Stein’s Method for Cauchy Approximation

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Abstract: In this paper, we use Stein’s method to find the necessary and sufficient conditions for Cauchy distribution and give a bound for Cauchy approximation. The random variables which considered need not to be independent.

Keywords: Stein’s method, Cauchy approximation.

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1 Introduction

Stein[11] introduced a powerful and general method for obtaining an explicit bound for the error in the normal approximation to the distribution of a sum of dependent random variables. Stein’s method has been applied with much success in the area of normal approximation (See, for example, Erickson[7], Bolthausen[5], Baldi, Rinott and Stein[1] and Barbour[2]). This method was extended from the normal distribution to the Poisson distribution by Chen[6]. Chen’s work has resulted in advances in the theory of Poisson approximation and has helped to develop and improve upon a body of interesting applications and examples. (For theoretical developments, see Barbour and Eagleson[3,4], Holst and Janson[8]). Many authors developed Stein’s method to other approximations, for examples, Pekőz[10] applied this method to geometric approximation.

In 2002, Neammanee[9] applied Stein’s method for Cauchy approximation to the distribution function of sums of independent random variables. In this paper, we further develop the Stein’s technique to find the bound in more general situation, i.e. we need not to assume independence of random variables. We organize this paper as follows. Main results are stated in section 2 while proof of main results is given in section 3.

2 Main results

Let $X_1, X_2, \ldots, X_n$ be random variables. Neammanee[9] used Stein’s method to find a bound between the distribution function $F_n$ of $X_1 + X_2 + \cdots + X_n$ and the Cauchy distribution $F$,

$$F(x) = \frac{1}{\pi} \int_{-\infty}^{x} \frac{1}{1 + t^2} dt. \quad (2.1)$$
In his work, he assumes the independence of $X_n$'s but it is not necessary in this paper. To apply Stein’s method we need the following construction.

Let $\mathcal{H} = \{ h : \mathbb{R} \to \mathbb{R} | \int_{-\infty}^{\infty} \frac{|h(x)|}{1+x^2} \, dx < \infty \}$ and for each $h \in \mathcal{H}$,

$$Cau(h) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{h(x)}{1+x^2} \, dx.$$  \hspace{1cm} (2.2)

From Neammanee[9], we know that the Stein’s equation for Cauchy distribution $F$, is

$$f'(w) - \frac{2wf(w)}{1+w^2} = h(w) - Cau(h)$$ \hspace{1cm} (2.3)

and

$$f_{w_0}(w) = \begin{cases} \pi(1+w^2)F(w)(1-F(w_0)) & \text{if } w \leq w_0 \\ \pi(1+w^2)F(w_0)(1-F(w)) & \text{if } w \geq w_0 \end{cases}$$  \hspace{1cm} (2.4)

is a solution of (2.3) when we choose $h(w) = I_{(-\infty,w_0]}$ and

$$I_{(-\infty,w_0]}(w) = \begin{cases} 1 & \text{if } w \leq w_0 \\ 0 & \text{if } w > w_0 \end{cases}.$$

The main results are the followings.

**Theorem 2.1.** Let $X_1, X_2, \ldots, X_n$ be random variables such that $EX_i = 0$, $EX_i^4 < \infty$ and $S_i$ be a subset of $\{1,2,\ldots,n\}$ for all $i = 1,2,\ldots,n$. Let $W = X_1 + X_2 + \cdots + X_n$. Then for all $w_0 \in \mathbb{R}$,

$$\left| P(W \leq w_0) - F(w_0) \right| \leq 3 \left[ \sqrt{E[1 - \frac{2}{1+W^2} \sum_{i=1}^{n} \sum_{j \in S_i} X_i X_j]^2} ight. + 4\pi \min \left\{ E \sum_{i=1}^{n} |X_i| \sum_{j \in S_i} X_j, \left( E \sum_{i=1}^{n} X_i^2 E \left| \sum_{i=1}^{n} X_i \sum_{j \in S_i} X_j \right|^2 \right) F(w_0)(1-F(w_0)) \right\} + 2\pi E \sum_{i=1}^{n} |E[X_i | j \notin S_i]| X_i \right] + 12(\pi + 1) E \sum_{i=1}^{n} |X_i| \left( \sum_{j \in S_i} X_j \right)^2$$

where $E^B X$ is the conditional expectation of $X$ with respect to $B$.

In the case of $X_n$'s are independent we have the following corollary and example.
Corollary 2.2. Let $X_1, X_2, \ldots, X_n$ be independent random variables such that $EX_i = 0$, $EX_i^4 < \infty$ for all $i = 1, 2, \ldots, n$. Let $W = X_1 + \cdots + X_n$. Then for all $w_0 \in \mathbb{R}$,

$$\left| P(W \leq w_0) - F(w_0) \right| \leq 3 \sqrt{E \left[ 1 - \frac{2}{1 + W^2} \sum_{i=1}^{n} X_i^2 \right]^2} + 4\pi \min \left\{ \sum_{i=1}^{n} EX_i^2, \sqrt{n \left( \sum_{i=1}^{n} EX_i^2 \right) \left( \sum_{i=1}^{n} EX_i^4 \right)} \right\} F(w_0)(1 - F(w_0)) + 12(\pi + 1) \sum_{i=1}^{n} E |X_i|^3.$$

Example 2.3. Let $Y_1, Y_2, \ldots, Y_n$ be identically independent random variables with zero means, $EY_i^2 = \frac{1}{2}$ and $E|Y_i|^5 < \infty$. Let $X_i = \frac{Y_i}{\sqrt{n}}$ and $W = X_1 + X_2 + \cdots + X_n$. Then for all $w_0 \in \mathbb{R}$,

$$\left| P(W \leq w_0) - F(w_0) \right| < C \frac{1}{\sqrt{n}} + C \min \left\{ \frac{1}{2}, \frac{\sqrt{2}}{\sqrt{2}} \sqrt{E|Y_i|^4} \right\} F(w_0)(1 - F(w_0)).$$

We note that Corollary 2.2 and Example 2.3 are main results of Neammanee[9]. Moreover, we also use Stein’s method to find the necessary and sufficient conditions in order to a random variable $W$ be Cauchy.

Theorem 2.4. Let $W$ be a random variable. Then $W$ has a Cauchy distribution if and only if for all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f'$ exists a.e., continuous a.e. and $E|f'(W)| < \infty$, we have

$$Ef'(W) = 2E \frac{Wf(W)}{1 + W^2}.$$

Throughout this paper, $C$ stands for an absolute constant with possibly different values in different places.

3 Proof of Main Results

To prove Theorem 2.1, we first introduce the basic assumption.

Basic assumption: Let $(\tilde{\Omega}, \tilde{\mathcal{B}}, \tilde{P})$ be a probability space and let $\mathcal{B}$ and $\mathcal{C}$ be sub-$\sigma$-algebras of $\tilde{\mathcal{B}}$. The random variable $G$ is $\tilde{\mathcal{B}}$-measurable and random variable $\tilde{W}$ is $\mathcal{C}$-measurable. Assuming

$$E|G| < \infty,$$

we define $W = E^\delta G$. 
Lemma 3.1. In addition to the basic assumption, let \( f: \mathbb{R} \to \mathbb{R} \) be such that \(|f(w)| \leq C(1 + w^2)\) for some \( C > 0 \). Then

\[
E \frac{W f(W)}{1 + W^2} = E G \left[ \frac{f(W)}{1 + W^2} - \frac{f(\tilde{W})}{1 + \tilde{W}^2} \right] + E(E^C G) \frac{f(\tilde{W})}{1 + \tilde{W}^2},
\]

(3.1)

Moreover,

\[
P(W \leq w_0) = F(w_0) + E \left[ f'_{w_0}(W) - 2G \left( \frac{f_{w_0}(W)}{1 + W^2} - \frac{f_{w_0}(\tilde{W})}{1 + \tilde{W}^2} \right) \right] - 2E(E^C G) \frac{f_{w_0}(\tilde{W})}{1 + \tilde{W}^2}
\]

(3.2)

for \( w_0 \in \mathbb{R} \).

Proof. Since \( E G f(\tilde{W}) = EE^C G f(\tilde{W}) = E(E^C G) \frac{f(\tilde{W})}{1 + \tilde{W}^2} \), so

\[
E \frac{W f(W)}{1 + W^2} = E(E^B G) \frac{f(W)}{1 + W^2}
\]

\[
= EE^B \left( \frac{G f(W)}{1 + W^2} \right)
\]

\[
= E \frac{G f(W)}{1 + W^2}
\]

\[
= EG \left[ \frac{f(W)}{1 + W^2} - \frac{f(\tilde{W})}{1 + \tilde{W}^2} \right] + E(E^C G) \frac{f(\tilde{W})}{1 + \tilde{W}^2}
\]

Then (3.1) holds and (3.2) follows from (3.1) and (2.3) when \( h = I_{(-\infty, w_0]} \). \( \square \)

Lemma 3.2. Under the basic assumption for any \( w_0 \in \mathbb{R} \), we have

\[
P(W \leq w_0) = F(w_0) + Ef'_{w_0}(W) \left[ 1 - 2G(W - \tilde{W}) \right] + 4E \frac{G(W - \tilde{W}) W f_{w_0}(W)}{(1 + W^2)^2}
\]

\[
+ 2EG \int_{-\infty}^{\infty} (\tilde{W} - w)[I(w \leq \tilde{W}) - I(w \leq W)] \left( \frac{f'_{w_0}(w)}{1 + w^2} \right) dw
\]

\[
- 4EG \int_{-\infty}^{\infty} (\tilde{W} - w)[I(w \leq \tilde{W}) - I(w \leq W)] \left( \frac{w f_{w_0}(w)}{(1 + w^2)^2} \right) dw
\]

\[
- 2E \frac{(E^C G) f_{w_0}(\tilde{W})}{1 + \tilde{W}^2}
\]

Proof. Let \( w_0 \in \mathbb{R} \). For \( W < \tilde{W} \), we see that
\[
\frac{f_{w_0}(\tilde{W})}{1 + W^2} - \frac{f_{w_0}(W)}{1 + W^2} - \frac{(\tilde{W} - W)f'_{w_0}(W)}{1 + W^2} + \frac{2(\tilde{W} - W)Wf_{w_0}(W)}{(1 + W^2)^2} \\
= \int_{W}^{\tilde{W}} \left[ \frac{f_{w_0}(w)}{1 + w^2} - \frac{f'_{w_0}(W)}{1 + W^2} + \frac{2Wf_{w_0}(W)}{(1 + W^2)^2} \right] dw \\
= \int_{W}^{\tilde{W}} \left[ \frac{f_{w_0}(w)}{1 + w^2} - \frac{2wf_{w_0}(w)}{1 + w^2} - \frac{f'_{w_0}(W)}{1 + W^2} + \frac{2Wf_{w_0}(W)}{(1 + W^2)^2} \right] dw \\
= \int_{W}^{\tilde{W}} \int_{W}^{w} \left( \frac{f_{w_0}(y)}{1 + y^2} \right)' dy dw - 2 \int_{W}^{\tilde{W}} \int_{W}^{w} \left( \frac{yf_{w_0}(y)}{1 + y^2} \right)' dy dw \\
= \int_{W}^{\tilde{W}} \int_{y}^{\tilde{W}} \left( \frac{f_{w_0}(y)}{1 + y^2} \right)' dw dy - 2 \int_{W}^{\tilde{W}} \int_{y}^{\tilde{W}} \left( \frac{yf_{w_0}(y)}{1 + y^2} \right)' dw dy \\
= \int_{\tilde{W}}^{W} (w - \tilde{W}) \left( \frac{f_{w_0}(w)}{1 + w^2} \right)' dw - 2 \int_{\tilde{W}}^{W} (w - \tilde{W}) \left( \frac{wf_{w_0}(w)}{(1 + w^2)^2} \right)' dw \\
\]

and by the same argument we can show that

\[
\frac{f_{w_0}(\tilde{W})}{1 + W^2} - \frac{f_{w_0}(W)}{1 + W^2} - \frac{(\tilde{W} - W)f'_{w_0}(W)}{1 + W^2} + \frac{2(\tilde{W} - W)Wf_{w_0}(W)}{(1 + W^2)^2} \\
= \int_{\tilde{W}}^{W} (w - \tilde{W}) \left( \frac{f_{w_0}(w)}{1 + w^2} \right)' dw - 2 \int_{\tilde{W}}^{W} (w - \tilde{W}) \left( \frac{wf_{w_0}(w)}{(1 + w^2)^2} \right)' dw \\
\]

for \( \tilde{W} < W \).

So

\[
\frac{f_{w_0}(\tilde{W})}{1 + W^2} - \frac{f_{w_0}(W)}{1 + W^2} - \frac{(\tilde{W} - W)f'_{w_0}(W)}{1 + W^2} + \frac{2(\tilde{W} - W)Wf_{w_0}(W)}{(1 + W^2)^2} \\
= \int_{-\infty}^{\tilde{W}} (\tilde{W} - w)I(w \leq \tilde{W}) - I(w \leq W) \left( \frac{f'_{w_0}(w)}{1 + w^2} \right)' dw \\
- 2 \int_{-\infty}^{\tilde{W}} (\tilde{W} - w)I(w \leq \tilde{W}) - I(w \leq W) \left( \frac{wf_{w_0}(w)}{(1 + w^2)^2} \right)' dw. \quad (3.3)
\]
By Lemma 3.1, we have
\[
P(W \leq w_0) = F(w_0) + E\left[ f'_{w_0}(W) - 2 \frac{G(W - \tilde{W}) f'_{w_0}(W)}{1 + W^2} + 2 \frac{G(W - \tilde{W}) f''_{w_0}(W)}{1 + W^2} \right]
\]
\[+ 4 \frac{G(W - \tilde{W})W f_{w_0}(W)}{(1 + W^2)^2} - 4 \frac{G(W - \tilde{W}) W f_{w_0}(W)}{(1 + W^2)^2} - 2G\left( f_{w_0}(W) + \frac{f_{w_0}(\tilde{W})}{1 + W^2} \right) - 2E\left( \frac{E^c G)}{f_{w_0}(\tilde{W})} \right)\]
\[= F(w_0) + Ef'_{w_0}(W) \left[ 1 - \frac{2G(W - \tilde{W})}{1 + W^2} \right]
\]
\[+ 2EG \left[ \left( \frac{f_{w_0}(\tilde{W}) - f_{w_0}(W)}{1 + W^2} - \frac{(W - W)f'_{w_0}(W)}{1 + W^2} + \frac{2(W - W)W f_{w_0}(W)}{(1 + W^2)^2} \right) \right]
\]
\[= F(w_0) + Ef'_{w_0}(W) \left[ 1 - \frac{2G(W - \tilde{W})}{1 + W^2} \right]
\]
\[+ 2EG \int_{-\infty}^{\infty} (\tilde{W} - w) I(w \leq \tilde{W}) - I(w \leq W) \left( \frac{f'_{w_0}(w)}{1 + w^2} \right) dw
\]
\[- 4EG \int_{-\infty}^{\infty} (\tilde{W} - w) I(w \leq \tilde{W}) - I(w \leq W) \left( \frac{f_{w_0}(w)}{1 + w^2} \right) dw
\]
\[+ 2EG \frac{G(W - \tilde{W})W f_{w_0}(W)}{(1 + W^2)^2} - 2E\left( \frac{E^c G)}{f_{w_0}(\tilde{W})} \right)\]
\[
(3.4)
\]
where we have used (3.3) in the last equality.

The following lemma is the properties of $f_{w_0}$ which we need in the proof of Theorem 2.1.

**Lemma 3.3.** For any real numbers $w_0$ and $w$,
1. $|f_{w_0}(w)/(1 + w^2)| \leq \pi F(w_0)(1 - F(w_0))$
2. $|f'_{w_0}(w)| \leq 3$
3. $|f''_{w_0}(w)| \leq 3 + 2\pi$
4. $|f'_{w_0}(w)/(1 + w^2)| \leq 6 + 2\pi$
5. $|w f_{w_0}(w)/(1 + w^2)^2| \leq 3 + 5\pi$.

**Proof.** See [9].

### 3.1 Proof of Theorem 2.1

Let $I$ be a random variable, uniformly distributed over \{1, 2, \ldots, n\} independent of \{X_1, X_2, \ldots, X_n\}. Let $\mathcal{B}$ be a $\sigma$-algebra in which the random variables
I and \( \{X_1, X_2, \ldots, X_n\} \) are measurable, \( \mathcal{B} \) the sub-\( \sigma \)-algebra of \( \overline{\mathcal{B}} \) generated by \( \{X_1, X_2, \ldots, X_n\} \) and \( \mathcal{C} \) the \( \sigma \)-algebra generated by \( I \) and \( \{X_j \mid j \notin S_I\} \). Let

\[
G = nX_I.
\]

Then

\[
E^G = E^G nX_I = \sum_{i=1}^{n} X_i = W.
\]

So the basic assumption is satisfied. Let

\[
\tilde{W} = W - \sum_{j \in S_I} X_j.
\]

To prove the theorem, let \( w_0 \in \mathbb{R} \). By Lemma 3.2, we obtain

\[
|P(W \leq w_0) - F(w_0)|
\]

\[
\leq \sup_{w \in \mathbb{R}} |f'_w(w)| E \left( 1 - 2 \frac{G(W - \tilde{W})}{1 + W^2} \right) + 4E \left| \frac{G(W - \tilde{W})W f_{w_0}(W)}{(1 + W^2)^2} \right| + 2 \sup_{w \in \mathbb{R}} |f_{w_0}(w)| E|E^G|
\]

\[
+ 4 \sup_{w \in \mathbb{R}} \left| \left( \frac{w f_{w_0}(w)}{(1 + w^2)^2} \right)' \right| E|G| \int_{-\infty}^{w} |\tilde{W} - w||I(w \leq \tilde{W}) - I(w \leq W)|dw
\]

\[
\leq \sup_{w \in \mathbb{R}} |f'_w(w)| \left( 1 - 2 \frac{2}{1 + W^2} E^G(W - \tilde{W}) \right)^2 + 4E \left| \frac{G(W - \tilde{W})W f_{w_0}(W)}{(1 + W^2)^2} \right|
\]

\[
+ 2 \sup_{w \in \mathbb{R}} \left| \left( \frac{f_{w_0}(w)}{1 + w^2} \right)' \right| E|E^G| + \left( \sup_{w \in \mathbb{R}} \left| \left( \frac{f_{w_0}(w)}{1 + w^2} \right)' \right| + 2 \sup_{w \in \mathbb{R}} \left| \left( \frac{w f_{w_0}(w)}{(1 + w^2)^2} \right)' \right| \right) E|G|(\tilde{W} - W)^2.
\]

Hence, by Lemma 3.3,

\[
|P(W \leq w_0) - F(w_0)| \leq 3 \sqrt{E \left[ 1 - 2 \frac{2}{1 + W^2} E^G(W - \tilde{W}) \right]^2} + 4E \left| \frac{G(W - \tilde{W})W f_{w_0}(W)}{(1 + W^2)^2} \right|
\]

\[
+ 2\pi E|E^G| + 12(\pi + 1) E|G|(\tilde{W} - W)^2.
\]

We see that

\[
E|E^G| = E|E^G nX_I| = E \left| E^C \sum_{i=1}^{n} X_i \right| \leq E \sum_{i=1}^{n} |E^{\{X_j \mid j \notin S_i\}} X_i|,
\]

\[
E|G|(\tilde{W} - W)^2 = E \left| \sum_{i=1}^{n} X_i \left( \sum_{j \in S_i} X_j \right)^2 \right| \leq E \sum_{i=1}^{n} |X_i| \left( \sum_{j \in S_i} X_j \right)^2,
\]

where

\[
G = nX_I.
\]
\[
\sqrt{E\left[1 - \frac{2}{1+W^2}E^B G(W - \widehat{W})\right]^2} = \sqrt{E\left[1 - \frac{2}{1+W^2}E^B n \sum_{j \in S_i} X_j\right]^2} \\
= \sqrt{E\left[1 - \frac{2}{1+W^2} \sum_{i=1}^{n} E^B X_i \sum_{j \in S_i} X_j\right]^2} \\
= \sqrt{E\left[1 - \frac{2}{1+W^2} \sum_{i=1}^{n} \sum_{j \in S_i} X_i X_j\right]^2}.
\]

By Lemma 3.3(1), we have
\[
E \left| \frac{G(\widehat{W} - W)W_{f_{w_0}}(W)}{(1+W^2)^2} \right| \leq \pi F(w_0)(1 - F(w_0))E|G||W - \widehat{W}| \\
= \pi F(w_0)(1 - F(w_0))E \left| \sum_{i=1}^{n} X_i \left( \sum_{j \in S_i} X_j \right) \right| \\
\leq \pi F(w_0)(1 - F(w_0))E \sum_{i=1}^{n} |X_i| \left( \sum_{j \in S_i} X_j \right)
\]
and
\[
E \left| \frac{G(\widehat{W} - W)W_{f_{w_0}}(W)}{(1+W^2)^2} \right| \leq \pi F(w_0)(1 - F(w_0))\sqrt{E|G|^2|W - \widehat{W}|^2 \sqrt{EW^2}} \\
= \pi F(w_0)(1 - F(w_0)) \sqrt{E \left| \sum_{i=1}^{n} X_i \sum_{j \in S_i} X_j \right|^2} \sqrt{E \sum_{i=1}^{n} X_i^2}.
\]

This complete the proof. \(\square\)

### 3.2 Proof of Corollary 2.2.

Follows from Theorem 2.1 by choosing \(S_i = \{i\}\), so
\[
E \left| \sum_{i=1}^{n} X_i \sum_{j \in S_i} X_j \right|^2 = E \left| \sum_{i=1}^{n} X_i^2 \right|^2 \leq n \sum_{i=1}^{n} EX_i^4,
\]
and the assumption that \(X_1, X_2, \ldots, X_n\) are independent implies
\[
|E^{(X_j \not \in S_i)} X_i| = |E^{(X_j \not \not \in i)} X_i| = |EX_i| = 0.
\]
3.3 Proof of Example 2.3.

See[9].

3.4 Proof of Theorem 2.4.

To prove the necessity, we assume that $W$ has a Cauchy distribution and let $f : \mathbb{R} \to \mathbb{R}$ be such that $f'$ exists a.e., continuous a.e. and $E|f'(W)| < \infty$. Then

$$2E\frac{Wf(W)}{1 + W^2}$$

$$= \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{wf(w)}{1 + w^2} dw$$

$$= \frac{2}{\pi} \left\{ \int_{-\infty}^{0} \frac{wf(w)}{1 + w^2} dw + \int_{0}^{\infty} \frac{wf(w)}{1 + w^2} dw \right\}$$

$$= \frac{2}{\pi} \left\{ \int_{-\infty}^{0} \frac{w}{(1 + w^2)^2} \left( f(0) - \int_{w}^{0} f'(t) dt \right) dw + \int_{0}^{\infty} \frac{w}{(1 + w^2)^2} \left( f(0) + \int_{0}^{w} f'(t) dt \right) dw \right\}$$

$$= \frac{2}{\pi} \left\{ \int_{-\infty}^{0} \int_{w}^{0} \frac{(-w)f'(t)}{(1 + w^2)^2} dt dw + \int_{0}^{\infty} \int_{0}^{w} \frac{wf'(t)}{(1 + w^2)^2} dt dw + f(0) \int_{-\infty}^{\infty} \frac{w}{(1 + w^2)^2} dw \right\}$$

$$= \frac{2}{\pi} \left\{ \int_{-\infty}^{0} f'(t) \int_{t}^{\infty} \frac{(-w)}{(1 + w^2)^2} dw dt + \int_{0}^{\infty} f'(t) \int_{t}^{\infty} \frac{w}{(1 + w^2)^2} dw dt \right\}$$

$$= \frac{1}{\pi} \left\{ \int_{-\infty}^{0} \frac{f'(t)}{1 + t^2} dt + \int_{0}^{\infty} \frac{f'(t)}{1 + t^2} dt \right\}$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f'(t)}{1 + t^2} dt$$

$$= Ef'(W).$$

Conversely, let $w_0 \in \mathbb{R}$. By Lemma 3.3(2), we see that $E|f'_{w_0}(W)| < \infty$. Then

$$0 = E\left[ f'_{w_0}(W) - 2\frac{Wf_{w_0}(W)}{1 + W^2} \right]$$

$$= E[I_{(-\infty, w_0)}(W) - Cau(I_{(-\infty, w_0)})]$$

$$= P(W \leq w_0) - F(w_0)$$

where we have used Stein’s equation (2.3) in the first equality. Hence $W$ has a Cauchy distribution. \qed
References


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