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Mathematical Simulation of a Groundwater Management in a Drought Area Using an Implicit Finite Difference Method

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Abstract : The groundwater management is required to solve the problem of lack water resources in many drought areas for agricultural usage. In this study, we propose a groundwater flow model and a groundwater management model that provide the pumping rates and the injection rates respectively. The groundwater model is providing the hydraulic head that gives the groundwater level. The implicit finite difference method is used to approximate the groundwater flow directions. The objective of groundwater flow management model is the minimum cost of injection rates. These are then subjected to optimal management of the water injection stations to achieve minimum cost. The numerical experiments are also given.

Keywords : groundwater management; groundwater model; implicit method.

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1 Introduction

Groundwater modeling is a powerful tool for water resources management, groundwater protection and remediation. The models are decided by maker to predict the behavior of a groundwater system prior to implementation of a remediation plan. The significance of the utilization of water resources continues to grow due to the increasing require of water for irrigation as well as drinking, agriculture, commercial and industrial proposes. Although, the amount of groundwater resources have been decaying due population growth, uncontrolled and unplanned urbanization, industrialization, and agricultural activities. Hence, the sustainable management planning must be developed for the groundwater systems. The management planning have to limited in the case of legal well drilling and limited-pumpings. On the other hand, the partial differential equations governing the system is solved by model. The groundwater model are solved by analytical and numerical solution techniques. Analytical methods is not suitable application to require much data and their application is limited to simple problem. Numerical methods can solve more complex problem than analytical solutions. Now, rapid development of computer processors and increasing speed, numerical modeling has become tools more effective an easy to use. The finite difference method and the finite element are most tools used numerical modeling approaches. Each method has its advantages and limitations. Selecting numerical modeling approach depend on the problem of concern and the objectives of modeling. Most of groundwater modeling has the aquifer systems with the heterogeneous structure. In the case of the steady-state groundwater model solutions can be obtained by the simply basic techniques. On the other hand, the case of transient ground water model is solved by the advanced techniques due to the difficult in terms of time dimension in the governing equations. Theoretical solution of the governing equation of groundwater model need general assumptions such as ideal solution domains and homogeneous geometries.

Groundwater models can be simple, analytical solutions of one-dimensional is like solutions of spreadsheet models [1], for very complicated three-dimensional models. It is always introduced to start with a simple model, as long as the model concept satisfies modeling objectives, and then the model complex can be increased [2]. The finite difference [3–5] and finite elements [6–8] methods are the most popular numerical solution techniques. A simulation/optimization model is proposed for the identification of unknown groundwater well locations and pumping rates for two-dimensions and model is combined with genetic algorithm based optimization model [9]. A useful spreadsheet for two and three dimensional steady-state and transient groundwater numerical simulation is proposed in [10].

In this research, the objective of groundwater flow management model is the minimum cost of injection rates. These are then subjected to optimal management of the water injection stations to achieve minimum cost. The numerical experiments are also given.

2 The Governing Equation of Groundwater Steady-Flow Model

Mathematical models of groundwater flow are all based on the water balance principle. The combination between the mass balance equation and Darcy's law produce the governing equation for groundwater flow. The general equation that governs two-dimensional groundwater steady-flow in isotropic, homogeneous porous media and vetically average [11],

$$\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} = 0, \tag{2.1}$$

where $H(x, y)$ is hydraulic head (metre). We will introduces the affected term as sources and sinks due to the external inputs and outputs. Consequently, the Eq.(2.1) becomes

$$\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} + W = 0, \tag{2.2}$$

where W is sinks and/or sources (1/day). The boundary conditions are specified, for all $(x, y) \in [0, L] \times [0, M]$ where L and M are positive constants which represent the dimension of the regtangular domain.

$$\frac{\partial H}{\partial n} = B_N \qquad \text{for all } 0 \le x \le L \text{ and } y = M, \tag{2.3}$$

$$\frac{\partial H}{\partial n} = B_S \qquad \text{for all } 0 \le x \le L \text{ and } y = 0, \tag{2.4}$$

$$H = B_W \qquad \text{for all } x = 0 \text{ and } 0 \le x \le M, \tag{2.5}$$

$$H = B_E \qquad \text{for all } x = L \text{ and } 0 \le x \le M. \tag{2.6}$$

and the source terms W that represented by the rate of pumping well in each point,

$$W(x_s, y_s) = Q(x_s, y_s) = Q_s \qquad \text{for all } s = 1, 2, 3, ..., p, \tag{2.7}$$

where s is a number of pumping wells.

3 Numerical Techniques

Due to the groundwater steady-flow model is independent of time, the implicit finite difference method is used to solve the groundwater flow model. The method is suitable for the model due to the linear systems of equations will be constructed. It is possible to implement with the groundwater management model. Taking the central difference scheme in space into terms of equation (2.2), then

$$\frac{\partial^2 H}{\partial x^2} \approx \frac{H_{i-1,j} - 2H_{i,j} + H_{i+1,j}}{(\Delta x)^2}, \tag{3.1}$$

$$\frac{\partial^2 H}{\partial y^2} \approx \frac{H_{i,j-1} - 2H_{i,j} + H_{i,j+1}}{(\Delta y)^2}, \tag{3.2}$$

$$W_{i,j} = \pm \frac{Q_{i,j}}{\Delta x \Delta y H_{i,j}}. \tag{3.3}$$

Substituting Eq.(3.1) - (3.3) into Eq.(2.2), for $1 < i < I-1$ and $1 < j < J-1$,

$$-(2a+2b) H_{i,j} + aH_{i-1,j} + aH_{i+1,j} + bH_{i,j-1} + bH_{i,j+1} = -W_{i,j}, \tag{3.4}$$

where $a = \frac{1}{(\Delta x)^2}$ and $b = \frac{1}{(\Delta y)^2}$. For $i = 1$ and $j = 1$, substituting the unknown value on the south boundary by forward difference approximation, $H_{1,0} = H_{1,1}$, into Eq.(3.4), we obtain

$$-(2a+b) H_{1,1} + aH_{2,1} + bH_{1,2} = -W_{1,1} - aB_W. \tag{3.5}$$

For $1 < i < I-1$ and $j = 1$, substituting the unknown value on the south boundary by forward difference approximation, $H_{i,0} = H_{i,1}$, into Eq.(3.4), we obtain

$$-(2a+b) H_{i,1} + aH_{i-1,1} + aH_{i+1,1} + bH_{i,2} = -W_{i,1}. \tag{3.6}$$

For $i = I-1$ and $j = 1$, substituting the unknown value on the south boundary by forward difference approximation,$H_{I-1,0} = H_{I-1,1}$, into Eq.(3.4), we obtain

$$-(2a+b) H_{I-1,1} + aH_{I-2,1} + bH_{I-1,2} = -W_{I-1,1} - aB_E. \tag{3.7}$$

For $i = 1$ and $j = J-1$, substituting the unknown value on the north boundary by backward difference approximation, $H_{1,J} = H_{1,J-1}$, into Eq.(3.4), we obtain

$$-(2a+b) H_{1,J-1} + aH_{2,J-1} + bH_{1,J-2} = -W_{1,J-1} - aB_W. \tag{3.8}$$

For $1 < i < I-1$ and $j = J-1$, substituting the unknown value on the north boundary by backward difference approximation, $H_{i,J} = H_{i,J-1}$, into Eq.(3.4), we obtain

$$-(2a+b) H_{i,J-1} + aH_{i-1,J-1} + aH_{i+1,J-1} + bH_{i,J-2} = -W_{i,J-1}. \tag{3.9}$$

For $i = I-1$ and $j = J-1$, substituting the unknown value on the north boundary by backward difference approximation, $H_{I-1,J} = H_{I-1,J-1}$, into Eq.(3.4), we obtain

$$-(2a+b) H_{I-1,J-1} + aH_{I-2,J-1} + bH_{I-1,J-2} = -W_{I-1,J-1} - aB_E. \tag{3.10}$$

The equations (3.4) - (3.10) can be written in matrix form as follow,

$$AH = B, \tag{3.11}$$

where

$$A = \begin{bmatrix} A_1 & A_3 & & & \\ A_3 & A_2 & A_3 & & \\ & \ddots & \ddots & \ddots & \\ & & A_3 & A_2 & A_3 \\ & & & A_3 & A_1 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} -(2a+b) & a & & & \\ a & -(2a+b) & a & & \\ & \ddots & \ddots & \ddots & \\ & & a & -(2a+b) & a \\ & & & a & -(2a+b) \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -(2a+2b) & a & & & \\ a & -(2a+2b) & a & & \\ & \ddots & \ddots & \ddots & \\ & & a & -(2a+2b) & a \\ & & & a & -(2a+2b) \end{bmatrix},$$

$$A_3 = \begin{bmatrix} b & & & & \\ & b & & & \\ & & \ddots & & \\ & & & b & \\ & & & & b \end{bmatrix},$$

$$H = \begin{bmatrix} H_{1,1} \\ H_{2,1} \\ \vdots \\ H_{I-2,J-1} \\ H_{I-1,J-1} \end{bmatrix},$$

and

$$B = \begin{bmatrix} -W_{1,1} - aB_W \\ -W_{2,1} \\ \vdots \\ -W_{I-2,1} \\ -W_{I-1,1} - aB_W \\ -W_{1,2} - aB_W \\ \vdots \\ -W_{i,j} \\ \vdots \\ -W_{I-1,J-2} - aB_E \\ -W_{1,J-1} - aB_W \\ -W_{2,J-1} \\ \vdots \\ -W_{I-2,J-1} \\ -W_{I-1,J-1} - aB_E \end{bmatrix}.$$

4 A Groundwater Management Model

The objective function is the total cost of all pump injection in the considered system,

$$C = \sum_{s=1}^{m} W_s Q_s, \tag{4.1}$$

where s is the number of pumping wells, W_s is the cost of water pumping for each well s ($Baht/m^3$) and Q_s is the injection rate for each well s (m^3/day). The constraint are

$$H_s \leq H_{ST_s}, \tag{4.2}$$

where H_s are the hydraulic head at monitoring point that measuring water requirement for each zone s and H_{ST_s} are the standard water requirement for each zone s. The upper bound of the injection rate for each pumping well are,

$$Q_s \leq Q_{\max_s}, \tag{4.3}$$

the lower bound of the injection rate for each pumping well are,

$$Q_s \geq Q_{\min_s}, \tag{4.4}$$

and the hydraulic head at monitoring point s and the injection rate at pumping wells are non-negative, that are

$$H_s,\, Q_s \geq 0, \tag{4.5}$$

where $Q_{\min_s}$ and $Q_{\max_s}$ are the lower and upper bounds of the water injection rate for each point s, respectively. The optimal cost of them is solved by using the simplex method.

5 Numerical Experiments

We consider the area width 2400 m and length 2400 m which is between a pair of two rivers. The area is meshed by 100 grids points with grid space is 240 m. The boundary conditions of the area are specified Eqs.(2.3) - (2.6) where $B_N = 0$, $B_S = 0$, $B_W = 20$ and $B_E = 19$. The four injection wells are injecting the water to the underground. The injection wells have the difference lower bound of injection rates, difference upper bound of injection rates and the difference cost of injection wells for each zone as Table 1. There are 8 monitoring point for measuring water requirement. The hydraulic head at monitoring point have the difference standard water requirement each point as Table 2.

Table 1: The injection rates and the cost of each pumping wells.

Position coordinate (x, y)	$Q(1440m, 480m)$	$Q(480m, 720m)$
Lower (m^3/day)	165	175
Upper (m^3/day)	250	230
Cost $(Bath/m^3)$	1.5	1.9
Position coordinate (x, y)	$Q(1680m, 1440m)$	$Q(720m, 1680m)$
Lower (m^3/day)	180	190
Upper (m^3/day)	300	270
Cost $(Bath/m^3)$	1.8	1.6

Table 2: The standard water requirement (SWR) for each monitoring points.

Position coordinate (x, y) (m, m)	(240,240)	(960,480)	(1920,720)	(720,1200)
SWR (m)	22	23	24	24
Position coordinate (x, y) (m, m)	(1200,1200)	(240,1440)	(1200,1920)	(1920,1920)
SWR (m)	25	23	25	24

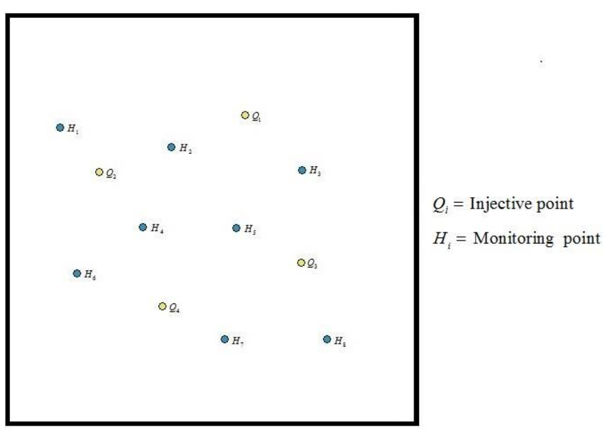

Figure 1: Simulation of groundwater management.

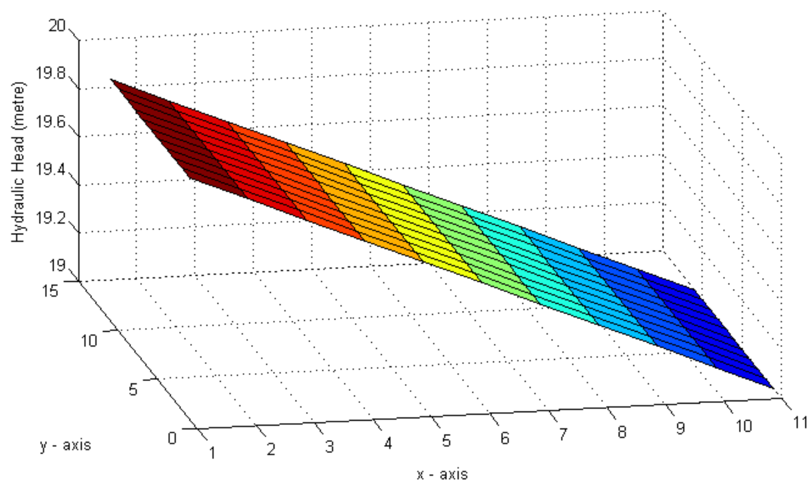

Figure 2: The surface graph before optimal control of cost.

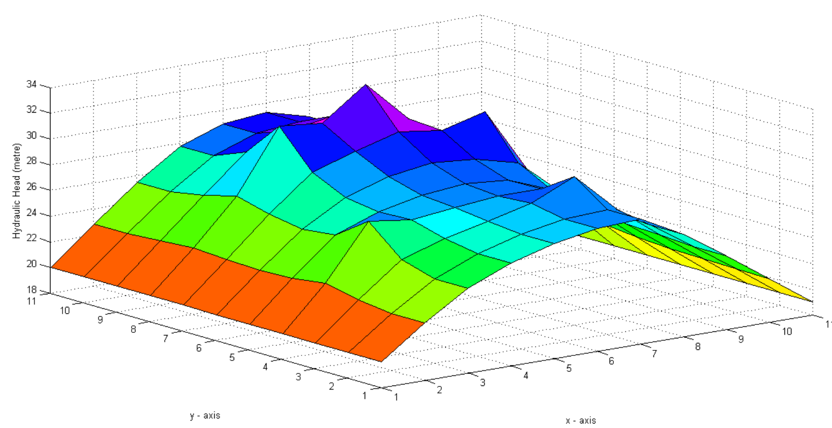

Figure 3: The surface graph after optimal control of cost.

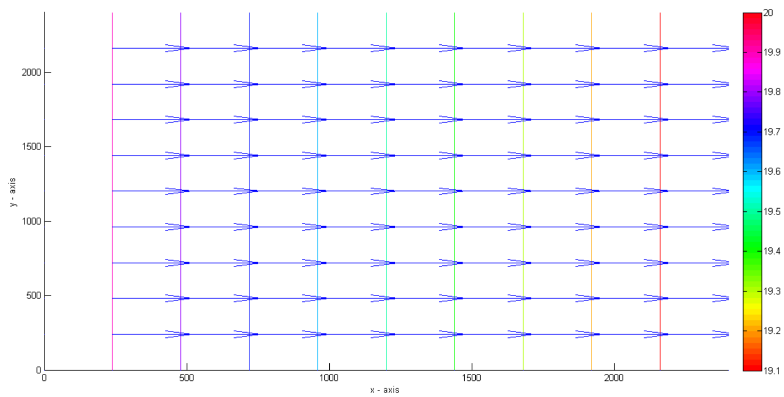

Figure 4: The contour graph and direction flow before optimal control of cost.

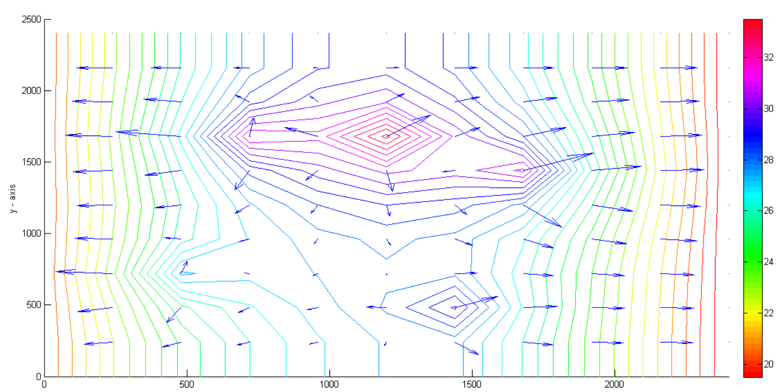

Figure 5: The contour graph and direction flow after optimal control of cost.

Table 3: Table of the hydraulic head before optimal control of cost.

y, x	0	240	480	720	960	1200
0	20.0000	19.9000	19.8000	19.7000	19.6000	19.5000
240	20.0000	19.9000	19.8000	19.7000	19.6000	19.5000
480	20.0000	19.9000	19.8000	19.7000	19.6000	19.5000
720	20.0000	19.9000	19.8000	19.7000	19.6000	19.5000
960	20.0000	19.9000	19.8000	19.7000	19.6000	19.5000
1200	20.0000	19.9000	19.8000	19.7000	19.6000	19.5000
1440	20.0000	19.9000	19.8000	19.7000	19.6000	19.5000
1680	20.0000	19.9000	19.8000	19.7000	19.6000	19.5000
1920	20.0000	19.9000	19.8000	19.7000	19.6000	19.5000
2160	20.0000	19.9000	19.8000	19.7000	19.6000	19.5000
2400	20.0000	19.9000	19.8000	19.7000	19.6000	19.5000
y, x	1440	1680	1920	2160	2400	
0	19.4000	19.3000	19.2000	19.1000	19.0000	
240	19.4000	19.3000	19.2000	19.1000	19.0000	
480	19.4000	19.3000	19.2000	19.1000	19.0000	
720	19.4000	19.3000	19.2000	19.1000	19.0000	
960	19.4000	19.3000	19.2000	19.1000	19.0000	
1200	19.4000	19.3000	19.2000	19.1000	19.0000	
1440	19.4000	19.3000	19.2000	19.1000	19.0000	
1680	19.4000	19.3000	19.2000	19.1000	19.0000	
1920	19.4000	19.3000	19.2000	19.1000	19.0000	
2160	19.4000	19.3000	19.2000	19.1000	19.0000	
2400	19.4000	19.3000	19.2000	19.1000	19.0000	

Table 4: Table of the hydraulic head after optimal control of cost.

y, x	0	240	480	720	960	1200
0	20.0000	22.4668	24.6064	25.9931	26.9362	27.5723
240	20.0000	22.4668	24.6064	25.9931	26.9362	27.5723
480	20.0000	22.7940	25.3593	26.4366	27.2432	28.0755
720	20.0000	23.3498	27.6001	27.1510	27.5245	27.9041
960	20.0000	23.0050	25.7905	27.0426	27.7999	28.1345
1200	20.0000	22.8798	25.5142	27.4292	28.4979	29.0074
1440	20.0000	23.0000	25.9574	28.6619	29.7552	30.6658
1680	20.0000	23.1628	26.6534	31.5058	31.1952	33.6724
1920	20.0000	22.9979	25.9875	28.7726	29.8473	30.4711
2160	20.0000	22.8414	25.5262	27.7496	28.9502	29.2537
2400	20.0000	22.8414	25.5262	27.7496	28.9502	29.2537
y, x	1440	1680	1920	2160	2400	
0	27.7052	25.9609	23.7603	21.4090	19.0000	
240	27.7052	25.9609	23.7603	21.4090	19.0000	
480	29.5824	26.4172	23.9110	21.4667	19.0000	
720	27.8818	26.2143	24.0000	21.5467	19.0000	
960	27.8265	26.5582	24.3280	21.7200	19.0000	
1200	28.7315	27.8639	25.0340	22.0051	19.0000	
1440	30.2281	31.1318	25.9390	22.2665	19.0000	
1680	30.3834	28.5222	25.3236	22.1220	19.0000	
1920	29.1111	27.2499	24.7114	21.8977	19.0000	
2160	28.3398	26.6547	24.3745	21.7574	19.0000	
2400	28.3398	26.6547	24.3745	21.7574	19.0000	

Table 5: The optimal injection rates of minimum cost in the systems.

Position Coordinate (x, y)	Injection Rate (m^3/day)
$Q(1440m, 480m)$	165
$Q(480m, 720m)$	175
$Q(1680m, 1440m)$	239.48
$Q(720m, 1680m)$	214.81

6 Discussion

The injective point and monitoring point are locate in the considered area as show in Figure 1. The monitored hydraulic head without and within controlled cost are shown in Figure 2 and Figure3 respectively. The vector fields of groundwater flow velocity between both case are are shown in Figure 4 and Figure 5 respectively. The hydraulic head before and after control of cost are shown in Table 3 and Table 4 respectively. The optimum injection rate at minimum cost of groundwater control is shown in Table 5.

7 Conclusion

We have established the groundwater management model, First, we will measure hydraulic head from the groundwater steady-flow model by using an implicit finite difference method. It will turn out that the system of linear equations is generated. We employ the system of linear equations to construct the groundwater management model to investigate the optimal least cost of the water injections in the system under the limitation conditions were required. Although, the water requirement and the injection cost of each monitoring points are unequal among injective stations. These are then subjected to the optimal presure of the groundwater injection station to achieve minimum cost. We have established a simulation process by means of which hydraulic head levels can be increase to agreed requirement levels at least cost.

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