On the Hyers-Ulam-Rassias Stability of an $n$-Dimensional Additive Functional Equation

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Abstract: In this paper, we prove the Hyers-Ulam-Rassias stability of the following $n$-dimensional additive functional equation

$$f\left(\sum_{i=1}^{n} x_i\right) = \sum_{i=1}^{n} f(x_i) + \sum_{i=1}^{n} f(x_i - x_{i-1})$$

where $x_0 \equiv x_n$ and $n > 1$.

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1 Introduction

In 1940 S.M. Ulam [8] proposed the famous Ulam stability problem of linear mappings. In 1941 D.H. Hyers [1] considered the case of approximately additive mappings $f : E \to E'$ where $E$ and $E'$ are Banach spaces and $f$ satisfies inequality $\|f(x + y) - f(x) - f(y)\| \leq \varepsilon$ for all $x, y \in E$. It was shown that the limit $L(x) = \lim_{n \to \infty} 2^{-n} f(2^n x)$ exists for all $x \in E$ and that $L : E \to E'$ is the unique additive mapping satisfying $\|f(x) - L(x)\| \leq \varepsilon$. The stability problem of various functional equations has been studied by a number of authors ([2]-[7]) since then.

In this paper, we propose an $n$-dimensional additive functional equation

$$f\left(\sum_{i=1}^{n} x_i\right) = \sum_{i=1}^{n} f(x_i) + \sum_{i=1}^{n} f(x_i - x_{i-1})$$

where $x_0 \equiv x_n$ and $n > 1$, and investigate its Hyers-Ulam-Rassias stability.

2 The Solution

The following theorem establishes the equivalence of the proposed functional equation and the Cauchy functional equation.
**Theorem 2.1.** Let $X$ and $Y$ be vector spaces. A mapping $f : X \to Y$ satisfies the functional equation

$$f \left( \sum_{i=1}^{n} x_i \right) = \sum_{i=1}^{n} f(x_i) + \sum_{i=1}^{n} f(x_i - x_{i-1}) \quad (2.1)$$

where $x_0 \equiv x_n$ and $n > 1$, for all $x_1, x_2, \ldots, x_n \in X$ if and only if it satisfies the Cauchy functional equation

$$f(x + y) = f(x) + f(y) \quad (2.2)$$

for all $x, y \in X$.

**Proof.** Suppose a mapping $f : X \to Y$ satisfies the Cauchy functional equation. Then it is straightforward to show that

$$f \left( \sum_{i=1}^{n} x_i \right) = \sum_{i=1}^{n} f(x_i),$$

and

$$\sum_{i=1}^{n} f(x_i - x_{i-1}) = \sum_{i=1}^{n} (f(x_i) - f(x_{i-1})) = 0.$$

Hence, $f$ satisfies (2.1).

Suppose a mapping $f : X \to Y$ satisfies (2.1). Setting $x_1 = x_2 = \cdots = x_n = 0$, we can see that $f(0) = 0$. Settings $x_1 = x$ and $x_2 = x_3 = \cdots = x_n = 0$, we have

$$f(x) = f(x) + (f(x) + f(-x)),$$

which shows that $f(-x) = -f(x)$ for all $x \in X$.

If $n = 2$, then (2.1) reduces to

$$f(x + y) = f(x) + f(y) + f(x - y) + f(y - x).$$

Noting the oddness of $f$, the above equation simplifies to the Cauchy functional equation.

If $n > 2$, then we substitute $(x_1, x_2, \ldots, x_n) = (x, y, 0, \ldots, 0)$ and (2.1) becomes

$$f(x + y) = f(x) + f(y) + (f(x - y) + f(y) + f(-x)) .$$

Again, by the oddness of $f$, the above equation reduces to

$$f(x + y) - f(x - y) = 2f(y).$$

Observing that the right-hand side is independent of $x$, we have

$$f \left( \left( x + \frac{y}{2} \right) + \frac{y}{2} \right) - f \left( \left( x + \frac{y}{2} \right) - \frac{y}{2} \right) = 2f \left( \frac{y}{2} \right) = f \left( \frac{y}{2} + \frac{y}{2} \right) - f \left( \frac{y}{2} - \frac{y}{2} \right) ,$$

or $f(x + y) = f(x) + f(y)$ as desired. This completes the proof. \qed
On the Hyers-Ulam-Rassias Stability of an $n$-Dimensional Additive Functional Equation

3 The Hyers-Ulam-Rassias Stability

The following theorem gives a condition for which a linear mapping exists near an approximately linear mapping.

**Theorem 3.1.** Let $p \neq 1$ be a positive real number and $\theta \geq 0$ be a real number. Let $X$ be a real vector space and $Y$ be a Banach space. If a mapping $f : X \to Y$ satisfies the inequality

$$\left\| \sum_{i=1}^{n} f(x_i) + \sum_{i=1}^{n} f(x_i - x_{i-1}) - f \left( \sum_{i=1}^{n} x_i \right) \right\| \leq \delta + \theta \sum_{i=1}^{n} \|x_i\|^p$$  \hspace{1cm} (3.1)

for all $x_1, x_2, \ldots, x_n \in X$, and $\delta = 0$ when $p > 1$, then there exists a unique linear mapping $L : X \to Y$ such that $L$ satisfies the functional equation given by (2.1), and satisfies the inequality

$$\|f(x) - L(x)\| \leq \frac{4n}{2n-1} \delta + \frac{3\theta}{2 - 2^p} \|x\|^p$$  \hspace{1cm} (3.2)

for all $x \in X$. The mapping $L$ is given by

$$L(x) = \begin{cases} \lim_{m \to \infty} 2^{-m} f(2^m x) & \text{if } 0 < p < 1 \\ \lim_{m \to \infty} 2^m f(2^{-m} x) & \text{if } p > 1 \end{cases}$$  \hspace{1cm} (3.3)

for all $x \in X$.

**Proof.** Setting $x_1 = x_2 = \cdots = x_n = 0$ in (3.1), we have $\|(2n - 1)f(0)\| \leq \delta$. Thus,

$$\|f(0)\| \leq \frac{\delta}{2n - 1}.$$  

Setting $x_1 = x$ and $x_2 = x_3 = \cdots = x_n = 0$, we have

$$\|(f(x) + (n - 1)f(0)) + (f(x) + f(-x) + (n - 2)f(0)) - f(x)\| \leq \delta + \theta \|x\|^p,$$

which simplifies to

$$\|(2n - 3)f(0) + f(x) + f(-x)\| \leq \delta + \theta \|x\|^p.$$  

If $n > 2$, we set $x_1 = x_2 = x$ and $x_3 = x_4 = \cdots = x_n = 0$, then

$$\|(2f(x) + (n - 2)f(0)) + (f(x) + f(-x) + (n - 2)f(0)) - f(2x)\| \leq \delta + 2\theta \|x\|^p,$$

or simply

$$\|f(0) + ((2n - 3)f(0) + f(x) + f(-x)) + 2f(x) - f(2x)\| \leq \delta + 2\theta \|x\|^p,$$
Thus,
\[\|2f(x) - f(2x)\| \leq \|f(0)\| + (\delta + \theta\|x\|^p) + (\delta + 2\theta\|x\|^p) = \|f(0)\| + 2\delta + 3\theta\|x\|^p.\]

If \(n = 2\), we set \(x_1 = x_2 = x\) in (3.1), then
\[\|2f(x) + 2f(0) - f(2x)\| \leq \delta + 2\theta\|x\|^p.\]

It follows that
\[\|2f(x) - f(2x)\| \leq 2\|f(0)\| + \delta + 2\theta\|x\|^p.\]

Hence, we determine that
\[
\|2f(x) - f(2x)\| \leq 2\|f(0)\| + 2\delta + 3\theta\|x\|^p \\
\leq \left(\frac{2}{2n-1} + 2\right)\delta + 3\theta\|x\|^p \\
\leq \frac{4n}{2n-1}\delta + 3\theta\|x\|^p.
\]

We first consider the case when \(0 < p < 1\). Rewrite the above inequality as
\[\|f(x) - 2^{-1}f(2x)\| \leq \frac{4n}{4n-2}\delta + \frac{3\theta}{2}\|x\|^p. \tag{3.4}\]

For every positive integer \(m\),
\[
\|f(x) - 2^{-m}f(2^m x)\| = \left\| \sum_{i=0}^{m-1} 2^{-i} f(2^i x) - 2^{-i} f(2^{i+1} x) \right\| \\
\leq \sum_{i=0}^{m-1} \left\| 2^{-i} f(2^i x) - 2^{-i} f(2^{i+1} x) \right\| \\
= \sum_{i=0}^{m-1} 2^{-i} \left\| f(2^i x) - 2^{-1} f(2^i 2^i x) \right\|. 
\]

Applying (3.4) with appropriate values of \(x\)'s, we have
\[
\|f(x) - 2^{-m}f(2^m x)\| \leq \frac{4n}{4n-2}\delta \sum_{i=0}^{m-1} 2^{-i} + \frac{3\theta}{2} \|x\|^p \sum_{i=0}^{m-1} 2^{(p-1)}.\]

Consider the sequence \(\{2^{-m}f(2^m x)\}\). For all positive integers \(k < l\), we have
\[
\|2^{-k} f(2^k x) - 2^{-l} f(2^l x)\| = 2^{-k} \| f(2^k x) - 2^{-(l-k)} f(2^{l-k}, 2^k x) \| \\
\leq 2^{-k} \left( \frac{4n}{4n-2} \delta \sum_{i=0}^{l-k-1} 2^{-i} + \frac{3\theta}{2} \|2^k x\|^p \sum_{i=0}^{l-k-1} 2^{(p-1)} \right) \\
\leq \frac{4n}{4n-2} 2^{-k} \delta \sum_{i=0}^{\infty} 2^{-i} + \frac{3\theta}{2} 2^{-k(1-p)} \|x\|^p \sum_{i=0}^{\infty} 2^{(p-1)}.\]
On the Hyers-Ulam-Rassias Stability of an n-Dimensional Additive Functional Equation

The right-hand side of the above inequality approaches 0 as $k$ tends to infinity. Since $Y$ is a Banach space and $\{2^{-m}f(2^m x)\}$ is a Cauchy sequence, we let

$$L(x) = \lim_{m \to \infty} 2^{-m}f(2^m x).$$

It follows that

$$\|f(x) - L(x)\| \leq \frac{4n}{4n - 2} \delta \sum_{i=0}^{\infty} 2^{-i} \|x\|^p \sum_{i=0}^{\infty} 2^{i(p-1)}$$

$$= \frac{4n}{2n - 1} \delta + \frac{3\theta}{2 - 2^p} \|x\|^p.$$

From (3.1), we have

$$2^{-m} \left| \sum_{i=1}^{n} f(2^m x_i) + \sum_{i=1}^{n} f(2^m x_i - 2^m x_{i-1}) \right| = f \left( \sum_{i=1}^{n} 2^m x_i \right)$$

$$\leq 2^{-m} \left( \delta + \theta \sum_{i=1}^{n} \|2^m x_i\|^p \right) = 2^{-m} \delta + 2^{-m(1-p)} \theta \sum_{i=1}^{n} \|x_i\|^p.$$

As $m$ tends to infinity, the right-hand side of the above inequality approaches 0; hence,

$$\sum_{i=1}^{n} L(x_i) + \sum_{i=1}^{n} L(x_i - x_{i-1}) = L \left( \sum_{i=1}^{n} x_i \right)$$

which reveals that $L$ satisfies (2.1).

To prove the uniqueness of $L$, suppose there is a mapping $L' : X \to Y$ such that $L'$ satisfies (2.1) and the condition (3.2). Then, the linearity of (2.1) as asserted by Theorem 2.1 implies

$$\|L(x) - L'(x)\| = 2^{-m} \|L(2^m x) - L'(2^m x)\|$$

$$\leq 2^{-m} \|L(2^m x) - f(2^m x)\| + 2^{-m} \|L'(2^m x) - f(2^m x)\|$$

$$\leq 2^{-m} \cdot 2 \left( \frac{4n}{2n - 1} \delta + \frac{3\theta}{2 - 2^p} \|x\|^p \right).$$

The right-hand side of the above inequality approaches 0 as $m$ tends to infinity; so, we conclude that $L(x) = L'(x)$ for all $x \in X$.

The proof for the case when $p > 1$ starts by replacing (3.4) with

$$\|f(x) - 2f(2^{-1}x)\| \leq \frac{4n}{2n - 1} \delta + 3\theta \|2^{-1}x\|^p,$$

and the rest of the proof can be reproduced accordingly.

The following corollary gives the Hyers-Ulam stability of the functional equation given by (2.1).
Corollary 3.2. Let $X$ be a real vector space and $Y$ be a Banach space. If a mapping $f : X \to Y$ satisfies the inequality
\[
\left\| \sum_{i=1}^{n} f(x_i) + \sum_{i=1}^{n} f(x_i - x_{i-1}) - f \left( \sum_{i=1}^{n} x_i \right) \right\| \leq \delta
\]
for all $x_1, x_2, \ldots, x_n \in X$, then there exists a unique linear mapping $L : X \to Y$ such that $L$ satisfies the functional equation given by (2.1), and satisfies the inequality
\[
\|f(x) - L(x)\| \leq \frac{4n}{2n-1} \delta
\]
for all $x \in X$.

Proof. Letting $\theta = 0$ and $p = \frac{1}{2}$ in Theorem 3.1, we immediately obtain the desired result.

References


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