Some Characterizations of Anti-Fuzzy (Generalized) Bi-Ideals of Semigroups

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Abstract : Our aim in this paper is to characterize anti-fuzzy subsemigroups, anti-fuzzy generalized bi-ideals and anti-fuzzy bi-ideals of a semigroup $S$. We define certain subsets of $S$, $[0,1]$ and $S \times [0,1]$. The relationships between sets of anti-fuzzy points and the certain subsets of $S \times [0,1]$ are investigated. Some interesting characterizations of anti-fuzzy subsemigroups, anti-fuzzy generalized bi-ideals and anti-fuzzy bi-ideals of semigroups are investigated by using the certain subsets of $S$, $[0,1]$ and $S \times [0,1]$.

Keywords : semigroups; anti-fuzzy points; anti-fuzzy subsemigroups; anti-fuzzy generalized bi-ideals; anti-fuzzy bi-ideals.

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1 Introduction

The fundamental concept of a fuzzy set, introduced by Zadeh [1], plays a major role in mathematics with wide applications in many other branches e.g. theoretical physics, computer science, control engineering, information science, measure theory. Rosenfeld [2] gave definitions of a fuzzy subgroupoid and a fuzzy subgroup, and obtained some properties of them. Since then, many fuzzy algebraic struc-
On anti-fuzzy algebraic structures, Biswas [13] introduced the concept of anti-fuzzy subgroups of groups and lower level sets of fuzzy subsets, and also showed that a fuzzy subset $f$ of a group $G$ is an anti-fuzzy subgroup of $G$ if and only if for every $\alpha \in [0, 1]$, a lower level set $L(f : \alpha) = \{x \in G \mid f(x) \leq \alpha\}$ is either empty or a subgroup of $G$. The concept of lower level sets of fuzzy subsets is one of mathematical methods for studying anti-fuzzy algebraic structures, some papers used the concept of lower level sets seen in [14–23]. Modifying and applying Biswas’ idea, concepts of many types of anti-fuzzy algebraic structures have been introduced and studied extensively by many authors. For example, Shabir and Nawas [22] in 2009 introduced the concept of an anti-fuzzy (generalized) bi-ideal of any semigroup $S$ and characterized anti-fuzzy (generalized) bi-ideals by using lower level sets. Moreover, they characterized semigroups in terms of anti-fuzzy (generalized) bi-ideals. Khan and Asif [18], the continuation of the work carried out by Shabir and Nawas, introduced anti-fuzzy interior ideals of $S$ and characterized semigroups by the properties of anti-fuzzy (generalized) bi-ideals and anti-fuzzy interior ideals. Khan et al. [24] gave relationships between anti-fuzzy (generalized) bi-ideals and anti-fuzzy right ideals on semilattice of left groups. Characterizations of semilattices of left (right) groups are investigated by using anti-fuzzy (generalized) bi-ideals and anti-fuzzy one-sided ideals [24]. Due to these possibilities of applications, semigroups and related structures are studied via anti-fuzzy generalized bi-ideals and anti-fuzzy bi-ideals.

Our propose of this work is to promote and develop anti-fuzzy algebraic structures by studying anti-fuzzy semigroup theory. We define the certain subsets of $S$, $[0, 1]$ and $S \times [0, 1]$ and investigate their properties. In particular, we define a certain subset $L(R : \alpha)$ of $S$ where $R$ is a subset of $S \times [0, 1]$ and this set is a general concept of the lower level set of a fuzzy set. We also describe relationship between sets of anti-fuzzy points and the certain subsets of $S \times [0, 1]$. Some interesting characterizations of anti-fuzzy subsemigroups, anti-fuzzy generalized bi-ideals and anti-fuzzy bi-ideals of semigroups are investigated by using the certain subsets of $S$, $[0, 1]$ and $S \times [0, 1]$. Moreover, we show that any fuzzy subset of $S$ is an anti-fuzzy (generalized) bi-ideal if and only if there exists the unique chain of (generalized) bi-ideals of $S$ together with two special conditions.

## 2 Preliminaries

In this section, we give basic definitions and results, which will be used in the next sections. A semigroup is an algebraic system $(S, \cdot)$ consisting of a nonempty set $S$ together with an associative binary operation “$\cdot$”. Throughout this paper, $S$ stands for a semigroup. For nonempty subsets $A$ and $B$ of $S$, we denote $AB = \{ab \mid a \in A, b \in B\}$. A nonempty subset $A$ of $S$ is called a subsemigroup of $S$ if $AA \subseteq A$. A nonempty subset $A$ of $S$ is called a generalized bi-ideal of $S$ if $ASA \subseteq A$. A subsemigroup $A$ of $S$ is called a bi-ideal of $S$ if $ASA \subseteq A$. By the
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above definitions, it is obvious that every bi-ideal of $S$ is a generalized bi-ideal, but the converse is not true in general.

A function $f$ from $S$ to the real closed interval $[0, 1]$ is called a fuzzy subset (or fuzzy set) of $S$. For $x \in S$, define $F_x = \{(y, z) \in S \times S \mid x = yz\}$. Let $f$ and $g$ be fuzzy subsets of $S$, then their anti-product $f \cdot g$ is defined by for all $x \in S$

$$(f \cdot g)(x) = \begin{cases} \inf \{\max \{f(y), g(z)\} \mid (y, z) \in F_x\}, & \text{if } F_x \neq \emptyset; \\ 1, & \text{otherwise.} \end{cases}$$

For $x \in S$, a fuzzy subset $f$ of $S$ of the form

$$f(y) = \begin{cases} \alpha \in [0, 1], & \text{if } x = y; \\ 1, & \text{otherwise} \end{cases}$$

for all $y \in S$ is called an anti-fuzzy point with support $x$ and value $\alpha$ and is denoted by $x^\alpha$. We denote by $AFP(S)$ the set of all anti-fuzzy points of $S$, that is,

$$AFP(S) = \{x^\alpha \mid x \in S, \alpha \in [0, 1]\}.$$ 

Then $(AFP(S), \cdot)$ is a semigroup and we conveniently denote it by $AFP(S)$. Indeed, we see that for all $x^\alpha, y^\beta, z^\gamma \in AFP(S)$, $x^\alpha \cdot y^\beta = (xy)^\max\{\alpha, \beta\}$ and $(x^\alpha \cdot y^\beta) \cdot z^\gamma = (xyz)^\max\{\alpha, \beta, \gamma\} = x^\alpha \cdot (y^\beta \cdot z^\gamma)$. For all $A, B \subseteq AFP(S)$, we define the product of two sets $A$ and $B$ as $A \cdot B = \{x^\alpha \cdot y^\beta \mid x^\alpha \in A, y^\beta \in B\}$. For every fuzzy subset $f$ of $S$, let $\hat{f} = \{x^\alpha \in AFP(S) \mid f(x) \leq \alpha\}$. Note that $\hat{f}$ is empty if and only if $f(x) = 1$ for all $x \in S$.

**Definition 2.1.** A fuzzy subset $f$ of a semigroup $S$ is called an anti-fuzzy subsemigroup of $S$ if $f(ab) \leq \max\{f(a), f(b)\}$ for all $a, b \in S$.

**Definition 2.2.** A fuzzy subset $f$ of a semigroup $S$ is called an anti-fuzzy generalized bi-ideal of $S$ if $f(axb) \leq \max\{f(a), f(b)\}$ for all $a, b, x \in S$.

**Definition 2.3.** An anti-fuzzy subsemigroup $f$ of a semigroup $S$ is called an anti-fuzzy bi-ideal of $S$ if $f(axb) \leq \max\{f(a), f(b)\}$ for all $a, b, x \in S$.

Define a binary operation “$\circ$” on $[0, 1]$ as follows: for all $(x, \alpha), (y, \beta) \in S \times [0, 1]$

$$(x, \alpha) \circ (y, \beta) = (xy, \max\{\alpha, \beta\}).$$

(2.1)

Then $(S \times [0, 1], \circ)$ is a semigroup. Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be subsets of $S \times [0, 1]$. Define the multiplication $\mathcal{R}_1 \circ \mathcal{R}_2$ of $\mathcal{R}_1$ and $\mathcal{R}_2$ as follows:

$$\mathcal{R}_1 \circ \mathcal{R}_2 = \{(a, \alpha) \circ (b, \beta) \mid (a, \alpha) \in \mathcal{R}_1 \text{ and } (b, \beta) \in \mathcal{R}_2\}.$$ 

(2.2)

For every subsemigroup $A$ of $S$ and nonempty subset $\Delta$ of $[0, 1]$, we have $(A \times \Delta, \circ)$ is a subsemigroup of $(S \times [0, 1], \circ)$. In what follows, let $S \times \Delta$ denote the semigroup $(S \times \Delta, \circ)$. Let $f$ be a fuzzy subset of $S$, $A \subseteq S$, $\alpha \in [0, 1]$, $\Delta \subseteq [0, 1]$ and
$R \subseteq S \times [0,1]$. We give the certain subsets of $S$, $[0,1]$ and $S \times [0,1]$ as the following.

$$[A \times \Delta]_f = \{(x, \alpha) \in A \times \Delta \mid f(x) \leq \alpha\}. \quad (2.3)$$

$$L(R : \alpha) = \{x \in S \mid (x, \beta) \in R \text{ and } \beta \leq \alpha \text{ for some } \beta \in [0,1]\}. \quad (2.4)$$

$$(Imf)^\alpha = \{\beta \in Imf \mid \beta \leq \alpha\}. \quad (2.5)$$

In particular, if $R$ is a fuzzy subset of $S$, then

$$L(R : \alpha) = \{x \in S \mid R(x) \leq \alpha\}.$$ 

If $\alpha, \beta \in [0,1]$ and $\alpha \leq \beta$, then $L(R : \alpha) \subseteq L(R : \beta)$ and hence the set $\{L(R : \alpha) \mid \alpha \in [0,1]\}$ is a chain of subsets of $S$ under the inclusion relation “$\subseteq$”.

**Proposition 2.4.** Let $f$ be a fuzzy subset of a semigroup $S$. Then the following statements are true.

(i) $(Imf)^\alpha \subseteq Imf$ for all $\alpha \in [0,1]$.

(ii) $L(f : \alpha) = \bigcup_{\gamma \in (Imf)^\alpha} f^{-1}(\gamma) = f^{-1}((Imf)^\alpha)$ for all $\alpha \in [0,1]$.

(iii) $[S \times \Delta]_f = \bigcup_{\gamma \in \Delta} (L(f : \gamma) \times \{\gamma\})$ for all $\Delta \subseteq [0,1]$.

(iv) If $\Delta \subseteq [0,1]$ and $R = [S \times \Delta]_f$, then $L(R : \alpha) = L(f : \alpha)$ for all $\alpha \in \Delta$.

**Proposition 2.5.** Let $S$ be a semigroup, $\Delta$ be a nonempty subset of $[0,1]$ and $R$ be a subsemigroup of $S \times \Delta$. Then $L(R : \alpha)$ is either empty or a subsemigroup of $S$ for all $\alpha \in \Delta$.

**Proposition 2.6.** Let $S$ be a semigroup, $\Delta$ be a nonempty subset of $[0,1]$ and $R$ be a generalized bi-ideal of $S \times \Delta$. Then $L(R : \alpha)$ is either empty or a generalized bi-ideal of $S$ for all $\alpha \in \Delta$.

**Proposition 2.7.** Let $S$ be a semigroup, $\Delta$ be a nonempty subset of $[0,1]$ and $R$ be a bi-ideal of $S \times \Delta$. Then $L(R : \alpha)$ is either empty or a bi-ideal of $S$ for all $\alpha \in \Delta$.

### 3 Anti-Fuzzy Subsemigroups of Semigroups

In this section, we characterize anti-fuzzy subsemigroups of a semigroup $S$ by using the certain subsets of $S$, $[0,1]$, $AFP(S)$ and $S \times [0,1]$.

For the following theorem, we discuss characterizations of anti-fuzzy subsemigroups of $S$ via the certain subsets of $[0,1]$ and $S \times [0,1]$.

**Theorem 3.1.** Let $f$ be a fuzzy subset of a semigroup $S$. Then the following statements are equivalent.
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(i) $f$ is an anti-fuzzy subsemigroup of $S$.

(ii) For every subsemigroup $A$ of $S$ and $\Delta \subseteq [0,1]$, we have $[A \times \Delta]_f$ is either empty or a subsemigroup of $S \times \Delta$.

(iii) $[S \times \Delta]_f$ is a subsemigroup of $S \times \Delta$ where $\text{Im}f \subseteq \Delta \subseteq [0,1]$.

(iv) For all $a, b \in S$, $(\text{Im}f)^f(ab) \subseteq (\text{Im}f)^f(a) \cup (\text{Im}f)^f(b)$.

Proof. $(i \Rightarrow ii)$ Let $A$ be a subsemigroup of $S$, $\Delta \subseteq [0,1]$ and $(a, \alpha), (b, \beta) \in [A \times \Delta]_f$. Then $f(a) \leq \alpha, f(b) \leq \beta$ and $\max\{\alpha, \beta\} \in \Delta$. Since $f$ is an anti-fuzzy subsemigroup of $S$ and $A$ is a subsemigroup of $S$, we have $ab \in A$ and

$$f(ab) \leq \max\{f(a), f(b)\} \leq \max\{\alpha, \beta\}.$$ 

Thus $(a, \alpha) \circ (b, \beta) \in [A \times \Delta]_f$. Hence $[A \times \Delta]_f$ is a subsemigroup of $S \times \Delta$.

$(ii \Rightarrow iii)$ It is obvious.

$(iii \Rightarrow iv)$ Suppose that $\alpha \in (\text{Im}f)^f(ab)$ and $\alpha \notin (\text{Im}f)^f(a) \cup (\text{Im}f)^f(b)$ for some $a, b \in S$, $\alpha \in [0,1]$. Then $\max\{f(a), f(b)\} < \alpha \leq f(ab)$. By the statement $(iii)$ and $(a, f(a)), (b, f(b)) \in [S \times \text{Im}f]_f$, we have $(a, f(a)) \circ (b, f(b)) \in [S \times \text{Im}f]_f$. Hence $f(ab) \leq \max\{f(a), f(b)\}$. It is a contradiction. Therefore $(\text{Im}f)^f(ab) \subseteq (\text{Im}f)^f(a) \cup (\text{Im}f)^f(b)$ for all $a, b \in S$.

$(iv \Rightarrow i)$ It is straightforward. \hfill $\Box$

By using and applying Theorem 3.1 we have Corollary 3.2.

**Corollary 3.2.** Let $f$ be a fuzzy subset of a semigroup $S$. Then the following statements are equivalent.

(i) $f$ is an anti-fuzzy subsemigroup of $S$.

(ii) $[S \times [0,1]]_f$ is either empty or a subsemigroup of $S \times [0,1]$.

(iii) $[S \times \text{Im}f]_f$ is a subsemigroup of $S \times \text{Im}f$.

(iv) $[S \times [0,1]]_f$ is a subsemigroup of $S \times [0,1]$.

**Example 3.3.** Let $S = \{a, b, c, d\}$ and define a binary operation “·” on $S$ as follows:

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Then $(S, \cdot)$ is a semigroup. Let $f$ be a fuzzy subset of $S$ such that

$$f(a) = f(b) = 0.1, \quad f(c) = 0.5, \quad f(d) = 0.7.$$ 

Thus, by routine calculations, we can check that $[S \times \text{Im}f]_f = \{(a, 0.1), (a, 0.5), (a, 0.7), (b, 0.1), (b, 0.5), (b, 0.7), (c, 0.5), (c, 0.7), (d, 0.7)\}$ is a subsemigroup of $S \times \text{Im}f$. By Corollary 3.2 $(iii \Rightarrow i)$, we have $f$ is an anti-fuzzy subsemigroup of $S$. 

Proposition 3.4. Let $f$ be a fuzzy subset of a semigroup $S$. Then $[S \times [0,1)]_{f}$ is a subsemigroup of $S \times [0,1)$ if and only if $\bar{f}$ is a subsemigroup of $\text{AFP}(S)$.

Proof. It is straightforward. \hfill $\square$

Theorem 3.5. Let $f$ be a fuzzy subset of a semigroup $S$. Then $f$ is an anti-fuzzy subsemigroup of $S$ if and only if $\bar{f}$ is either empty or a subsemigroup of $\text{AFP}(S)$.

Proof. It follows from Corollary 3.2(i) $\iff$ ii) and Proposition 3.4. \hfill $\square$

In the following theorem, we characterize anti-fuzzy subsemigroups of a semigroup by chain of subsemigroups of $S$.

Theorem 3.6. Let $f$ be a fuzzy subset of a semigroup $S$. Then $f$ is an anti-fuzzy subsemigroup of $S$ if and only if there exists the unique chain $\{A_{\alpha} \mid \alpha \in \text{Im}f\}$ of subsemigroups of $S$ such that

i) $f^{-1}(\alpha) \subseteq A_{\alpha}$ for all $\alpha \in \text{Im}f$ and

ii) for all $\alpha, \beta \in \text{Im}f$, if $\alpha < \beta$ then $A_{\alpha} \subset A_{\beta}$ and $A_{\alpha} \cap f^{-1}(\beta) = \emptyset$.

Proof. ($\Rightarrow$) For each $\alpha \in \text{Im}f$, we choose $A_{\alpha} = L(f : \alpha)$. By Proposition 2.4(iv), Proposition 2.3 and Theorem 3.1(i) $\Rightarrow$ iii), we get $\{A_{\alpha} \mid \alpha \in \text{Im}f\}$ is a chain of subsemigroups of $S$. By Proposition 2.4(ii), we have the conditions i) and ii). Suppose that $\{B_{\alpha} \mid \alpha \in \text{Im}f\}$ is a chain of subsemigroups of $S$ with the conditions i) and ii). Let $\alpha \in \text{Im}f$ and $a \in B_{\alpha}$. If $\alpha < f(a)$ then by the condition ii), we have $B_{\alpha} \cap f^{-1}(f(a)) = \emptyset$. Since $a \in f^{-1}(f(a))$, we get $a \in B_{\alpha} \cap f^{-1}(f(a))$. It is a contradiction. Thus $f(a) \leq \alpha$, so $a \in L(f : \alpha) = A_{\alpha}$. Hence $B_{\alpha} \subseteq A_{\alpha}$. Let $a \in A_{\alpha}$. Then $f(a) \leq \alpha$. By the conditions i) and ii), we get

$$a \in f^{-1}(f(a)) \subseteq B_{f(a)} \subseteq B_{\alpha}.$$

Hence $A_{\alpha} \subseteq B_{\alpha}$. Therefore $A_{\alpha} = B_{\alpha}$.

($\Leftarrow$) Let $(a, \alpha), (b, \beta) \in [S \times \text{Im}f]_{f}$. Then $f(a) \leq \alpha$, $f(b) \leq \beta$ and $\max\{\alpha, \beta\} \in \text{Im}f$. Suppose that $\max\{\alpha, \beta\} < f(ab)$. By the condition ii), we have $A_{\max\{\alpha, \beta\}} \cap f^{-1}(f(ab)) = \emptyset$. Since $f(a) \leq \max\{\alpha, \beta\}$ and by the conditions i) and ii), we have

$$a \in f^{-1}(f(a)) \subseteq A_{f(a)} \subseteq A_{\max\{\alpha, \beta\}}.$$

In the same way, we have $b \in A_{\max\{\alpha, \beta\}}$. Since $\{A_{\alpha} \mid \alpha \in \text{Im}f\}$ is a chain of subsemigroups of $S$, we get $ab \in A_{\max\{\alpha, \beta\}}$. Then $ab \in A_{\max\{\alpha, \beta\}} \cap f^{-1}(f(ab)) = \emptyset$. It is a contradiction. Thus $f(ab) \leq \max\{\alpha, \beta\}$. Hence $(a, \alpha) \circ (b, \beta) \in [S \times \text{Im}f]_{f}$. Therefore $[S \times \text{Im}f]_{f}$ is a subsemigroup of $S \times \text{Im}f$. By Corollary 3.2(iii) $\Rightarrow$ i), we have $f$ is an anti-fuzzy subsemigroup of $S$. \hfill $\square$

In the proof of Theorem 3.6, the unique chain of subsemigroups of $S$, satisfying conditions i) and ii), is the set $\{L(f : \alpha) \mid \alpha \in \text{Im}f\}$. Next, we consider one formula of an anti-fuzzy subsemigroup $f$ of a semigroup where $\text{Im}f$ is finite.
Corollary 3.7. Let $f$ be a fuzzy subset of a semigroup $S$ and $\text{Im} f = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ such that $\alpha_1 < \alpha_2 < \ldots < \alpha_n$. Then $f$ is an anti-fuzzy subsemigroup of $S$ if and only if $\{L(f : \alpha_i) | i \in \{1, 2, \ldots, n\}\}$ is the chain of subsemigroups of $S$ such that

$$f(x) = \begin{cases} 
\alpha_n & \text{if } x \in L(f : \alpha_n) \setminus L(f : \alpha_{n-1}) \\
\alpha_{n-1} & \text{if } x \in L(f : \alpha_{n-1}) \setminus L(f : \alpha_{n-2}) \\
\vdots & \\
\alpha_2 & \text{if } x \in L(f : \alpha_2) \setminus L(f : \alpha_1) \\
\alpha_1 & \text{if } x \in L(f : \alpha_1)
\end{cases}$$

for all $x \in S$.

Proof. Apply Theorem 3.6.

Corollary 3.8. Let $f$ be a fuzzy subset of a semigroup $S$ and $\text{Im} f \subseteq \Delta \subseteq [0, 1]$. The following statements are equivalent.

(i) $f$ is an anti-fuzzy subsemigroup of $S$.

(ii) There exists a subsemigroup $R$ of $S \times \Delta$ such that $L(R : \alpha) = L(f : \alpha)$ for all $\alpha \in \Delta$.

(iii) $L(f : \alpha)$ is either empty or a subsemigroup of $S$ for all $\alpha \in \Delta$.

Proof. $(i \Rightarrow ii)$ Choose $R = [S \times \Delta]_f$ and use Theorem 3.1 $(i \Rightarrow iii)$ and Proposition 2.4 $(iv)$.

$(ii \Rightarrow iii)$ It follows from Proposition 2.5.

$(iii \Rightarrow i)$ Apply Theorem 3.6.

4 Anti-Fuzzy (Generalized) Bi-Ideals of Semigroups

In this section, characterizations of anti-fuzzy generalized bi-ideals and anti-fuzzy bi-ideals of a semigroup $S$ are investigated by using the certain subsets of $S$, $[0, 1]$, $\text{AFP}(S)$ and $S \times [0, 1]$.

In the following theorem, we characterize anti-fuzzy generalized bi-ideals of a semigroup $S$ by the certain subsets of $[0, 1]$ and $S \times [0, 1]$.

Theorem 4.1. Let $f$ be a fuzzy subset of a semigroup $S$. Then the following statements are equivalent.

(i) $f$ is an anti-fuzzy generalized bi-ideal of $S$.

(ii) For every generalized bi-ideal $A$ of $S$ and $\Delta \subseteq [0, 1]$, we have $[A \times \Delta]_f$ is either empty or a generalized bi-ideal of $S \times \Delta$.

(iii) $[S \times \Delta]_f$ is a generalized bi-ideal of $S \times \Delta$ where $\text{Im} f \subseteq \Delta \subseteq [0, 1]$.

(iv) For all $a, b, x \in S$, $(\text{Im} f)^{f(AXB)} \subseteq (\text{Im} f)^{f(a)} \cup (\text{Im} f)^{f(b)}$. 
Proof. (i ⇒ ii) Let $A$ be a generalized bi-ideal of $S$, $\Delta \subseteq [0,1]$, $(x, \gamma) \in S \times \Delta$ and $(a, \alpha), (b, \beta) \in [A \times \Delta]_f$. Then $f(a) \leq \alpha$, $f(b) \leq \beta$ and $\max \{\alpha, \beta, \gamma\} \in \Delta$. Since $f$ is a fuzzy generalized bi-ideal of $S$ and $A$ is a generalized bi-ideal of $S$, we get $axb \in A$ and

$$f(axb) \leq \max \{f(a), f(b)\} \leq \max \{\alpha, \beta\} \leq \max \{\alpha, \beta, \gamma\}.$$ 

Thus $(a, \alpha) \circ (x, \gamma) \circ (b, \beta) \in [A \times \Delta]_f$. Hence $[A \times \Delta]_f$ is a generalized bi-ideal of $S \times \Delta$.

(ii ⇒ iii) It is obvious.

(iii ⇒ iv) Suppose that $\alpha \in (Im f)^f(axb)$ and $\alpha \notin (Im f)^f(a) \cup (Im f)^f(b)$ for some $a, b, x \in S$ and $\alpha \in [0,1]$. Then $\max \{f(a), f(b)\} < \alpha \leq f(axb)$. Since $(a, f(a)), (b, f(b)) \in [S \times Im f]_f$, $(x, f(a)) \in S \times Im f$ and the statement (iii), we have $(a, f(a)) \circ (x, f(a)) \circ (b, f(b)) \in [S \times Im f]_f$. Thus $f(axb) \leq \max \{f(a), f(b)\}$. It is a contradiction. Hence $(Im f)^f(axb) \subseteq (Im f)^f(a) \cup (Im f)^f(b)$ for all $a, b, x \in S$. (iv ⇒ i) It is straightforward.

By using and applying Theorem 4.1, we get Corollary 4.2.

**Corollary 4.2.** Let $f$ be a fuzzy subset of a semigroup $S$. Then the following statements are equivalent.

(i) $f$ is an anti-fuzzy generalized bi-ideal of $S$.

(ii) $[S \times [0,1]]_f$ is either empty or a generalized bi-ideal of $S \times [0,1]$.

(iii) $[S \times Im f]_f$ is a generalized bi-ideal of $S \times Im f$.

(iv) $[S \times [0,1]]_f$ is a generalized bi-ideal of $S \times [0,1]$.

**Example 4.3.** Let $S = \{a, b, c, d\}$ be the semigroup under the same binary operation in Example 3.3. Let $f$ be a fuzzy subset of $S$ such that $f(a) = 0.3$, $f(b) = 0.5$, $f(c) = 0.4$, $f(d) = 0.6$. Then $[S \times Im f]_f = \{(a, 0.3), (a, 0.4), (a, 0.5), (a, 0.6), (b, 0.5), (b, 0.6), (c, 0.4), (c, 0.5), (c, 0.6), (d, 0.6)\}$ is a generalized bi-ideal of $S \times Im f$. By Corollary 4.2 (i ⇒ iii), we get $f$ is an anti-fuzzy generalized bi-ideal of $S$.

**Proposition 4.4.** Let $f$ be a fuzzy subset of a semigroup $S$. Then $[S \times [0,1]]_f$ is a generalized bi-ideal of $S \times [0,1]$ if and only if $f$ is a generalized bi-ideal of $AFP(S)$.

**Proof.** It is straightforward.

**Theorem 4.5.** Let $f$ be a fuzzy subset of a semigroup $S$. Then $f$ is an anti-fuzzy generalized bi-ideal of $S$ if and only if $f$ is either empty or a generalized bi-ideal of $AFP(S)$.

**Proof.** It follows from Corollary 4.2 (i ⇒ ii) and Proposition 4.4.
In the following theorem, we characterize anti-fuzzy generalized bi-ideal of a semigroup $S$ by chain of generalized bi-ideals of $S$.

**Theorem 4.6.** Let $f$ be a fuzzy subset of a semigroup $S$. Then $f$ is an anti-fuzzy generalized bi-ideal of $S$ if and only if there exists the unique chain $\{A_\alpha \mid \alpha \in \text{Im} f\}$ of generalized bi-ideals of $S$ such that

i) $f^{-1}(\alpha) \subseteq A_\alpha$ for all $\alpha \in \text{Im} f$ and

ii) for all $\alpha, \beta \in \text{Im} f$, if $\alpha < \beta$ then $A_\alpha \subset A_\beta$ and $A_\alpha \cap f^{-1}(\beta) = \emptyset$.

**Proof.** $(\Rightarrow)$ Choose $A_\alpha = L(f : \alpha)$ for all $\alpha \in \text{Im} f$. By Proposition 2.4(iv), Proposition 2.6 and Theorem 4.3(i $\Rightarrow$ iii), we get $\{A_\alpha \mid \alpha \in \text{Im} f\}$ is a chain of generalized bi-ideals of $S$ satisfying the conditions i) and ii). For the proof of uniqueness, it is similar to the proof of Theorem 3.6.

$(\Leftarrow)$ Let $(a, \alpha), (b, \beta) \in [S \times \text{Im} f]_f$ and $(x, \gamma) \in S \times \text{Im} f$. Then max{$\alpha, \beta, \gamma$} $\in \text{Im} f$ and

$$\text{max}\{f(a), f(b)\} \leq \text{max}\{\alpha, \beta\} \leq \text{max}\{\alpha, \beta, \gamma\}.$$

Suppose that max{$\alpha, \beta, \gamma$} $< f(axb)$. By the condition ii), we get $A_{\text{max}\{\alpha, \beta, \gamma\}} \cap f^{-1}(f(axb)) = \emptyset$. Since $f(a) \leq \text{max}\{\alpha, \beta, \gamma\}$ and by the conditions i) and ii), we have

$$a \in f^{-1}(f(a)) \subseteq A_f(a) \subseteq A_{\text{max}\{\alpha, \beta, \gamma\}}.$$

Similarly, we have $b \in A_{\text{max}\{\alpha, \beta, \gamma\}}$. Since $A_{\text{max}\{\alpha, \beta, \gamma\}}$ is a generalized bi-ideal of $S$, we have $axb \in A_{\text{max}\{\alpha, \beta, \gamma\}}$. Then $axb \in A_{\text{max}\{\alpha, \beta, \gamma\}} \cap f^{-1}(f(axb)) = \emptyset$. It is a contradiction. Thus $f(axb) \geq \text{max}\{\alpha, \beta, \gamma\}$. Hence $(a, \alpha) \circ (x, \gamma) \circ (b, \beta) \in [S \times \text{Im} f]_f$. Therefore $[S \times \text{Im} f]_f$ is a generalized bi-ideal of $S \times \text{Im} f$. By Corollary 4.2(iii $\Rightarrow$ i), we get $f$ is an anti-fuzzy generalized bi-ideal of $S$.

In the proof of Theorem 4.6, the unique chain of generalized bi-ideals of $S$, satisfying conditions i) and ii), is the set $\{L(f : \alpha) \mid \alpha \in \text{Im} f\}$. Next, we consider one formula of an anti-fuzzy generalized bi-ideal $f$ of $S$ where $\text{Im} f$ is finite.

**Corollary 4.7.** Let $f$ be a fuzzy subset of a semigroup $S$ and $\text{Im} f = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ such that $\alpha_1 < \alpha_2 < \ldots < \alpha_n$. Then $f$ is an anti-fuzzy generalized bi-ideal of $S$ if and only if $\{L(f : \alpha_i) \mid i \in \{1, 2, \ldots, n\}\}$ is the chain of generalized bi-ideals of $S$ such that

$$f(x) = \begin{cases} 
\alpha_n & \text{if } x \in L(f : \alpha_n) \setminus L(f : \alpha_{n-1}) \\
\alpha_{n-1} & \text{if } x \in L(f : \alpha_{n-1}) \setminus L(f : \alpha_{n-2}) \\
\vdots \\
\alpha_2 & \text{if } x \in L(f : \alpha_2) \setminus L(f : \alpha_1) \\
\alpha_1 & \text{if } x \in L(f : \alpha_1)
\end{cases}$$

for all $x \in S$.

**Proof.** Apply Theorem 4.6.

\[\square\]
Corollary 4.8. Let \( f \) be a fuzzy subset of a semigroup \( S \) and \( \text{Im} f \subseteq \Delta \subseteq [0,1] \). The following statements are equivalent.

(i) \( f \) is an anti-fuzzy generalized bi-ideal of \( S \).

(ii) There exists a generalized bi-ideal \( R \) of \( S \times \Delta \) such that \( L(R : \alpha) = L(f : \alpha) \) for all \( \alpha \in \Delta \).

(iii) \( L(f : \alpha) \) is either empty or a generalized bi-ideal of \( S \) for all \( \alpha \in \Delta \).

Proof. (i \( \Rightarrow \) ii) Choose \( R = [S \times \Delta]_f \) and use Theorem 4.1(i) and Proposition 2.4(iv).

(ii \( \Rightarrow \) iii) It follows from Proposition 2.6.

(iii \( \Rightarrow \) i) Apply Theorem 4.6. \( \square \)

In the following two results, we characterize anti-fuzzy bi-ideal of a semigroup \( S \) by using the certain subsets of \( S, [0,1], \text{AFP}(S) \) and \( S \times [0,1] \).

Theorem 4.9. Let \( f \) be a fuzzy subset of a semigroup \( S \). Then the following statements are equivalent.

(i) \( f \) is an anti-fuzzy bi-ideal of \( S \).

(ii) \( [A \times \Delta]_f \) is either empty or a bi-ideal of \( S \times \Delta \) for every bi-ideal \( A \) of \( S \) and every subset \( \Delta \) of \( [0,1] \).

(iii) \( [S \times \Delta]_f \) is a bi-ideal of \( S \times \Delta \) where \( \text{Im} f \subseteq \Delta \subseteq [0,1] \).

(iv) \( \tilde{f} \) is either empty or a bi-ideal of \( \text{AFP}(S) \).

(v) There exists the unique chain \( \{A_{\alpha} \mid \alpha \in \text{Im} f\} \) of bi-ideals of \( S \) such that

\[
\begin{align*}
\text{a)} \quad f^{-1}(\alpha) & \subseteq A_{\alpha} \text{ for every } \alpha \in \text{Im} f
\text{b)} \quad \text{for every } \alpha, \beta \in \text{Im} f, \text{ if } \alpha < \beta \text{ then } A_{\alpha} \subseteq A_{\beta} \text{ and } A_{\alpha} \cap f^{-1}(\beta) = \emptyset.
\end{align*}
\]

(vi) Choosing \( \text{Im} f \subseteq \Delta \subseteq [0,1] \), we have \( L(f : \alpha) \) is either empty or a bi-ideal of \( S \) for every \( \alpha \in \Delta \).

(vii) Choosing \( \text{Im} f \subseteq \Delta \subseteq [0,1] \), there exists a bi-ideal \( R \) of \( S \times \Delta \) such that \( L(R : \alpha) = L(f : \alpha) \) for every \( \alpha \in \Delta \).

(viii) \( (\text{Im} f)^{f(a \times b)} \cup (\text{Im} f)^{f(ab)} \subseteq (\text{Im} f)^{f(a)} \cup (\text{Im} f)^{f(b)} \) for every \( a, b, x \in S \).

Corollary 4.10. Let \( f \) be a fuzzy subset of a semigroup \( S \) and \( \text{Im} f = \{\alpha_1, \alpha_2, ..., \alpha_n\} \) such that \( \alpha_1 < \alpha_2 < ... < \alpha_n \). Then \( f \) is an anti-fuzzy bi-ideal of \( S \) if and only if \( \{L(f : \alpha_{i}) \mid i \in \{1,2,...,n\}\} \) is the chain of bi-ideals of \( S \) such that

\[
f(x) = \begin{cases} 
\alpha_n & \text{if } x \in L(f : \alpha_n) \setminus L(f : \alpha_{n-1}) \\
\alpha_{n-1} & \text{if } x \in L(f : \alpha_{n-1}) \setminus L(f : \alpha_{n-2}) \\
& \ldots \\
\alpha_2 & \text{if } x \in L(f : \alpha_2) \setminus L(f : \alpha_1) \\
\alpha_1 & \text{if } x \in L(f : \alpha_1)
\end{cases}
\]

for all \( x \in S \).
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References


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