A Set-Valued Fixed Point Theorem for Nonexpansive Mappings in Partially Ordered Ultrametric and Non-Archimedean Normed Spaces

Hamid Mamghaderi and Hashem Parvaneh Masiha

Faculty of Mathematics, K. N. Toosi University of Technology, P.O. Box 16315-1618, Tehran, Iran
e-mail: h.mamghaderi@mail.kntu.ac.ir (H. Mamghaderi); m.asiha@kntu.ac.ir (H. P. Masiha)

Abstract In this paper, a set-valued fixed point theorem for a class of generalized nonexpansive mappings on partially ordered ultrametric spaces and partially ordered non-Archimedean normed spaces is proved.

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1. Introduction and Preliminaries

We begin by recalling definition of an ultrametric space. The classical definition goes back over fifty years [1]. A metric space \((X, d)\) is called an ultrametric space if the metric \(d\) satisfies the strong triangle inequality; namely for all \(x, y, z \in X\) \(d(x, y) \leq \max\{d(x, z), d(y, z)\}\). A non-Archimedean normed space [1] \((X, \|\cdot\|)\) is said to be spherically complete if every shrinking collection of balls in \(X\) has a nonempty intersection [1]. In 1993, Petals proved a fixed point theorem on non-Archimedean normed space using a contractive condition [2]. This result is extended by Kubiaczyk (1996) from single valued to set-valued contractive mapping [3]. Also for nonexpansive set-valued mappings, some fixed point theorems are proved.

In this paper, we investigate the existence of a fixed point for set-valued nonexpansive mappings in partially ordered ultrametric spaces and non-Archimedean normed spaces and we also give more constructive proof for our theorem and obtain a useful conclusion. It would be interesting to study the conclusions that obtained by Xu et al. [4] and Mursaleen et al. [5] in 2016 and compare with our results. Therefore, we can find out the importance of our results and get such results for \(p\)-adic fuzzy non-Archimedean numbers and the applications of Schwarz lemma involving the boundary fixed point in non-Archimedean complex analysis.

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2. MAIN RESULTS

In the following proposition, we prove that if \((X, d)\) is an ultrametric space, then the Hausdorff metric \(H [6] \) on \(CB(X)\) is also an ultrametric. Where \(H(A, B) = \max \{\delta(A, B), \delta(B, A)\}\), \(\delta(A, B) = \sup_{a \in A} d(a, B)\) and \(d(a, B) = \inf_{b \in B} d(a, b)\).

**Proposition 2.1.** Let \((X, d)\) be an ultrametric space. Then the metric \(H\) is an ultrametric on \(CB(X)\).

**Proof.** Let \(a \in A\). Then for each \(b \in B\), \(d(a, B) \leq d(a, b)\), which implies that \(d(a, B) \leq \max\{d(a, c), d(c, b)\}\) for all \(b \in B\), \(c \in C\). Because \(b \in B\) was arbitrary, for each \(\varepsilon > 0\) we can choose \(b \in B\) such that \(d(c, b) \leq d(c, B) + \varepsilon\) and hence
\[
\begin{align*}
d(a, B) & \leq \max\{d(a, c), d(c, B) + \varepsilon\} \quad (c \in C), \\
d(a, B) & \leq \max\{d(a, c), \delta(C, B) + \varepsilon\} \quad (c \in C).
\end{align*}
\]
Similarly, because \(c \in C\) was arbitrary, we can choose \(c \in C\) such that \(d(a, c) \leq d(a, C) + \varepsilon\) and hence
\[
\begin{align*}
d(a, B) & \leq \max\{d(a, C) + \varepsilon, \delta(C, B) + \varepsilon\}, \\
d(a, B) & \leq \max\{\delta(A, C) + \varepsilon, \delta(C, B) + \varepsilon\}, \\
d(a, B) & \leq \max\{\delta(A, C), \delta(C, B)\} + \varepsilon.
\end{align*}
\]
Because \(a \in A\) and \(\varepsilon > 0\) were arbitrary, we conclude that \(\delta(A, B) \leq \max\{\delta(A, C), \delta(C, B)\}\). Therefore, \(H(A, B) \leq \max\{H(A, C), H(C, B)\}\).

**Definition 2.2.** Let \((X, \preceq)\) be a partially ordered set and suppose that there exists an ultrametric \(d\) in \(X\) such that \((X, d)\) is an ultrametric space, and \(T : X \to CB(X)\) is a mapping. A closed ball \(B(x, r)\) is said to be a partially \(T\)-invariant ball if for any \(u \in B(x, r)\) comparable to \(x\), there exists \(v \in Tu\) such that \(d(x, v) \leq r\). Also, the closed ball \(B(x, r)\) is called minimal partially \(T\)-invariant ball if \(B(x, r)\) is partially \(T\)-invariant and \(d(u, Tu) = r\) for any \(u \in B(x, r)\) comparable to \(x\).

**Theorem 2.3.** Let \((X, \preceq)\) be a partially ordered set and suppose that there exists a metric \(d\) in \(X\) such that \((X, d)\) is an ultrametric space and a mapping \(T : X \to CB(X)\). Suppose also that the following properties are satisfied:

1. (C1) \(H(Tx, Ty) \leq d(x, y)\) for every comparable \(x, y \in X\).
2. (C2) If \(d(x, y) < 1\) for some \(x \in X\) and some \(y \in Tx\), then \(x \preceq y\);
3. (C3) There exist \(x_0 \in X\) and \(x_1 \in Tx_0\) such that \(d(x_0, x_1) < 1\);
4. (C4) If \(\{x_n\}\) is a non-decreasing sequence in \(X\) and \(\{B(x_n, r_n)\}\) is a descending sequence of closed balls in \(X\), then there exists a subsequence \(\{x_{n_k}\}\) of \(\{x_n\}\) such that \(\{x_n\}\) has an upper bound \(z \in \bigcap_{k=1}^{\infty} B(x_{n_k}, r_{n_k})\), and set \(X_T = \{x \in X : \) there exists \(x' \in Tx\) such that \(d(x, x') < 1\}\). Then for any \(x \in X_T\), the ball \(B(x, d(x, Tx))\) contains either a fixed point of \(T\) or a minimal partially \(T\)-invariant closed ball.

**Proof.** Let \(z \in X\), put \(r = d(z, Tz)\) and pick \(u \in B(z, r)\) comparable to \(z\). Then
\[
d(z, Tu) \leq \{d(z, Tz), H(Tu, Tz)\} \leq \max\{d(z, Tz), d(u, z)\} = d(z, Tz).
\]
Thus, there exists $v \in Tu$ such that $d(z,v) \leq d(z,Tz)$. Therefore, every ball in $X$ of the form $B(z,d(z,az))$ is partially $T$-invariant. Now, let $x_0 \in X_T$ and put $x_1 = x_0$, $r_1 = d(x_1,Tx_1)$ and $\lambda_1 = \inf \{d(x,Tx) : x \in B(x_1,r_1) \cap Tx_1, x_1 \leq x \}$. If $x \in B(x_1,r_1)$ and $x_1 \leq x$, then
\[
d(x,Tx) \leq \max \{d(x,Tx_1), H(Tx_1,Tx)\} \\
\leq \max \{d(x,Tx_1), d(x_1,x)\} \\
\leq \max \{d(x_1,x), d(x_1,Tx_1), d(x_1,x)\} \leq d(x_1,Tx_1).
\]
Hence $\lambda_1 \leq r_1$. Suppose $\{\epsilon_n\}$ is a sequence of positive numbers such that $\lim_{n \to \infty} \epsilon_n = 0$. If $r_1 = \lambda_1$, then the proof is completed because in this case either $r_1 = \lambda_1 = 0$ therefore $x_1$ is a fixed point of $T$ in $B(x_1,r_1)$ or $B(x_1,r_1)$ is minimal partially $T$-invariant. Otherwise, if $x \in B(x_1,r_1) \cap Tx_1$ and $x_1 \leq x$, then
\[
d(x,Tx) \leq \max \{d(x,Tx_1), H(Tx_1,Tx)\} \\
\leq H(Tx_1,Tx_1) < d(x_1,x) \leq d(x_1,Tx_1).
\]
Hence $\lambda_1 < r_1 < 1$. Choose an $x_2 \in B(x_1,r_1)$ such that $x_1 \leq x_2$, $x_2 \in Tx_1$ and $r_2 = d(x_2,Tx_2) < \min \{\lambda_1 + \varepsilon_1, r_1\}$. Let $\lambda_2 = \inf \{d(x,Tx) : x \in B(x_2,r_2) : x_2 \leq x, x \in Tx_2\}$. Choose $x_2 \in B(x_1,r_1)$ such that there exists a path in $\tilde{G}$ between $x_1$ and $x_2$ and $\varepsilon_2 = d(x_2,Tx_2) < \min \{\lambda_1, \lambda_2 + \varepsilon_1\}$. With the same argument, if $r_2 = \lambda_2$, then $B(x_2,r_2)$ is minimal partially $T$-invariant. Otherwise, we have $\lambda_2 < r_2$, and choose an $x_3 \in B(x_2,r_2)$ such that $x_2 \leq x_3$, $x_3 \in Tx_3$ and $r_3 = d(x_3,Tx_3) < \min \{\lambda_2, r_2 + \varepsilon_2\}$. Having defined $x_n \in X$, let
\[
\lambda_n = \inf \{d(x,Tx) : x \in B(x_n,r_n) \cap Tx_n, x_n \leq x\}.
\]
Then we have $\lambda_n \leq r_n$, and choose an $x_{n+1} \in B(x_n,r_n)$ such that $x_n \leq x_{n+1}$, $x_{n+1} \in Tx_{n+1}$ and $r_{n+1} = d(x_{n+1},Tx_{n+1}) < \min \{r_n, \lambda_n + \varepsilon_n\}$. The sequence $\{x_n\}$ is non-decreasing and $\{B(x_n,r_n)\}$ is a descending sequence of non-trivial closed balls. Thus by assumption, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $z \in \bigcap_{k=1}^{\infty} B(x_{n_k},r_{n_k})$ such that $x_{n_k} \leq z$ for all $k \geq 1$. Since $\{r_n\}$ is non-increasing, it follows that $r = \lim_{n \to \infty} r_n$ exists. Also, $\{\lambda_n\}$ is non-decreasing and bounded above and so $\lambda := \lim_{n \to \infty} \lambda_n$ also exists, too. Hence $d(z,Tz) \leq \max \{d(z,x_{n_k}), d(x_{n_k},Tz)\} \leq r_{n_k}$, for all $n \geq 1$. Moreover, $\lambda_{n_k} \leq d(z,Tz) \leq r \leq r_{n_k+1} \leq \lambda_{n_k} + \varepsilon_{n_k}$ for all $k \geq 1$. Letting $k \to \infty$, we see that $d(z,Tz) = \lambda = r$. Set $a = \inf \{d(x,Tx) : x \in B(z,d(z,Tz)) \cap Tx,z \leq x\}$. Since $z \in B(x_n,r_n)$ and $x_n \leq z$ for all $n \geq 1$, it follows that $d(x,Tx) \leq d(z,Tz) \leq r_n$ for all $x \in B(z,d(z,Tz))$. Hence $a \leq r_n$ for all $n \geq 1$. Moreover, since every closed ball in $X$ is partially $T$-invariant, we have $\lambda_n \leq a$ for all $n \geq 1$. Thus,
\[
a = \inf \{d(x,Tx) : x \in B(z,d(z,Tz)) \cap Tx,z \leq x\} = r = d(z,Tz).
\]
If $r = 0$, then $z$ is a fixed point of $T$ in $B(x,d(x,Tx))$, if not, then the closed ball $B(z,d(z,Tz))$ is minimal partially $T$-invariant. Therefore the proof is completed.

**Corollary 2.4.** Theorem 2.3 remains valid if the partially ordered ultrametric space $(X,d)$ is replaced by a partially ordered non-Archimedean normed space over a non-Archimedean valued field $K$.  

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REFERENCES


