finite.

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1 Introduction

An element $a$ of a semigroup $S$ is called regular if $a = aba$ for some $b \in S$, and $S$ is called a regular semigroup if every element of $S$ is regular.

For a nonempty set $X$, let $T(X)$ be the full transformation semigroup on $X$, that is, $T(X)$ is the semigroup, under composition, of all mappings $\alpha : X \rightarrow X$. The image of $x$ under $\alpha \in T(X)$ is written by $x\alpha$, and the range (image) of $\alpha \in T(X)$ is denoted by $\text{ran } \alpha$. It is well-known that $T(X)$ is a regular semigroup ([1], page 4 or [2], page 63).

A mapping $\varphi$ from a poset $X$ into a poset $Y$ is said to be order-preserving if

$$\forall x, x' \in X, \ x \leq x' \text{ in } X \implies x\varphi \leq x'\varphi \text{ in } Y.$$  

The posets $X$ and $Y$ are said to be order-isomorphic if there is an order-preserving bijection $\varphi$ from $X$ onto $Y$ such that $\varphi^{-1} : Y \rightarrow X$ is order-preserving.

If $X$ is a poset, let $OT(X)$ be the subsemigroup of $T(X)$ consisting of all order-preserving mappings, that is,

$$OT(X) = \{ \alpha \in T(X) \mid \alpha \text{ is order-preserving } \}.$$
It is known from [1, page 203] that \( OT(X) \) is regular if \( X \) is a finite chain. In 2000, Y. Kemprasit and T. Changphas [3] extended this result to any chain which is order-isomorphic to a subset of \( \mathbb{Z} \), the set of integers with their natural order. It was also proved in [3] that if \( X \) is an interval in \( \mathbb{R} \), the set of real numbers with usual order, then \( OT(X) \) is regular if and only if \( X \) is closed and bounded. In [4], the authors generalized full order-preserving transformation semigroups by using sandwich multiplication and investigated their regularity and also provided some isomorphism theorems.

The results in [3] motivate us to consider the regularity of \( OT(\mathcal{A} \times \mathcal{A}, \leq_d) \) where \( \phi \neq \mathcal{A} \subseteq \mathbb{Z} \) and \( \leq_d \) is the dictionary partial order, that is,

\[
(a, b) \leq_d (c, d) \text{ if and only if } (i) \quad a < c \text{ or } (ii) \quad a = c \text{ and } b \leq d.
\]

Then \( (\mathcal{A} \times \mathcal{A}, \leq_d) \) is a chain. If \( \mathcal{A} \) is finite, then \( (\mathcal{A} \times \mathcal{A}, \leq_d) \) is a finite chain, and hence \( OT(\mathcal{A} \times \mathcal{A}, \leq_d) \) is a regular semigroup. If \( \mathcal{A} \) is infinite, then \( \mathcal{A} \times \mathcal{A} \) is countably infinite, so \( (\mathcal{A} \times \mathcal{A}, \leq_d) \) is not order-isomorphic to any interval in \( \mathbb{R} \). It will be shown that for an infinite subset \( \mathcal{A} \) of \( \mathbb{Z} \), \( (\mathcal{A} \times \mathcal{A}, \leq_d) \) is not order-isomorphic to any subset of \( \mathbb{Z} \). Our main purpose is to show that for any \( \phi \neq \mathcal{A} \subseteq \mathbb{Z} \), \( OT(\mathcal{A} \times \mathcal{A}, \leq_d) \) is regular if and only if \( \mathcal{A} \) is finite.

\section{Main Results}

Let \( \mathbb{Z}^+ \) and \( \mathbb{Z}^- \) denote the set of positive integers and the set of negative integers, respectively. If \( \mathcal{A} \subseteq \mathbb{Z} \) is infinite, then \( \mathcal{A} \) has one of following properties:

(i) \( \mathcal{A} \) is bounded below but not bounded above.
(ii) \( \mathcal{A} \) is bounded above but not bounded below.
(iii) \( \mathcal{A} \) is neither bounded below nor bounded above.

Then (i), (ii) and (iii) imply respectively that

(i) \( \mathcal{A} = \{ a_i \mid i \in \mathbb{Z}^+ \} \) where \( a_1 < a_2 < a_3 < \ldots \),
(ii) \( \mathcal{A} = \{ a_i \mid i \in \mathbb{Z}^- \} \) where \( a_{-1} > a_{-2} > a_{-3} > \ldots \) and
(iii) \( \mathcal{A} = \{ a_i \mid i \in \mathbb{Z} \} \) where \( \ldots < a_{-2} < a_{-1} < a_0 < a_1 < a_2 < \ldots \).

We first show that for an infinite subset \( \mathcal{A} \) of \( \mathbb{Z} \), the chain \( (\mathcal{A} \times \mathcal{A}, \leq_d) \) is not order-isomorphic to a subset of \( \mathbb{Z} \).

For convenience, if \( S_1 \) and \( S_2 \) are subsets of a chain, we write \( S_1 < S_2 \) if \( x < y \) for all \( x \in S_1 \) and \( y \in S_2 \).

\begin{proposition}
For an infinite subset \( \mathcal{A} \) of \( \mathbb{Z} \), \( (\mathcal{A} \times \mathcal{A}, \leq_d) \) is not order-isomorphic to a subset of \( \mathbb{Z} \).
\end{proposition}

\begin{proof}
First, we recall that for a sequence \( (x_n) \) in \( S \subseteq \mathbb{Z} \), if \( x_1 < x_2 < x_3 < \ldots \), then \( \{ x_n \mid n \in \mathbb{Z}^+ \} \) has no upper bound in \( S \). Also, if \( x_1 > x_2 > x_3 > \ldots \), then \( \{ x_n \mid n \in \mathbb{Z}^+ \} \) has no lower bound in \( S \).

\textbf{Case 1}: \( \mathcal{A} = \{ a_i \mid i \in \mathbb{Z}^+ \} \) where \( a_1 < a_2 < \ldots \) or \( \mathcal{A} = \{ a_i \mid i \in \mathbb{Z} \} \)
where \( \ldots < a_{-1} < a_0 < a_1 < \ldots \). Then \((a_2, a_1)\) is an upper bound of \(\{ (a_1, a_i) \mid i \in \mathbb{Z}^+ \}\) and \((a_1, a_1) <_d (a_1, a_2) <_d (a_1, a_3) <_d \ldots \). Hence we have that \((A \times A, \leq_d)\) is not order-isomorphic to a subset of \(\mathbb{Z}\).

**Case 2:** \(A = \{ a_i \mid i \in \mathbb{Z}^- \}\) where \(a_{-1} > a_{-2} > \ldots\). Then \((a_{-2}, a_{-1})\) is a lower bound of \(\{ (a_{-1}, a_i) \mid i \in \mathbb{Z}^- \}\) and \((a_{-1}, a_{-1}) >_d (a_{-1}, a_{-2}) >_d (a_{-1}, a_{-3}) >_d \ldots\). It follows that \((A \times A, \leq_d)\) is not order-isomorphic to a subset of \(\mathbb{Z}\). \(\square\)

To obtain the main result, the following fact is needed.

**Lemma 2.2.** If \(\alpha\) and \(\beta\) are elements of \(T(X)\) such that \(\alpha = \alpha \beta \alpha\), then \(\text{ran}(\beta \alpha) = \text{ran} \alpha\) and \(x \beta \alpha = x\) for all \(x \in \text{ran} \alpha\).

**Proof.** Since \(\text{ran} \alpha = \text{ran}(\alpha \beta \alpha) \subseteq \text{ran}(\beta \alpha) \subseteq \text{ran} \alpha\), we have \(\text{ran}(\beta \alpha) = \text{ran} \alpha\). If \(x \in X\), then \(x \alpha = x \alpha \beta \alpha = (x \alpha) \beta \alpha\). This implies that \(x \beta \alpha = x\) for all \(x \in \text{ran} \alpha\). \(\square\)

**Theorem 2.3.** Let \(\phi \neq A \subseteq \mathbb{Z}\). Then \(OT(A \times A, \leq_d)\) is a regular semigroup if and only if \(A\) is finite.

**Proof.** Suppose that \(A\) is infinite. Let \(c \in A\) be a fixed element and define \(\alpha : A \times A \to A \times A\) by

\[(\{x\} \times A) \alpha = \{(c, x)\}\] for all \(x \in A\),

that is,

\[(x, y) \alpha = (c, x)\] for all \(x, y \in A\).

Then we have

\[\text{ran} \alpha = \{c\} \times A.\] (2.1)

Since \(\{x\} \times A \not< \{y\} \times A\) and \((c, x) <_d (c, y)\) for all \(x, y \in A\) with \(x < y\), we deduce that \(\alpha \in OT(A \times A, \leq_d)\). To show that \(\alpha\) is not regular in \(OT(A \times A, \leq_d)\), suppose on the contrary that \(\alpha = \alpha \beta \alpha\) for some \(\beta \in OT(A \times A, \leq_d)\). By (1) and Lemma 2.2,

\[(c, x) \beta \alpha = (c, x)\] for all \(x \in A\). (2.2)

**Case 1:** \(A = \{ a_i \mid i \in \mathbb{Z}^+ \}\) where \(a_1 < a_2 < \ldots\) or \(A = \{ a_i \mid i \in \mathbb{Z} \}\) where \(\ldots < a_{-1} < a_0 < a_1 < \ldots\). Then \(e < f\) for some \(e \in A\). Hence \((c, x) <_d (c, e)\) for all \(x \in A\). This implies that \((c, x) \beta \alpha \leq_d (c, e) \beta \alpha\) for all \(x \in A\). By (2),

\[(c, x) \leq_d (c, e) \beta \alpha\] for all \(x \in A\).

Since \((c, e) \beta \alpha \in \text{ran} \alpha\), by (1), \((c, e) \beta \alpha = (c, f)\) for some \(f \in A\). Consequently,

\[(c, x) \leq_d (c, f)\] for all \(x \in A\),
so $x \leq f$ for all $x \in A$. This is a contradiction since $A$ has no maximum.

Case 2 : $A = \{ a_i | i \in \mathbb{Z}^-\}$ where $a_{i-1} > a_{i-2} > \ldots$. Then $r < c$ for some $r \in A$. Thus $(r, r) \leq_d (c, x)$ for all $x \in A$ which implies that $(r, r) \beta \alpha \leq_d (c, x) \beta \alpha$ for all $x \in A$. Therefore we have from (2) that

$$(r, r) \beta \alpha \leq_d (c, x) \quad \text{for all } x \in A.$$

But $(r, r) \beta \alpha \in \text{ran } \alpha$, so by (1), $(r, r) \beta \alpha = (c, s)$ for some $s \in A$. Hence

$$(c, s) \leq_d (c, x) \quad \text{for all } x \in A.$$

This implies that $s \leq x$ for all $x \in A$ which is a contradiction since $A$ has no minimum.

This proves that if $OT(A \times A, \leq_d)$ is regular, then $A$ is finite. The converse holds because $(A \times A, \leq_d)$ is a finite chain if $A$ is finite.

Remark 2.4. (1) The given proof of Proposition 2.1 uses the basic fact of $\mathbb{Z}$. As mentioned previously, if a chain $X$ is order-isomorphic to a subset of $\mathbb{Z}$, then $OT(X)$ is a regular semigroup. Then Proposition 2.1 can be referred as a corollary of Theorem 2.3.

(2) From the proof of Theorem 2.3, we define $\alpha \in OT(A \times A, \leq_d)$ depending on a given $c \in A$ where $A$ is an infinite subset of $\mathbb{Z}$. Then $\alpha$ can be written as $\alpha_c$. Observe that $\alpha_{c_1} \neq \alpha_{c_2}$ if $c_1 \neq c_2$ in $A$. Since each $\alpha_c$ is a nonregular element of $OT(A \times A, \leq_d)$, we deduce that $OT(A \times A, \leq_d)$ has an infinite number of nonregular elements. Since every constant mapping from $A \times A$ into $A \times A$ is a regular element of $OT(A \times A, \leq_d)$, it follows that $OT(A \times A, \leq_d)$ also contains an infinite number of regular elements.

References


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