Endo-Regularity of Cycle Book Graphs

N. Pipattanajinda and Sr. Arworn

Abstract: A graph $G$ is endo-regular (endo-completely-regular, endo-orthodox) if the monoid of all endomorphisms on $G$ is regular (completely regular, orthodox). In this paper, we characterized endo-regular (endo-completely-regular, endo-orthodox) cycle book graphs.

Keywords: Cycle book graph, Endomorphism, Regular, Completely regular, Orthodox.

2000 Mathematics Subject Classification: 05C25; 05C38

1 Introduction and Preliminaries

In [2], W. Li characterized regular endomorphisms on arbitrary graphs. The characterizations of endo-regular and endo-orthodox connected bipartite graphs were explicitly found in [6] and [1], respectively. A characterization of endo-completely regular of even cycles in found in [4]. And in [5], J. Thamkeaw and Sr. Arworn showed that every odd cycle book graphs are endo-regular.

As usual we denote by $V(G)$ and $E(G)$ the vertex set and the edge set of the graph $G$, respectively, where $V(G) \neq \emptyset$ and $E(G) \subseteq \{\{u, v\} | u \neq v \in V(G)\}$. The distance between $u$ and $v$, $d(u, v)$ is the smallest length of $u - v$ path in $G$. The greatest distance between any two vertices of a connected graph $G$ is called the diameter of $G$ and is denoted by $diam(G)$. The graph with vertex set $\{0, 1, \ldots, n\}$ and edge set $\{\{0, 1\}, \{1, 2\}, \ldots, \{n - 1, n\}\}$ is called a path of length $n$, denoted by $P_n$. Therefore, the path $P_n$ has $n + 1$ vertices and $n$ edges. The graph with vertex set $\{0, 1, \ldots, n - 1\}$, such that $n \geq 3$, and edge set $\{\{i, i + 1\}| i = 0, 1, \ldots, n - 1\}$ (with addition modulo $n$) is called a cycle of length $n$, denoted by $C_n$. Therefore, the cycle $C_n$ has $n$ vertices and $n$ edges.

A (graph) homomorphism from a graph $G$ to a graph $H$ is a mapping $f : V(G) \rightarrow V(H)$ which preserves edges, i.e. $\forall u, v \in V(G), \{u, v\} \in E(G)$ implies $\{f(u), f(v)\} \in E(H)$. A homomorphism $f$ is called an isomorphism if $f$ is bijective and $f^{-1}$ is also a homomorphism. A homomorphism (resp. isomorphism)
phism) \( f \) from \( G \) to itself is called an endomorphism (resp. automorphism) of \( G \). The sets of all endomorphisms and automorphisms of \( G \) are denoted by \( \text{End}(G) \) and \( \text{Aut}(G) \), respectively.

An element \( a \) of a semigroup \( S \) (or a monoid \( S \)) is called an idempotent of \( S \) if \( a^2 = a \). A regular element of \( S \) is an element \( a \in S \) such that \( a = aa'a \) for some \( a' \in S \), such \( a' \) is called a pseudo inverse to \( a \). A semigroup \( S \) is called regular if every element of \( S \) is regular. An element \( a' \in S \) such that \( a = aa'a \) and \( a' = a'aa' \) is called an inverse to \( a \). A regular element \( a \) of \( S \) is called completely regular if there exists a pseudo inverse \( a' \) to \( a \) such that \( aa' = a'a \). In this case we call \( a' \) a commuting pseudo inverse to \( a \). A semigroup \( S \) is called completely regular if every element of \( S \) is completely regular. A regular semigroup \( S \) is called orthodox if the set of all idempotent elements of \( S \) (\( \text{Idpt}(S) \)) forms a semigroup under the operation of \( S \).

We called a graph \( G \) endo-regular (endo-completely-regular, endo-orthodox), if the monoid \( \text{End}(G) \) is regular (completely regular, orthodox).

The following lemmas are useful for this paper.

**Lemma 1.1.** [6] Let \( G \) be a connected bipartite graph. Then \( G \) is endo-regular if and only if \( G \) is one of following graphs:

1. completely bipartite graph \( K_{m,n} \), (including \( K_1, K_2 \), cycle \( C_4 \) and tree \( T \) with \( \text{diam}(T) = 2 \)).
2. tree \( T \) with \( \text{diam}(T) = 3 \).
3. cycle \( C_6 \) and \( C_8 \).
4. path with 5 vertices, i.e. \( P_5 \).

**Lemma 1.2.** [5] Every odd cycle book graph is endo-regular.

**Lemma 1.3.** [4] Every even cycle is not endo-completely-regular.

**Lemma 1.4.** [3] A semigroup \( S \) is completely regular if and only if \( S \) is a union of (disjoint) groups.

**Lemma 1.5.** [1] Let \( G \) be a bipartite graph. Then \( G \) is endo-orthodox if and only if \( G \) is one of the following graphs: \( K_1, K_2, P_2, P_3, C_4, 2K_1 \) and \( K_1 \cup K_2 \).

### 2 Endo-Regularity of Cycle Book Graphs

For each \( i = 1, 2, \ldots, m \), let \( G_i \) be a graph which isomorphic to cycle \( C_n \) with the following vertex set \( V(G_i) = \{0_i, 1_i, 2_i, \ldots, (n - 1)_i\} \), and edge set \( E(G_i) = \{\{x_i, (x + 1)_i\} | x = 0, 1, 2, \ldots, n - 1 \} \) where \( 0_i = 1_i = 1 \) for all \( i = 1, \ldots, m \) and \( + \) is the addition modulo \( n \). (Note that for all \( i \neq j \), \( V(G_i) \cap V(G_j) = \{0, 1\} \)).

Let \( B_n(m) \) be a \( C_n \) book graph of \( m \) page with the vertex set \( V(B_n(m)) = \bigcup_{i=1}^{m} V(G_i) \) and the set \( E(B_n(m)) = \bigcup_{i=1}^{m} E(G_i) \).
This section is the main results, the characterization of endo-regular, endo-completely-regular and endo-orthodox of cycle book graphs.

**Theorem 2.1.** A cycle book graph is endo-regular if and only if it is an odd cycle book graph or one page cycle $C_4$, $C_6$, or $C_8$ book graph.

**Proof.** For any cycle book graph $B_n(m)$:

1. If $n$ is odd, then by Lemma 1.2, $B_n(m)$ is a endo-regular.
2. If $n$ is even, then $B_n(m)$ is bipartite graph. From Lemma 1.1, $B_n(m)$ is an endo-regular if and only if $B_n(m)$ is $C_4$, $C_6$, or $C_8$. 

From $K_3 \cong B_3(1)$ and $End(K_3)$ is a group, then $B_3(1)$ is endo-completely-regular and endo-orthodox. Next, we will characterized the endo-completely-regular and endo-orthodox cycle book graphs.

**Lemma 2.2.** For even cycle book graphs,

1. $B_{2n}(m)$ is not endo-completely-regular for all positive integers $m, n, n \geq 2$.
2. $B_{2n}(m)$ is endo-orthodox if and only if $n = 2$ and $m = 1$.

**Proof.** (1) From Theorem 2.1, $B_{2n}(m)$ is endo-regular if and only if $B_{2n}(m)$ is a cycle $C_{2n}$ where $n \in \{2, 3, 4\}$. But from Lemma 1.3, those $C_{2n}$ is not endo-completely-regular.

(2) Since $B_{2n}(m)$ are bipartite graphs and from Lemma 1.5, $B_{2n}(m)$ is endo-orthodox if and only if $B_{2n}(m)$ is a cycle $C_4$, i.e, $n = 2$ and $m = 1$.

For every one page odd cycle book graph $B_{2n+1}(1)$, they are isomorphic to $C_{2n+1}$. Then $End(B_{2n+1}(1))$ forms a group for all positive integers $n$. Therefore, all $B_{2n+1}(1)$ are endo-completely-regular and endo-orthodox.

For $B_{2n+1}(2)$:
Lemma 2.3. The endomorphism monoid of any odd cycle book graph $B_{2n+1}(2)$ is a union of (disjoint) groups, which is isomorphic to $(S_2 \times S_2) \cup G_1 \cup G_2$ when $G_1 \cong G_2 \cong Aut(C_{2n+1})$ and $S_2$ is the permutation group of order 2.

Proof. Consider the following subsets of $End(B_3(2))$: Let $S = \left\{ \begin{pmatrix} 0 & 1 & 2 \ 0 & 1 & 2 \ 0 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 2 \ 1 & 0 & 2 \ 1 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 \ 0 & 1 & 2 \ 1 & 0 & 2 \end{pmatrix} \right\}$.

Let $G_1 = \left\{ \begin{pmatrix} 0 & 1 & 2 \ 0 & 1 & 2 \ 0 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 \ 0 & 1 & 2 \ 1 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 0 \ 2 & 1 & 0 \ 1 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 \ 0 & 1 & 2 \ 2 & 0 & 1 \end{pmatrix} \right\}$.

And $G_2 = \left\{ \begin{pmatrix} 0 & 1 & 2 \ 0 & 1 & 2 \ 2 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 2 \ 0 & 1 & 2 \ 1 & 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 \ 0 & 1 & 2 \ 1 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 \ 0 & 1 & 2 \ 1 & 0 & 2 \end{pmatrix} \right\}$.

We found that those subset $S, G_1, G_2$ form groups with the composition, and $S \cong S_2 \times S_2$, and $G_1 \cong G_2 \cong Aut(C_3)$. Moreover, $End(B_3(2))$ is isomorphic to the monoid of the union of those groups, $S \cup G_1 \cup G_2$. Therefore, $End(B_3(2)) \cong S_2 \times S_2 \cup G_1 \cup G_2$.

Then by Lemma 1.4, $B_3(2)$ is endo-completely-regular.

For the other cycle book graphs $B_{2n+1}(2)$ where $n > 1$, we can define the subsets $S, G_1, G_2$ of the monoid $End(B_{2n+1}(2))$ by the same way as we did for the monoid of the book graph $B_3(2)$. Then $End(B_{2n+1}(2)) \cong S_2 \times S_2 \cup G_1 \cup G_2$.

\[\square\]

Corollary 2.4. Every odd cycle book graph of two pages $B_{2n+1}(2)$ is endo-completely-regular.

Corollary 2.5. Every odd cycle book graph of two pages $B_{2n+1}(2)$ is endo-orthodox.

Proof. Form Lemma 2.3, $End(B_{2n+1}(2)) \cong S \cup G_1 \cup G_2$, where $G_1 \cong G_2 \cong Aut(C_{2n+1})$ and $S \cong S_2 \times S_2$. Then there are only three idempotents in $End(B_{2n+1}(2))$ which are the identities elements of the group $S, G_1, G_2$, say $i, i_1, i_2$, respectively, i.e.

$Idpt(End(B_{2n+1}(2))) = \left\{ i = \begin{pmatrix} 0 & 1 & 2 \ 0 & 1 & 2 \ 0 & 1 & 2 \end{pmatrix}, i_1 = \begin{pmatrix} 0 & 1 & 2 \ 0 & 1 & 2 \ 0 & 1 & 2 \end{pmatrix}, i_2 = \begin{pmatrix} 0 & 1 & 2 \ 0 & 1 & 2 \ 0 & 1 & 2 \end{pmatrix} \right\}$.

Therefore $Idpt(B_{2n+1}(2)) = \{i, i_1, i_2\}$ and the table of composite:

<table>
<thead>
<tr>
<th>$\circ$</th>
<th>$i$</th>
<th>$i_1$</th>
<th>$i_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$i$</td>
<td>$i$</td>
<td>$i_1$</td>
<td>$i_2$</td>
</tr>
<tr>
<td>$i_1$</td>
<td>$i_1$</td>
<td>$i_1$</td>
<td>$i_1$</td>
</tr>
<tr>
<td>$i_2$</td>
<td>$i_2$</td>
<td>$i_2$</td>
<td>$i_2$</td>
</tr>
</tbody>
</table>
forms a monoid.

For $B_{2n+1}(m)$, $m \geq 3$,

**Lemma 2.6.** For any positive integer $m, n, m \geq 3$,

1. there exists $f \in \text{End}(B_{2n+1}(m))$ such that $f$ is not completely regular, and

2. there exist idempotents $g, h \in \text{End}(B_{2n+1}(m))$ such that $gh$ is not idempotent.

**Proof.** Consider $\text{End}(B_3(3))$,

(1) Let $f \in \text{End}(B_{2n+1}(3))$ be such that

$$f = \begin{pmatrix}
0 & 1 & 2_1 & 3_1 & \ldots & 2n_1 & 2_2 & 3_2 & \ldots & 2n_2 & 2_3 & 3_3 & \ldots & 2n_3 \\
0 & 1 & 2_1 & 3_1 & \ldots & 2n_1 & 2_1 & 3_1 & \ldots & 2n_1 & 2_2 & 3_2 & \ldots & 2n_2
\end{pmatrix}.$$

We will show that $f$ is not completely regular. Since $B_{2n+1}(3)$ is endo-regular, there exists pseudo-inverse $g \in \text{End}(B_{2n+1}(3))$ of $f$.

Then $g$ is one of the following forms

$$g_1 = \begin{pmatrix}
0 & 1 & 2_1 & 3_1 & \ldots & 2n_1 & 2_2 & 3_2 & \ldots & 2n_2 & 2_3 & 3_3 & \ldots & 2n_3 \\
0 & 1 & 2_1 & 3_1 & \ldots & 2n_1 & 2_3 & 3_3 & \ldots & 2n_3 & 2_1 & 3_1 & \ldots & 2n_1
\end{pmatrix},$$

or

$$g_2 = \begin{pmatrix}
0 & 1 & 2_1 & 3_1 & \ldots & 2n_1 & 2_2 & 3_2 & \ldots & 2n_2 & 2_3 & 3_3 & \ldots & 2n_3 \\
0 & 1 & 2_2 & 3_2 & \ldots & 2n_2 & 2_3 & 3_3 & \ldots & 2n_3 & 2_1 & 3_1 & \ldots & 2n_1
\end{pmatrix},$$

where $i \in \{1, 2, 3\}$.

Consider

$$fg_1 = \begin{pmatrix}
0 & 1 & 2_1 & 3_1 & \ldots & 2n_1 & 2_2 & 3_2 & \ldots & 2n_2 & 2_3 & 3_3 & \ldots & 2n_3 \\
0 & 1 & 2_1 & 3_1 & \ldots & 2n_1 & 2_2 & 3_2 & \ldots & 2n_2 & f(2_1) & f(3_1) & \ldots & f(2n_1)
\end{pmatrix},$$

$$g_1f = \begin{pmatrix}
0 & 1 & 2_1 & 3_1 & \ldots & 2n_1 & 2_2 & 3_2 & \ldots & 2n_2 & 2_3 & 3_3 & \ldots & 2n_3 \\
0 & 1 & 2_1 & 3_1 & \ldots & 2n_1 & 2_3 & 3_3 & \ldots & 2n_3 & 2_1 & 3_1 & \ldots & 2n_1
\end{pmatrix},$$

and

$$fg_2 = \begin{pmatrix}
0 & 1 & 2_1 & 3_1 & \ldots & 2n_1 & 2_2 & 3_2 & \ldots & 2n_2 & 2_3 & 3_3 & \ldots & 2n_3 \\
0 & 1 & 2_1 & 3_1 & \ldots & 2n_1 & 2_1 & 3_1 & \ldots & 2n_1 & f(2_1) & f(3_1) & \ldots & f(2n_1)
\end{pmatrix},$$

$$g_2f = \begin{pmatrix}
0 & 1 & 2_1 & 3_1 & \ldots & 2n_1 & 2_2 & 3_2 & \ldots & 2n_2 & 2_3 & 3_3 & \ldots & 2n_3 \\
0 & 1 & 2_2 & 3_2 & \ldots & 2n_2 & 2_3 & 3_3 & \ldots & 2n_3 & 2_1 & 3_1 & \ldots & 2n_1
\end{pmatrix}.$$

Thus, for any cases $fg \neq gf$, i.e. $f$ is not completely regular element.

(2) Let $g, h \in \text{End}(B_{2n+1}(3))$ such that

$$g = \begin{pmatrix}
0 & 1 & 2_1 & 3_1 & \ldots & 2n_1 & 2_2 & 3_2 & \ldots & 2n_2 & 2_3 & 3_3 & \ldots & 2n_3 \\
0 & 1 & 2_1 & 3_1 & \ldots & 2n_1 & 2_1 & 3_1 & \ldots & 2n_1 & 2_3 & 3_3 & \ldots & 2n_3
\end{pmatrix}$$

and

$$h = \begin{pmatrix}
0 & 1 & 2_1 & 3_1 & \ldots & 2n_1 & 2_2 & 3_2 & \ldots & 2n_2 & 2_3 & 3_3 & \ldots & 2n_3 \\
0 & 1 & 2_3 & 3_3 & \ldots & 2n_3 & 2_2 & 3_2 & \ldots & 2n_2 & 2_3 & 3_3 & \ldots & 2n_3
\end{pmatrix}.$$

Thus $g^2 = g$ and $h^2 = h$.

But $gh = \begin{pmatrix}
0 & 1 & 2_1 & 3_1 & \ldots & 2n_1 & 2_2 & 3_2 & \ldots & 2n_2 & 2_3 & 3_3 & \ldots & 2n_3 \\
0 & 1 & 2_3 & 3_3 & \ldots & 2n_3 & 2_1 & 3_1 & \ldots & 2n_1 & 2_3 & 3_3 & \ldots & 2n_3
\end{pmatrix}$.

$$(gh)^2 = \begin{pmatrix}
0 & 1 & 2_1 & 3_1 & \ldots & 2n_1 & 2_2 & 3_2 & \ldots & 2n_2 & 2_3 & 3_3 & \ldots & 2n_3 \\
0 & 1 & 2_3 & 3_3 & \ldots & 2n_3 & 2_1 & 3_1 & \ldots & 2n_1 & 2_3 & 3_3 & \ldots & 2n_3
\end{pmatrix}.$$
Theorem 2.8. The following statements are true:

1. A cycle book graph is endo-completely-regular if and only if it is an odd cycle book graph $B_{2k+1}(m)$ where $m \leq 2$.

2. A cycle book graph is endo-orthodox if and only if it is $B_4(1)$, or odd cycle book $B_{2k+1}(m)$ where $m \leq 2$.

Acknowledgement:
The work was partially supported by the Graduate School and Faculty of Science, Chiang Mai University, Chiang Mai, Thailand.

References


Nirutt Pipattanajinda, Srichan Arworn
Department of Mathematics, Faculty of Sciences, Chiang Mai University, Chiang Mai 50200, THAILAND
e-mail: nirutt.p@gmail.com, srichan28@yahoo.com