On the Residue of the Generalized Function $P^\lambda$

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Abstract: In this paper we study the residue of the generalized function $P^\lambda$ where

$$P \left( \sum_{i=1}^{p} x_i^2 \right)^m - \left( \sum_{j=p+1}^{p+q} x_j^2 \right)^m,$$

$p + q = n$ is a dimension of the Euclidean space $\mathbb{R}^n$, $m$ is a positive integer and $\lambda$ is a complex number. For the case $m = 1$ we obtain the special residue appear in the sense of I.M. Gel’fand and G.E. Shilov. Moreover for case $m = 2$ we obtain the residue of Fourier transform of the Diamond kernel related to the spectrum see 2, pages 715-723. And for case $m = 4$ obtain the residue of the Fourier transform of the distributional kernel related to the spectrum see 3.

Keywords: generalized function, Diamond kernel, distributional kernel.

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1 Introduction

I.M. Gel’fand and G.E. Shilov 1, p253-258 have studied the generalized function $P^\lambda$ where

$$P = \sum_{i=1}^{p} x_i^2 - \sum_{j=p+1}^{p+q} x_j^2$$

is quadratic form. $\lambda$ is a complex number and $p + q - n$ is the dimension of the Euclidean space $\mathbb{R}^n$. They found that $P^\lambda$ has two sets of singularities namely $\lambda = -1, -2, \ldots, -k, \ldots$ and $\lambda = -\frac{n}{2}, -\frac{n}{2} - 1, \ldots, -\frac{n}{2} - k, \ldots$, where $k$ is a positive integer. For singular point $\lambda = -k$, the generalized function $P^\lambda$ has a simple pole with residues

$$\operatorname{res}_{\lambda=-k} P^\lambda = \frac{(-1)^k}{k!} \delta^{(k)}(P)$$

for $p + q - n$ is odd with $q$ odd and $q$ even.

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Also for the singular point $\lambda = -\frac{n}{2} - k$ they obtain
\[
\text{res}_{\lambda = -\frac{n}{2} - k} P^\lambda \frac{\pi^{\frac{n}{2}}(-1)^{\frac{n}{2}}L^k \delta(x)}{2^{2k}k!\Gamma(\frac{n}{2} + k)} \tag{1.3}
\]
where
\[
L = \sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} - \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2},
\]
$p + q - n$ is odd with $p$ odd and $q$ even.

The purpose of this work is studying the residues of the generalized function $P^\lambda$ which is positive definite and defined by
\[
P^\lambda \left( \sum_{i=1}^{p} x_i^2 \right)^m = \left( \sum_{j=p+1}^{p+q} x_j^2 \right)^m - \lambda \tag{1.4}
\]
where $\lambda$ is a complex number and $m$ is a positive integer. Now (1.4) is generalized function of (1.1).

We obtain
\[
\text{res}_{\lambda = -k} P^\lambda = \frac{(-1)^{k}}{k!} \delta^{(k-1)}(P),
\]
and also
\[
\text{res}_{\lambda = -\frac{n}{2m} - k} < P^\lambda, \varphi > = \frac{1}{k!} \cdot \delta^k \Phi(-\frac{n}{2m} + k, u) \tag{1.5}
\]
where $\delta^{(k-1)}(P)$ defined by (2.15), $\Phi(-\frac{n}{2m} + k, u)$ defined by (3.10) with $\lambda = -\frac{n}{2m} - k$ and $u = \frac{n}{2m}$. In particular for $m - 1 \chi(1.5)$ reduces to (1.2), and (1.6) reduces to (1.3).

Moreover for case $m = 2$ we obtain the residue of the Fourier transform of the Diamond kernel related to the spectrum. See [2 pages 715-723]. And for case $m = 4$ we obtain the residue of the Fourier transform of the distributional kernel related to the spectrum. See [3]

2 Preliminaries

Definition 2.1 Given $\varphi(x)$ be any testing function in the Schwartz space $\mathcal{S}$. Define
\[
< P^\lambda, \varphi > = \int_{\mathbb{R}^p} P^\lambda \varphi(x) dx \tag{2.1}
\]
where $P^\lambda$ is defined by (1.4). Actually $P^\lambda \in \mathcal{S}$ - the space of tempered distribution.

Definition 2.2 Given the hyper-surface $P = 0$ where
\[
P^\lambda = \left( \sum_{i=1}^{p} x_i^2 \right)^m = \left( \sum_{j=p+1}^{p+q} x_j^2 \right)^m.
\]
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Let $\varphi \neq \delta$ be a dimension of $\mathbb{R}^n$ and $m$ is a positive integer

\[ p + q - n \text{ is a dimension of } \mathbb{R}^n \text{ and } m \text{ is a positive integer} \]

Let $\text{grad} P \neq 0$ that means there is no singular point on $P = 0$. Then we define

\[ < \delta^{(k)}(P), \varphi > = \int \delta^{(k)}(P) \varphi(x) \, dx \quad (2.2) \]

where $\delta^{(k)}$ is the Dirac-delta distribution with $k$-derivatives, $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $dx = dx_1 \, dx_2 \ldots dx_n$. In a sufficiently small neighborhood $U$ of any point $(x_1, x_2, \ldots, x_n)$ of the hyper-surface $P = 0$. We can introduce a new coordinate system such that $P + U$ becomes one of the coordinate hyper-surface. For this purpose we write $P = u_1$ and choose the remaining $u_i$ coordinates (with $i = 2, 3, \ldots, n$) for which the Jacobian $D(\xi_i, \xi_i) > 0$ where

\[ D(\xi_i, \xi_i) = \frac{\partial (x_1, x_2, \ldots, x_n)}{\partial (u_1, u_2, \ldots, u_n)} \]

Thus (2.2) can be written by

\[ < \delta^{(k)}(P), \varphi > = (-1)^k \int \frac{\delta^k}{\partial P^k} \{ \varphi D(\xi_i) \} \frac{\partial u_1}{\partial x_i} \, dx_2 \ldots dx_n \quad (2.3) \]

**Lemma 2.1** Given the hyper-surface

\[ P = \left( \sum_{i=1}^p x_i^2 \right)^m - \left( \sum_{j=p+1}^{p+q} x_j^2 \right)^m \quad (2.4) \]

$p + q - n$ is the dimension of $\mathbb{R}^n$ and $m$ is a positive integer.

If we transform to bipolar coordinates defined by

\[ x_1 = r \omega_1, x_2 = r \omega_2, \ldots, x_n = r \omega_n \]

and

\[ x_{p+1} = r \omega_{p+1}, x_{p+2} = r \omega_{p+2}, \ldots, x_{p+q} = r \omega_{p+q} \]

where $\sum_{i=1}^p \omega_i = 1, \sum_{j=p+1}^{p+q} \omega_j = 1$. Then $r^2 = \sum_{i=1}^p x_i^2$ and $s^2 = \sum_{j=p+1}^{p+q} x_j^2$ and (2.4) can be written by $P = r^{2m} - s^{2m}$. Then we obtain

\[ < \delta^{(k)}(P), \varphi > = \int_0^\infty \left( \frac{1}{2m} \frac{\partial}{\partial s} \right)^k \left( s^{2m-2} \psi(r, s) \right) \left( s^{2m-2} \psi(r, s) \right) \frac{r^{p-1}}{s^{m-1}} \, dr \quad (2.5) \]

\[ < \delta^{(k)}(P), \varphi > = (-1)^k \int_0^\infty \left( \frac{1}{2m} \frac{\partial}{\partial r} \right)^k \left( s^{2m-2} \psi(r, s) \right) \left( s^{2m-2} \psi(r, s) \right) \frac{s^{2m-2}}{r^{m-1}} \, ds \quad (2.6) \]

where $\psi(r, s) = \int \varphi d\Omega _p d\Omega _q$, $d\Omega _p$ and $d\Omega _q$ are the elements of surface area on the unit sphere in $\mathbb{R}^p$ and $\mathbb{R}^q$ respectively.
Proof. We have the element of volume
\[ ds = r^{p-1}s^{q-1} dr \, ds \, d\Omega_q \] (2.7)
where \( d\Omega_p \) and \( d\Omega_q \) are the elements of surface area on the unit sphere in \( \mathbb{R}^p \) and \( \mathbb{R}^q \) respectively. Now
\[ P = r^{2m} - s^{2m}. \] (2.8)
choose the coordinates to be \( p, r \) and the \( \omega_i \). Then (2.7) becomes
\[ ds = \frac{1}{2m} (r^{2m} - P)^{\frac{1}{2m}} r^{p-1} dP \, d\Omega_p \, d\Omega_q \] (2.9)
Thus by (2.3), we obtain
\[ < \delta^{(k)}(P), \varphi > = (-1)^k \int_{p=0}^{\infty} \left[ \frac{\partial}{\partial P} \left( \frac{1}{2m} (r^{2m} - P)^{\frac{1}{2m}} \right) \right] r^{p-1} \, d\Omega_p \, d\Omega_q. \] (2.10)
Further if we transform from \( P \) to \( s = (r^{2m} - P)^{\frac{1}{2m}} \), we have
\[ \frac{P}{s} = \frac{1}{2ms^{2m-1}}. \]
Thus (2.10) becomes
\[ < \delta^{(k)}(P), \varphi > = \int (\frac{1}{2ms^{2m-1}} \frac{\partial}{\partial s})^k \left\{ s^{q-2m} \frac{\varphi}{2m} \right\} \bigg|_{s=r} r^{p-1} \, d\Omega_p \, d\Omega_q \] (2.11)
Write
\[ \psi(r, s) = \int \varphi \, d\Omega_p \, d\Omega_q. \] (2.12)
Then (2.11) becomes
\[ < \delta^{(k)}(P), \varphi > = \int_0^\infty \left( \frac{1}{2ms^{2m-1}} \frac{\partial}{\partial s} \right)^k \left\{ s^{q-2m} \frac{\psi(r,s)}{2m} \right\} \bigg|_{s=r} r^{p-1} \, dr \] (2.13)
Similarly
\[ < \delta^{(k)}(P), \varphi > = (-1)^k \int_0^\infty \left( \frac{1}{2ms^{2m-1}} \frac{\partial}{\partial r} \right)^k \left\{ r^{p-2m} \frac{\psi(r,s)}{2m} \right\} \bigg|_{s=r} s^{q-1} \, ds. \] (2.14)
Thus we obtain (2.5) and (2.6) as required.

Now, we assume that \( \varphi \) vanishes in the neighborhood of the origin so that these integrals will converge for any \( k \).

Now for \( p - 1 + (q - 2m) \geq 2mk \) or \( k < \frac{1}{2m} (p + q - 2m) \) the integral in (2.13) converges for any \( \varphi(x) \in \mathcal{S} \). Similarly, for \( p - 1 + (q - 2m) \geq 2mk \) or \( k < \frac{1}{2m} (p + q - 2m) \) the integral in (2.14) also converges for any \( \varphi(x) \in \mathcal{S} \). Thus we take (2.13) and (2.14) to be the defining equation for \( \delta^{(k)}(P) \). If, on the other
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hand $k > \frac{1}{2m}(p + q - 2m)$ we shall define $\delta_1^{(k)}(P), \varphi >$ and $\delta_2^{(k)}(P), \varphi >$ as the regularization of (2.13) and (2.14) respectively.

We shall say that for $p > 1$ and $q > 1$ the generalized function $\delta_1^{(k)}(P)$, and $\delta_2^{(k)}(P)$, are defined by

$$
< \delta_1^{(k)}(P), \varphi > = \int_0^\infty \left( \frac{1}{2m} \frac{\partial}{\partial s} \right)_k \left\{ s^{q-2m} \psi(r,s) \right\} \frac{r^{p-1} ds}{r_{min}^r}
$$

for $k \geq \frac{1}{2m}(p + q - 2m)$ and

$$
< \delta_2^{(k)}(P), \varphi > = (-1)^k \int_0^\infty \left( \frac{1}{2m} \frac{\partial}{\partial r} \right)_k \left\{ r^{p-2m} \psi(r,s) \right\} \frac{s^{q-1} ds}{s_{min}^s}
$$

for $k > \frac{1}{2m}(p + q - 2m)$.

In particular for $m = 1$, $\delta_1^{(k)}(P)$ reduces to $\delta_{11}^{(k)}(P)$, and $\delta_2^{(k)}(P)$ reduces to $\delta_{21}^{(k)}(P)$. See [1, p.250]

3 Main Results

**Theorem 3.1** Let $P^\lambda$ be hyper-surface given by (1.4) for $p > 2m$ and $q > 2m$. Then the residues of $P^\lambda$ at the singular point $\lambda = -k$ and $\lambda = -\frac{n}{2m} - k$ are

$$
\text{res}_{\lambda = -k} P^\lambda = (-1)^{k-1} \frac{\delta_{11}^{(k-1)}(P)}{(k-1)!}
$$

and

$$
\text{res}_{\lambda = -\frac{n}{2m} - k} P^\lambda, \varphi > = \frac{1}{k!} \frac{\partial^k}{\partial u^k} \Phi\left(-\frac{n}{2m} - k, u \right)
$$

where $\delta_{11}^{(k-1)}(P)$ defined by (2.15), $\Phi(-\frac{n}{2m} - k, u)$ defined by (3.10), with $\lambda = -\frac{n}{2m} - k$ and $\phi = r^{2m}$. In particular, for $k = 1$ we have

$$
\text{res}_{\lambda = -\frac{n}{2m}} P^\lambda = \frac{1}{m!} \Gamma\left(\frac{1}{2}\right) \Gamma\left(1 - \frac{n}{2m}\right) \delta(x)
$$

**Proof.** From (2.1).

$$
< P^\lambda, \varphi > = \int_{r > 0} P^\lambda \varphi(r) dr
$$

$$
\int_{r > 0} (r^{2m} - s^{2m}) \varphi (r^{p-1} s^{q-1} dr ds d\Omega, d\Omega
$$

by (2.8) and changing to bipolar coordinates. Write

$$
\varphi(r,s) = \int \varphi d\Omega, d\Omega
$$

$$
< \delta_1^{(k)}(P), \varphi > = \int_0^\infty \left( \frac{1}{2m} \frac{\partial}{\partial s} \right)_k \left\{ s^{q-2m} \psi(r,s) \right\} \frac{r^{p-1} ds}{r_{min}^r}
$$

for $k \geq \frac{1}{2m}(p + q - 2m)$ and

$$
< \delta_2^{(k)}(P), \varphi > = (-1)^k \int_0^\infty \left( \frac{1}{2m} \frac{\partial}{\partial r} \right)_k \left\{ r^{p-2m} \psi(r,s) \right\} \frac{s^{q-1} ds}{s_{min}^s}
$$

for $k > \frac{1}{2m}(p + q - 2m)$.

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$$

where $\delta_{11}^{(k-1)}(P)$ defined by (2.15), $\Phi(-\frac{n}{2m} - k, u)$ defined by (3.10), with $\lambda = -\frac{n}{2m} - k$ and $\phi = r^{2m}$. In particular, for $k = 1$ we have

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\text{res}_{\lambda = -\frac{n}{2m}} P^\lambda = \frac{1}{m!} \Gamma\left(\frac{1}{2}\right) \Gamma\left(1 - \frac{n}{2m}\right) \delta(x)
$$

**Proof.** From (2.1).

$$
< P^\lambda, \varphi > = \int_{r > 0} P^\lambda \varphi(r) dr
$$

$$
\int_{r > 0} (r^{2m} - s^{2m}) \varphi (r^{p-1} s^{q-1} dr ds d\Omega, d\Omega
$$

by (2.8) and changing to bipolar coordinates. Write

$$
\varphi(r,s) = \int \varphi d\Omega, d\Omega
$$
We obtain
\[
< P^\lambda, \varphi > = \int_0^\infty \int_0^r (r^{2m} - s^{2m})^{\lambda} \psi(r, s) r^{p-1} s^{q-1} \, ds \, dr 
\]  
\tag{3.6}

Since \( \varphi \in \mathcal{D} \) the space of infinitely differentiable function with compact support Then \( \psi(r, s) \) as defined in (3.5) is infinitely differentiable function of \( r^{2m} \) and \( s^{2m} \) with compact support. Thus \( \psi(r, s) \in \mathcal{D} \). Put \( u = r^{2m}, v = s^{2m} \) in the integrand of (3.6) writing
\[
\psi(r, s) = \psi_1(u, v). 
\]  
\tag{3.7}

Then (3.6) becomes
\[
< P^\lambda, \varphi > = \frac{1}{4m^2} \int_0^\infty \int_0^u (u-v)^\lambda \psi_1(u, v) u^{p-1} v^{q-1} \, dv \, du 
\]  
\tag{3.8}

Write \( r = ut \) thus (3.8) becomes
\[
< P^\lambda, \varphi > = \frac{1}{4m^2} \int_0^\infty u^{\lambda+\frac{p}{2m}(p+q)-1} du \int_0^1 (1-t)^\lambda t^{\frac{q}{2m}-1} \psi_1(u, ut) \, dt. 
\]  
\tag{3.9}

Write
\[
\Phi(\lambda, u) = \frac{1}{4m^2} \int_0^1 (1-t)^\lambda t^{\frac{q}{2m}-1} \psi_1(u, ut) \, dt 
\]  
\tag{3.10}

Thus for \( u > 2m \), \( \Phi(\lambda, u) \) is regular for all \( \lambda \) except at the singularities
\[
\lambda = -1, -2, -3, \ldots, -k, \ldots, 
\]
where it has simple poles. At these poles we have
\[
\text{res} \left. \Phi(\lambda, u) = \frac{1}{4m^2} \left( \frac{-1}{(k-1)!} \frac{\partial^{k-1}}{\partial t^{k-1}} \left( t^{\frac{q}{2m}-1} \psi_1(u, ut) \right) \right) \right|_{t=1} \]  
\tag{3.11}

like the function \( < x^\lambda, \varphi > \), Sec. 1, p50.

Thus \( \text{res} \Phi(\lambda, u) \) is a functional concentrated on the space of \( F - \Omega \) cones.

Now (3.9) can also be written as
\[
< P^\lambda, \varphi > = \int_0^\infty u^{\lambda+\frac{p}{2m}(p+q)-1} \Phi(\lambda, u) \, du. 
\]  
\tag{3.12}

Even at the regular points of \( \Phi(\lambda, u) \) the integral of (3.12) have poles at
\[
\lambda = -\frac{n}{2m}, -\frac{n}{2m} - 1, \ldots, -\frac{n}{2m} - k, \ldots 
\]
where \( p + q - n \) is the dimension of \( \mathbb{R}^n \) and \( n \) is odd.

At these points we have
\[
\text{res} \left. \left( \frac{\partial^k}{\partial u^k} \Phi(-\frac{n}{2m} - k, u) \right) \right|_{u=0} \]  
\tag{3.13}
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These the residues of $< P^\lambda, \varphi >$ at $\lambda = -\frac{n}{2m} - k$ is a functional concentrated on the vertex of the cone.

Now consider the singular point $\lambda = -k$. By [3.11], [3.12], and also see [1, p255-256] we obtain

$$\text{res}_{\lambda = -k} \frac{(-1)^{k-1}}{(k-1)!} < \delta^+(k-1) \varphi >$$

where

$$< \delta^+(k-1)(P), \varphi > = \frac{1}{4m^2} \int_0^\infty \left[ \frac{1}{2ms^{2m-1}} \frac{\partial}{\partial s} \right]^{k-1} \left\{ s^{q-2m} \psi(r,s) \right\} \frac{1}{2m} r^{p-1} dr$$

Summarizing, we have the following. For odd $n$ for even $n$ if $k < \frac{1}{2m} n$, the generalized function $P^\lambda$ has simple poles at $\lambda = -k$ for positive integral values of $k$, where the residues are

$$\text{res}_{\lambda = -k} P^\lambda = \frac{(-1)^{k-1}}{(k-1)!} \delta^+(k-1)(P).$$

Now consider the singular point at $\lambda = -\frac{n}{2m} - k$, from [3.13] we have

$$\text{res}_{\lambda = -\frac{n}{2m} - k} < P^\lambda, \varphi > = \frac{1}{k!} \frac{\partial^k}{\partial u^k} \Phi \left( -\frac{n}{2m} - k, u \right)$$

for $n$ odd with $p$ odd and $q$ even. Thus, for $k = 0$ we have

$$\text{res}_{\lambda = -\frac{n}{2m}} < P^\lambda, \varphi > = \Phi \left( -\frac{n}{2m}, 0 \right)$$

By [3.10] we obtain

$$\text{res}_{\lambda = -\frac{n}{2m}} < P^\lambda, \varphi > - \frac{1}{4m^2} \int_0^1 (1-t)^{-\frac{n}{2m}} t^\frac{n}{2m} \psi_1(0,0) dt$$

$$- \frac{\psi_1(0,0)}{4m^2} \int_0^1 (1-t)^{-\frac{n}{2m}} t^\frac{n}{2m} \psi_1(0,0) dt$$

$$- \frac{\psi_1(0,0) \Gamma\left(\frac{n}{2m}\right) \Gamma\left(\frac{n}{2m} + 1\right)}{4m^2 \Gamma\left(-\frac{n}{2m} + 1\right)}$$

since

$$\int_0^1 (1-t)^{-\frac{n}{2m}} t^\frac{n}{2m} \psi_1(0,0) dt = \frac{\Gamma\left(\frac{n}{2m}\right) \Gamma\left(\frac{n}{2m} + 1\right)}{\Gamma\left(-\frac{n}{2m} + 1\right)}$$

Now

$$\psi_1(0,0) = \psi(0,0) = - \int \psi(0) d\Omega p d\Omega q$$

by [3.5].

$$\Omega_p \Omega_q \varphi(0).$$
and
\[ \Omega_{\varepsilon} = \frac{2\pi^{\frac{q}{2}}}{\Gamma\left(\frac{q}{2}\right)} \quad \text{and} \quad \Omega_{\varphi} = \frac{2\pi^{\frac{q}{2}}}{\Gamma\left(\frac{q}{2}\right)} \]

Thus
\[ \text{res}_{\lambda = -\frac{m}{2m}} P_{\lambda, \varphi} = \frac{1}{4m^{2}} \frac{\Gamma\left(\frac{q}{2m}\right)\Gamma\left(-\frac{m}{2m} + 1\right)}{\Gamma\left(\frac{q}{2m}\right)\Gamma\left(\frac{q}{2}\right)} \left(\frac{2\pi^{\frac{q}{2}}}{\Gamma\left(\frac{q}{2}\right)}\right) \varphi(0) \]  
(3.17)

Now for \( p \geq 2m \) and \( p \) is even then \( \Gamma\left(-\frac{p}{2m} - 1\right) \rightarrow \infty \) Thus \( \text{res}_{\lambda = -\frac{m}{2m}} < P_{\lambda, \varphi} > = 0 \) From (3.17),
\[ \text{res}_{\lambda = -\frac{m}{2m}} P_{\lambda, \varphi} = -\frac{1}{m^{2}} \frac{\Gamma\left(\frac{q}{2m}\right)\Gamma\left(-\frac{m}{2m} + 1\right)}{\Gamma\left(\frac{q}{2m}\right)\Gamma\left(\frac{q}{2}\right)} \frac{4\pi^{\frac{q}{2}}}{m^{2}} \delta(x), \varphi > \]
\[ -\frac{1}{m^{2}} \frac{\Gamma\left(\frac{q}{2m}\right)\Gamma\left(-\frac{m}{2m} + 1\right)}{\Gamma\left(\frac{q}{2m}\right)\Gamma\left(\frac{q}{2}\right)} \frac{\pi^{\frac{q}{2}}}{m^{2}} \delta(x), \varphi > . \]

Thus
\[ \text{res}_{\lambda = -\frac{m}{2m}} P_{\lambda} = \frac{1}{m^{2}} \frac{\Gamma\left(\frac{q}{2m}\right)\Gamma\left(1 - \frac{m}{2m}\right)\pi^{\frac{q}{2}}}{\Gamma\left(\frac{q}{2m}\right)\Gamma\left(\frac{q}{2}\right)} \delta(x) \]  
(3.18)

In particular, if \( m = 1 \) then (3.18) reduces to
\[ \text{res}_{\lambda = -\frac{m}{2m}} P_{\lambda} = \frac{(-1)^{q}\pi^{\frac{q}{2}}}{\Gamma\left(\frac{q}{2}\right)} \delta(x) \]

which appeared in [1, eq.(23), p.258]

Moreover for equations (3.1) and (3.2), if \( m \neq 0 \) we obtain
\[ \text{res}_{\lambda = -\frac{m}{2m}} (-1)^{k}K_{2k, 2k}(x, \varphi) \quad \frac{1}{2\pi^{n/2}} \text{res}_{\lambda = -\frac{m}{2m}} P_{\lambda} = \frac{(-1)^{k-1}}{2\pi^{n/2}(k-1)!} \delta^{(k-1)}(P), \]

and
\[ \text{res}_{\lambda = -\frac{m}{2m}} < P_{\lambda, \varphi} > = \frac{1}{k!} \left[ \frac{\partial^{k}}{\partial u^{k}} \Phi\left(-\frac{n}{4} - k, u\right) \right]_{u = 0} \]

where \( (-1)^{k}K_{2k, 2k}(x) \) is the Fourier transform of the Diamond kernel see [2, pages 715-723] And if \( m = 4 \) we obtain
\[ \text{res}_{\lambda = -\frac{m}{2m}} (-1)^{k}K_{2k, 2k, 2k, 2k}(x) = \text{res}_{\lambda = -\frac{m}{2m}} P_{\lambda} = \frac{(-1)^{k-1}}{(k-1)!} \delta^{(k-1)}(P) \]

and
\[ \text{res}_{\lambda = -\frac{m}{2m}} < P_{\lambda, \varphi} > = \frac{1}{k!} \left[ \frac{\partial^{k}}{\partial u^{k}} \Phi\left(-\frac{n}{8} - k, u\right) \right]_{u = 0} \]

where \( (-1)^{k}K_{2k, 2k, 2k, 2k}(x) \) is the Fourier transform of the distributional kernel of the operator \( \Pi^{k} \), defined by
\[ \Pi^{k} = \left( \sum_{i=1}^{p} \frac{\partial^{2}}{\partial x_{i}^{2}} \right)^{4} - \left( \sum_{j=p+1}^{p+q} \frac{\partial^{2}}{\partial x_{j}^{2}} \right)^{4} \quad p + q - n, \text{see [3], [4]} \]
On the Residue of the Generalized Function $P^\lambda$

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References


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