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Numerical Solution of Third Order Singularly Perturbed Boundary Value Problems Using Exponential Quartic Spline

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Abstract : In this paper, we develop a generalized scheme based on exponential quartic spline for solving third-order self adjoint singularly perturbed boundary value problems(SPBVP). The method is proved to be second-order convergent. We have shown that the proposed method is better than existing spline methods. Numerical illustrations are carried out to confirm the applicability and efficiency of the method.

Keywords : exponential quartic spline; third order SPBVP; boundary layers; convergence analysis.
2010 Mathematics Subject Classification : 65L10.

1 Introduction

Third-order self-adjoint singularly perturbed boundary value problem (SP-BVP) of the form:

$$\left.\begin{array}{c} Lz(t) \equiv -\epsilon z'''(t) + p(t)z(t) = q(t), \quad p(t) \geq 0, \quad t \in [a,b], \\ z(a) = \alpha_0, \quad z(b) = \alpha_1, \quad z'(a) = \alpha_2, \end{array}\right\} \tag{1.1}$$

where $\alpha_0, \alpha_1, \alpha_2$ are constants, ϵ is a small positive parameter $(0 < \epsilon < 1)$ and $p(t),\ q(t)$ are sufficiently smooth functions is considered. Various methods for approximating the solution of (1.1) have been developed such as [1–5].

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The problems of type (1.1) occurs in engineering and applied mathematics such as fluid mechanics, chemical reactor theory, convection-diffusion processes etc. There are many applications which include boundary layer problems, the modelling of steady and unsteady viscous flow problems with large Reynolds number and convective-heat transport problems with large Peclet number. Since the past two decades, many researchers are working in this field but they mostly confined to second-order problems [6–9]. Few authors have also discussed higher order problems such as [1–5].

Details of SPPs are given in O'Malley [10], Miller et al. [11], Roos et al. [12]. Valamarthi and Ramanujam [4] used boundary value technique for the solution of third-order SP ODEs. The analytical solution of third-order nonlinear SPPs were derived by Zaho [5]. Al-Said et al. [13] and Noor et al. [14] used cubic splines and quartic splines respectively for solving third-order obstacle problems. Akram [1] used quartic splines while Saini and Mishra [3] used quartic B-splines to solve third-order self-adjoint SPBVPs.

This paper proposes a new approach based on exponential quartic spline. Here, we construct spline that has an exponential and polynomial part . The advantage is that it is not only provide continuous approximations to $z(t)$, but also for its derivatives at every point of the range of integration. The proposed exponential spline function has the form:

$$T_4 = span\left\{1,\ t,\ t^2,\ e^{\tau t},\ e^{-\tau t}\right\},$$

where τ can be real or pure imaginary parmeter. Thus in each subinterval $t_i \leq t \leq t_{i+1}$, we have

$$span\left\{1,\ t,\ t^2,\ e^{\tau t},\ e^{-\tau t}\right\},$$

or

$$span\left\{1,\ t,\ t^2,\ t^3,\ t^4,\right\},\ when\ \tau \to 0.$$

We organised the paper into six sections. In Section 2, we describe exponential quartic spline for solving (1.1). In Section 3, we describe the method. Truncation error is carried out in section 4. Section 5 outlines the convergence analysis of exponential quartic spline. Finally, we concluded the numerical results of the proposed method alongwith comparison in Section 6.

2 Exponential Quartic Spline

To obtain the spline approximation of the third-order SPBVP (1.1), we divided the interval $[a, b]$ into n equal subintervals using the grid $t_i = a + ih,\ 0 \leq i \leq n$, where $h = (b - a)/n$. The method is developed by using the exponential quartic spline of the form:

$$E_i(t) \quad = \quad a_{0i}e^{\tau(t-t_i)} + a_{1i}e^{-\tau(t-t_i)} + a_{2i}(t - t_i)^2 + a_{3i}(t - t_i) + a_{4i}, \tag{2.1}$$

where $a_{0i}, a_{1i}, a_{2i}, a_{3i}, a_{4i}$ are real finite constants and τ is a free parameter which will be used to raise the accuracy of the method. If $\tau \to 0$, then $E_i(t)$ reduces to

quartic polynomial spline.
To obtain the coefficients of equation (2.1) in terms of $z_i,\ M_i,\ D_i$ and T_i, we define

$$\left.\begin{array}{c} E_i(t_i) = z_i, \quad E_i'(t_i) = D_i, \\ \\ E_i''(t_i) = M_i, \quad E_i'''(t_i) = T_i. \end{array}\right\} \tag{2.2}$$

We obtain the following expressions via simple calculation:

$$\begin{aligned}
a_{0i} &= \frac{T_{i+1} - T_i e^{-\theta}}{\tau^3(e^{\theta} - e^{-\theta})}, \\
a_{1i} &= \frac{T_{i+1} - T_i e^{\theta}}{\tau^3(e^{\theta} - e^{-\theta})}, \\
a_{2i} &= \frac{M_i - a_{0i}\tau^2 - a_{1i}\tau^2}{2}, \\
a_{3i} &= \frac{(z_{i+1} - z_i) - a_{0i}(e^{\theta} - \frac{\theta^2}{2} - 1) - a_{1i}(e^{-\theta} - \frac{\theta^2}{2} - 1) - \frac{h^2 M_i}{2}}{h}, \\
a_{4i} &= z_i - a_{0i} - a_{1i}, \quad \theta = \tau h \quad and \quad 0 \le i \le n-1.
\end{aligned}$$

Using the continuity of first and second derivatives at (t_i, z_i), that is, $E_{i-1}^{(k)}(t_i) = E_i^{(k)}(t_i);\ k = 1, 2$ we obtain the following result for $1 \le i \le n-1$:

$$\frac{M_i + M_{i+1}}{2} = \frac{(z_{i+1} - 2z_i + z_{i-1})}{h^2} + h(AT_{i+1} + BT_i + CT_{i-1}), \tag{2.3}$$

$$\frac{M_i - M_{i-1}}{2} = \frac{(z_{i+1} - 2z_i + z_{i-1})}{h^2} + h(DT_i + DT_{i-1}), \tag{2.4}$$

where

$$\begin{aligned}
A &= \frac{-\eta + \theta^2 + 2}{\theta^3 \xi}, \\
B &= \frac{-\theta^2\eta + 2\theta^2}{2\theta^3 \xi}, \\
C &= \frac{2\xi - \theta^2\eta - 4}{2\theta^3 \xi}, \\
D &= \frac{\eta - 2}{\theta \xi} \\
and \quad \eta &= e^{\theta} + e^{-\theta} \\
\xi &= e^{\theta} - e^{-\theta}
\end{aligned}$$

From above equations, we get the following relation:

$$-z_{i-2} + 3z_{i-1} - 3z_i + z_{i+1} = h^3(\mu\, T_{i-2} + \nu\, T_{i-1} + \nu\, T_i + \mu\, T_{i+1}),$$

$i = 2, 3, \cdots, n-1,$ (2.5)

where

$$\mu = \frac{\eta - \theta^2 - 2}{\theta^3 \xi},$$
$$\nu = \frac{\theta^2 \eta - \eta - \theta^2 - 2}{\theta^3 \xi}.$$

If $\theta \to 0$, then $(\mu,\ \nu) \to (\frac{1}{24},\ \frac{11}{24})$, then spline relation (2.5) reduces to the ordinary quartic spline relations.
The relation (2.5) gives $(n-2)$ equations in $(n-1)$ unknowns $z_i,\ i = 2(1)n-1$, one more equation is required. The required boundary equation is:

$$\sum_{k=0}^{3} u_k z_k + c\epsilon h z_0' + h^3 \sum_{k=0}^{3} v_k z_k''' + t_1 = 0, \qquad i = 1. \tag{2.6}$$

where u_k, c and v_k are arbitrary parameters.

2.1 Boundary Equations

To find the second-order boundary equations we have

$$(u_0, u_1, u_2, u_3, c, v_0, v_1, v_2, v_3)$$
$$= (1/4, 247/4, -397/4, 149/4, 25, 1/4, -709/24, -391/24, 0)$$

and the truncation error is

$$t_i = 4180 h^5 z_i^{(5)} + O(h^6), \qquad i = 1. \tag{2.7}$$

3 The Method

At the grid point t_i, the third-order SPBVP (1.1) can be discretized as

$$-\epsilon z'''(t_i) + p(t_i) z(t_i) = q(t_i).$$

By using spline's third derivative, we have

$$T_i = \frac{p_i z_i - q_i}{\epsilon},$$

$$T_{i-2} = \frac{p_{i-2} z_{i-2} - q_{i-2}}{\epsilon}, \quad T_{i-1} = \frac{p_{i-1} z_{i-1} - q_{i-1}}{\epsilon}, \quad T_{i+1} = \frac{p_{i+1} z_{i+1} - q_{i+1}}{\epsilon},$$

where $p_i = p(t_i)$ and $q_i = q(t_i)$.

Put the values of $T_j(j = i, i \pm 1, i-2)$ in equation (2.5), we get

$$(\epsilon + \mu h^3 p_{i-2}) z_{i-2} + (-3\epsilon + \nu h^3 p_{i-1}) z_{i-1} + (3\epsilon + \nu h^3 p_i) z_i + (-\epsilon + \mu h^3 p_{i+1}) z_{i+1}$$
$$= h^3(\mu q_{i-2} + \nu q_{i-1} + \nu q_i + \mu q_{i+1}), \quad i = 2(1)n-1. \tag{3.1}$$

4 Truncation Error

The local truncation error t_i of the scheme (2.5)can be obtained by writing in the form:

$$-z_{i-2} + 3z_{i-1} - 3z_i + z_{i+1} = h^3(\mu\ z'''_{i-2} + \nu\ z'''_{i-1} + \nu\ z'''_i + \mu\ z'''_{i+1}) + t_i,$$

$$i = 2(1)n-1. \tag{4.1}$$

Expanded the terms z'''_{i-2}, z'''_{i-1} etc using Taylor's series about t_i we get the expression for t_i as

$$t_i = [1 - 2\mu + 2\nu]\epsilon h^3 z'''_i + \left(\frac{-1}{2} - \mu + \nu\right)\epsilon h^4 z_i^{(4)} + \left(\frac{1}{4} - \frac{5\mu + \nu}{2}\right)\epsilon h^5 z_i^{(5)} + O(h^6). \tag{4.2}$$

For arbitrary choices of μ and ν we get second-order method.

5 Convergence Analysis

Let $Z = z(t_i)$ $\bar{Z} = (z_i)$, $C = (c_i)$, $T = (t_i)$, $E = (e_i) = Z - \bar{Z}$, $i = 1(1)n-1$ be an exact column vectors, where Z *and* $\bar{Z}$ are exact and approximate solution, T and E are local truncation error and discretization error respectively.
Thus, the system (3.1) can be written as:

$$(M + h^3 BF)\bar{Z} = C, \tag{5.1}$$

where

$$M = \begin{bmatrix} u_1\epsilon & u_2\epsilon & u_3\epsilon & & & & & \\ -3\epsilon & 3\epsilon & -\epsilon & & & & & \\ \epsilon & -3\epsilon & 3\epsilon & -\epsilon & & & & \\ & & & \ddots & & & & \\ & & & & \ddots & & & \\ & & & & & \epsilon & -3\epsilon & 3\epsilon & -\epsilon \\ & & & & & & \epsilon & -3\epsilon & 3\epsilon \end{bmatrix}, \tag{5.2}$$

$$B=\begin{bmatrix} v_1 & v_2 & v_3 & & & & & \\ \nu & \nu & \mu & & & & & \\ \mu & \nu & \nu & \mu & & & & \\ & & & \ddots & & & & \\ & & & & \ddots & & & \\ & & & & \mu & \nu & \nu & \mu \\ & & & & & \mu & \nu & \nu \end{bmatrix}, \tag{5.3}$$

$$F=\begin{bmatrix} p_1 & & & & & \\ & p_2 & & & & \\ & & \ddots & & & \\ & & & \ddots & & \\ & & & & p_{n-2} & \\ & & & & & p_{n-1} \end{bmatrix} \tag{5.4}$$

and $C=[c_1,\ c_2,\cdots,c_{n-1}]^T$
with

$$c_i=\begin{cases} -u_0\alpha_0\epsilon-c_1h\epsilon\alpha_2-h^3v_0(p_0\alpha_0-q_0)+h^3(v_1q_1+v_2q_2+v_3q_3), & i=1,\\ -\alpha_0\epsilon-h^3\mu(p_0\alpha_0-q_0)+h^3(\nu q_1+\nu q_2+\mu q_3), & i=2,\\ h^3(\mu q_{i-2}+\nu q_{i-1}+\nu q_i+\nu q_{i+1}), & 3\le i\le n-2,\\ \alpha_1\epsilon-h^3\mu(p_n\alpha_1-q_n)+h^3(\mu q_{n-3}+\nu q_{n-2}+\nu q_{n-1}), & i=n-1. \end{cases} \tag{5.5}$$

Consider the above system with exact solution $Z=[z(t_1),\ z(t_2),\cdots,z(t_{n-1})]^T$, we have

$$(M+h^3BF)Z=T(h)+C, \tag{5.6}$$

where
$T(h)=[t_1(h),\ t_2(h),\cdots,t_{n-1}(h)]^T$ defined as follows:

$$t_i=\begin{cases} 4180\epsilon h^5z_i^{(5)}+O(h^6), & i=1,\\ -\frac{1}{12}\epsilon h^5z_i^{(5)}+O(h^6), & 2\le i\le n-1. \end{cases} \tag{5.7}$$

Subtracting equation (5.1) from (5.6), we obtain the error equation

$$(M+h^3BF)(Z-\bar{Z})=T(h)$$

or

$$M_0 E = T(h), \tag{5.8}$$

where
$M_0 = (M + h^3 BF)$ and $E = Z - \bar{Z} = [e_1,\ e_2,\ \cdots, e_{n-1}]^T$.

The row sums $S_1,\ S_2, \cdots, S_{n-1}$ of M_0 are

$$S_i = \begin{cases} -\frac{1}{4}\epsilon + h^3(v_1 p_1 + v_2 p_2 + v_3 p_3), & i = 1, \\ \epsilon + h^3(-\nu p_1 - \nu p_2 - \mu p_3), & i = 2, \\ h^3(-\mu p_{i-2} - \nu p_{i-1} - \nu p_i - \mu p_{i+1}), & 3 \le i \le n-2, \\ -\frac{1}{4}\epsilon + h^3(-\mu p_{n-3} - \nu p_{n-2} - \nu p_{n-1}), & i = n-1. \end{cases} \tag{5.9}$$

M_0 becomes irreducible and monotone if h to be chosen as $O(\sqrt(\epsilon)$. Therefore M_0^{-1} exists and its elements are non-negative. Hence from (5.8), we get

$$E = M_0^{-1} T(h) \Longrightarrow \|E\| \le \|M_0^{-1}\| \cdot \|T(h)\|. \tag{5.10}$$

Let $a_{k,i}^{-1}$ is the $(k,i)^{th}$ element of the matrix M_0^{-1}.

We define

$$\|a_{k,i}^{-1}\| = \max_{1 \le k \le n} \sum_{i=1}^{n-1} |a_{k,i}^{-1}| \tag{5.11}$$

and

$$\|T\| = \max_{1 \le k \le n} |t_k|. \tag{5.12}$$

Using the theory of matrices, we have

$$\sum_{i=1}^{n-1} a_{k,i}^{-1} S_i = 1, \qquad k = 1(1)n-1. \tag{5.13}$$

Therefore,

$$a_{k,i}^{-1} \le \frac{1}{\min_{1 \le i \le n-1} S_i} = \frac{1}{h^3 Q_{i_o}}, \tag{5.14}$$

where $Q_{i_o} = \frac{1}{h^3} \min_i S_i > 0$, for some i_o between 1 to $n-1$.
From equation (5.7), (5.10) and (5.11), we obtain

$$e_i = \sum_{i=1}^{n-1} a_{k,i}^{-1} T_i(h), \qquad k = 1(1)n-1 \tag{5.15}$$

and therefore

$$|e_i| \leq \frac{Kh^2}{Q_{i_o}}, \qquad i = 1(1)n-1, \tag{5.16}$$

where K is a constant independent of h. Therefore $\|E\| = O(h^2)$.

The above results can be summarized as follows:

Theorem 5.1. *Let $z(t)$ be the exact solution of third-order SPBVP*(1.1) *and let z_i be the numerical solution obtained by scheme* (5.1)*. Then, for sufficiently small h,* (5.1) *gives a second order convergent solution.*

6 Numerical Examples

The developed method is tested on two SPBVPs and the results are compared with the existing methods. Computations are carried out by using MATLAB.

Example 6.1. Consider the third-order SPBVP discussed in [1, 3]:

$$\begin{gathered} -\epsilon z'''(t) + z(t) = q(t), \qquad t \in [0,1], \\ with \;\; z(0) = 0, \; z(1) = 0, \; z'(0) = 0, \end{gathered}$$

where

$$q(t) = 6\epsilon(1-t)^5 t^3 - 6\epsilon^2[6(1-t)^5 - 90(1-t)^4 t + 180(1-t)^3 t^2 - 60(1-t)^2 t^3].$$

The analytical solution is

$$z(t) = 6t^3\epsilon(1-t)^5.$$

Example 6.2. Consider the third-order SPBVP [3]:

$$\begin{gathered} -\epsilon z'''(t) + z(t) = 81\epsilon^2 \cos 3t + 3\epsilon \sin 3t, \qquad t \in [0,1], \\ with \;\; z(0) = 0, \; z(1) = 3\epsilon \sin 3, \; z'(0) = 9\epsilon. \end{gathered}$$

The exact solution is

$$z(t) = 3\epsilon \sin 3t.$$

The observed maximum absolute errors (MAE) are given in Tables 1 and 2.

Conclusion

The method based on exponential quartic spline is developed for the solution of third-order self-adjoint SPBVP. This method is second-order convergent and computationally efficient. Two examples are carried out for numerical illustrations.

Table 1: Observed MAE, Example 6.1.

$\epsilon \downarrow$	$\mathbf{n=10}$	$\mathbf{n=20}$	$\mathbf{n=40}$
Our method			
1/16	4.87×10^{-4}	1.86×10^{-5}	1.95×10^{-5}
1/32	1.95×10^{-4}	8.76×10^{-6}	8.63×10^{-6}
1/64	7.97×10^{-5}	4.00×10^{-6}	3.61×10^{-6}
Akram [1]			
1/16	2.9×10^{-3}	1.2×10^{-4}	6.4×10^{-6}
1/32	9.2×10^{-4}	3.8×10^{-5}	2.1×10^{-6}
1/64	1.4×10^{-4}	6.8×10^{-6}	4.6×10^{-7}
Saini and Mishra [3]			
1/16	4.7×10^{-4}	1.1×10^{-4}	2.6×10^{-5}
1/32	1.9×10^{-4}	4.7×10^{-5}	1.2×10^{-5}
1/64	8.0×10^{-5}	1.9×10^{-5}	4.8×10^{-6}

Table 2: Observed MAE, Example 6.2.

$\epsilon \downarrow$	$\mathbf{n=10}$	$\mathbf{n=20}$	$\mathbf{n=40}$
Our method			
1/16	2.32×10^{-4}	6.12×10^{-5}	1.52×10^{-5}
1/32	9.77×10^{-5}	2.59×10^{-5}	6.45×10^{-6}
1/64	3.78×10^{-5}	1.00×10^{-6}	2.50×10^{-6}
Saini and Mishra [3]			
1/16	2.4×10^{-4}	6.1×10^{-5}	1.5×10^{-5}
1/32	1.0×10^{-4}	2.6×10^{-5}	6.4×10^{-6}
1/64	4.0×10^{-5}	1.0×10^{-6}	2.5×10^{-6}

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References

[1] G. Akram, Quartic spline solution of a third order singularly perturbed boundary value problem, ANZIAM J. 53 (2012) 44-58.

[2] M. Cui, F. Geng, A computational method for solving third order singularly perturbed boundary value problem, Appl. Math. Comput. 198 (2008) 896-903.

[3] S. Saini, H.K. Mishra, A new quartic B-spline spline method for third-order self-adjoint singularly perturbed boundary value problems, Appl. Math. Sci. 9 (8) (2015) 399-408.

[4] S. Valarmathi, N. Ramanujam, Boundary value technique for finding numerical solution to boundary-value problems for third order singularly perturbed ordinary differential equations, Intern. J. Computer Math. 79 (6) (2002) 747-763.

[5] W. Zaho, Singular perturbation of boundary value problems for a class of third order nonlinear ordinary differential equations, J. Comput. Diff. Eqn. 88 (2) (1990) 265-278.

[6] T. Aziz, A. Khan, Quintic spline approach to the solution of a singularly perturbed boundary value problem, J. Optim. Theory Appl. 112 (3) (2002) 517-527.

[7] T. Aziz, A. Khan, A spline method for second-order singularly perturbed boundary-value problems, J. Comput. Appl. Math. 147 (2002) 445-452.

[8] A. Khan, I. Khan, T. Aziz, Sextic spline solution of singularly perturbed boundary-value problems, Appl. Math. Comput. 181 (2006) 432-439.

[9] A. Khan, P. Khandelwal, Non-polynomial sextic spline solution of singularly perturbed boundary-value problems, Int. J. Comput. Math. 91 (5) (2014) 1122-1135.

[10] R.E. O'Malley, Singular Perturbation Methods for Ordinary Differential Equations, Springer, New York, 1981.

[11] J.J.H. Miller, O'Riordan, G.I. Shishkin, Fitted Numerical Methods for Singular Perturbation Problems, World Scientific, Singapore, 1996.

[12] H.G. Roos, M. Stynes, L. Tobiska, Numerical Methods for Singularly Perturbed Differential Equations, Springer, New York, 1996.

[13] E.A. Al-Said, M.A. Noor, Cubic splines method for a system of third-order boundary-value problems, Appl. Math. Comput. 142 (2003) 195-204.

[14] M.A. Noor, E.A. Al-Said, Quartic splines solutions of third-order obstacle problems, Appl. Math. Comput. 153 (2004) 307-316.

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