Chromaticity of Complete 5-Partite Graphs with Certain Star or Matching Deleted

Ameen Shaman Ameen†, Yee Hock Peng‡, Haixing Zhao§, Gee Choon Lau♯ and Roslan Hasni†,1

†School of Mathematical Sciences, Universiti Sains Malaysia
11800 USM, Penang, Malaysia
e-mail: amensh66@yahoo.com
‡Department of Mathematics, and Institute for Mathematical Research
Universiti Putra Malaysia, 43400 Serdang, Malaysia
e-mail: yhpeng88@yahoo.com
§Department of Mathematics, Qinghai Normal University
Xining, Qinghai 810008, P.R. China
e-mail: haixingzhao@yahoo.com.cn
♯Faculty of Computer and Mathematical Sciences
University Teknologi MARA (Segamat Campus), 85010, Johor, Malaysia
e-mail: drlaugc@gmail.com

Abstract: Let $P(G, \lambda)$ be the chromatic polynomial of a graph $G$. Two graphs $G$ and $H$ are said to be chromatically equivalent, denoted by $G \sim H$, if $P(G, \lambda) = P(H, \lambda)$. We write $[G] = \{H | H \sim G\}$. If $[G] = \{G\}$, then $G$ is said to be chromatically unique. In this paper, we first characterize certain complete 5-partite graphs with $5n + 1$ vertices according to the number of 6-independent partitions of $G$. Using these results, we investigate the chromaticity of $G$ with certain star or matching deleted. As a by-product, many new families of chromatically unique complete 5-partite graphs with certain star or matching deleted are obtained.

Keywords: Chromatic polynomial; Chromatically closed; Chromatic uniqueness.

2010 Mathematics Subject Classification: 05C15.

Copyright © 2012 by the Mathematical Association of Thailand. All rights reserved.
1 Introduction

All graphs considered here are simple and finite. For a graph $G$, let $P(G, \lambda)$ be the chromatic polynomial of $G$. Two graphs $G$ and $H$ are said to be chromatically equivalent (or simply $\chi$–equivalent), symbolically $G \sim H$, if $P(G, \lambda) = P(H, \lambda)$. The equivalence class determined by $G$ under $\sim$ is denoted by $[G]$. A graph $G$ is chromatically unique (or simply $\chi$–unique) if $H \cong G$ whenever $H \sim G$, i.e., $[G] = \{G\}$ up to isomorphism. For a set $\mathcal{G}$ of graphs, if $[G] \subseteq \mathcal{G}$ for every $G \in \mathcal{G}$, then $\mathcal{G}$ is said to be $\chi$–closed. Many families of $\chi$–unique graphs are known (see [1–3]).

For a graph $G$, let $V(G)$, $E(G)$ and $t(G)$ be the vertex set, edge set and number of triangles in $G$, respectively. Let $S$ be a set of $s$ edges in $G$. Let $G - S$ (or $G - s$) be the graph obtained from $G$ by deleting all edges in $S$, and by $(S)$ the graph induced by $S$. Let $K(n_1, n_2, \ldots, n_t)$ be a complete $t$-partite graph. We denote by $K^{-}\sigma(n_1, n_2, \ldots, n_t)$ the family of graphs which are obtained from $K(n_1, n_2, \ldots, n_t)$ by deleting a set $S$ of some $s$ edges.

In [2–5], one can find many results on the chromatic uniqueness of bipartite and tripartite graphs. Also there are some results on the chromaticity of 4-partite graphs. However, there are very few 5-partite graphs known to be $\chi$–unique, see [6, 7].

Let $G$ be a complete 5-partite graph with $5n + 1$ vertices. In this paper, we characterize certain complete 5-partite graphs with $5n + 1$ vertices according to the number of 6-independent partitions of $G$. Using these results, we investigate the chromaticity of $G$ with certain star or matching deleted. As a by-product, many new families of chromatically unique complete 5-partite graphs with certain star or matching deleted are obtained.

2 Some Lemmas and Notations

For a graph $G$ and a positive integer $r$, a partition $\{A_1, A_2, \ldots, A_r\}$ of $V(G)$, where $r$ is a positive integer, is called an $r$-independent partition of $G$ if every $A_i$ is independent of $G$. Let $\alpha(G, r)$ denote the number of $r$-independent partitions of $G$. Then, we have $P(G, \lambda) = \sum_{r=1}^{p} \alpha(G, r)(\lambda)_r$, where $(\lambda)_r = \lambda(\lambda - 1)(\lambda - 2) \cdots (\lambda - r + 1)$ (see [8]). Therefore, $\alpha(G, k) = \alpha(H, k)$ for each $k = 1, 2, \ldots, n$, if $G \sim H$.

For a graph $G$ with $p$ vertices, the polynomial $\sigma(G, x) = \sum_{r=1}^{p} \alpha(G, r)x^r$ is called the $\sigma$-polynomial of $G$ (see [9]). Clearly, $P(G, \lambda) = P(H, \lambda)$ implies that $\sigma(G, x) = \sigma(H, x)$ for any graphs $G$ and $H$.

For disjoint graphs $G$ and $H$, $G \cup H$ denotes the disjoint union of $G$ and $H$. The join of $G$ and $H$ denoted by $G \vee H$ is defined as follows: $V(G \vee H) = V(G) \cup V(H)$; $E(G \vee H) = E(G) \cup E(H) \cup \{xy \mid x \in V(G), y \in V(H)\}$. For notations and terminology not defined here, we refer [10].

**Lemma 2.1** (Koh et al. [2], Brenti [9]). Let $G$ and $H$ be two disjoint graphs. Then
Lemma 2.2 (Brenti [9]). Let $G = K(n_1, n_2, n_3, ..., n_k)$ and $\sigma(G, x) = \sum_{r \geq 1} \alpha(G, r)x^r$, then $\alpha(G, r) = 0$ for $1 \leq r \leq t - 1$, $\alpha(G, t) = 1$ and $\alpha(G, t + 1) = \sum_{i=1}^{t-1} 2^{n_i - 1} - t$.

Let $x_1 \leq x_2 \leq x_3 \leq x_4 \leq x_5$ be positive integers and $\{x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}, x_{i_5}\} = \{x_1, x_2, x_3, x_4, x_5\}$. If there are two elements $x_{i_1}$ and $x_{i_2}$ in $\{x_1, x_2, x_3, x_4, x_5\}$ such that $x_{i_2} - x_{i_1} \geq 2$, then $H' = K(x_{i_1} + 1, x_{i_2} - 1, x_{i_3}, x_{i_4}, x_{i_5})$ is called an improvement of $H = K(x_1, x_2, x_3, x_4, x_5)$.

Lemma 2.3 (Zhao et al. [6]). Suppose $x_1 \leq x_2 \leq x_3 \leq x_4 \leq x_5$ and $H' = K(x_{i_1} + 1, x_{i_2} - 1, x_{i_3}, x_{i_4}, x_{i_5})$ is an improvement of $H = K(x_1, x_2, x_3, x_4, x_5)$, then

$$\alpha(H, 6) - \alpha(H', 6) = 2^{x_{i_2} - 2} - 2^{x_{i_1} - 1} \geq 2^{x_{i_1} - 1}.$$ 

For a graph $G$, let $q(G)$ be the number of edges in $G$.

Lemma 2.4 (Zhao et al. [6]). Let $G = K(n_1, n_2, n_3, n_4, n_5)$ and $S$ be a set of some $s$ edges of $G$. If $H \sim G - S$, then there is a graph $F = K(y_1, y_2, y_3, y_4, y_5)$ and a subset $S'$ of $E(F)$ of some $s'$ edges of $F$ such that $H = F - S'$ and $|S'| = s' = q(F) - q(G) + s$.

Let $G = K(n_1, n_2, n_3, n_4, n_5)$. For a graph $H = G - S$, where $S$ is a set of some $s$ edges of $G$, define $\alpha'(H) = \alpha(H, 6) - \alpha(G, 6)$. Clearly, $\alpha'(H) \geq 0$.

Lemma 2.5 (Zhao [7]). Let $G = K(n_1, n_2, n_3, n_4, n_5)$. Suppose that $\min \{n_i| i = 1, 2, 3, 4, 5\} \geq s + 1 \geq 1$ and $H = G - S$, where $S$ is a set of some $s$ edges of $G$, then

$$s \leq \alpha'(H) = \alpha(H, 6) - \alpha(G, 6) \leq 2^s - 1,$$

$\alpha'(H) = s$ iff the set of end-vertices of any $r \geq 2$ edges in $S$ is not independent in $H$, and $\alpha'(H) = 2^s - 1$ iff $S$ induces a star $K_{1,s}$ and all vertices of $K_{1,s}$ other than its center belong to a same $A_i$.

Let $K(A_1, A_2)$ be a complete bipartite graph with partite sets $A_1$ and $A_2$. We denote by $K^{-K_{1,s}}(A_i, A_j)$ the graph obtained from $K(A_i, A_j)$ by deleting $s$ edges that induce a star with its center in $A_i$. Note that $K^{-K_{1,s}}(A_i, A_j) \neq K^{-K_{1,s}}(A_j, A_i)$ if $|A_i| \neq |A_j|$ for $i \neq j$ (see [5]).

Lemma 2.6 (Dong et al. [5]). Let $K(n_1, n_2)$ be a complete bipartite graph with partite sets $A_1$ and $A_2$ such that $|A_i| = n_i$ for $i = 1, 2$. If $\min \{n_1, n_2\} \geq s + 2$, then every $K^{-K_{1,s}}(A_i, A_j)$ is $\chi$-unique, where $i \neq j$ and $i, j = 1, 2$. 
Theorem 3.1. Let \( G = K(n_1, n_2, n_3, n_4, n_5) \) be a complete 5-partite graph with partite sets \( A_i (i = 1, 2, \ldots, 5) \) such that \( |A_i| = n_i \). Let \( \langle A_i \cup A_j \rangle \) be the subgraph of \( G \) induced by \( A_i \cup A_j \), where \( i \neq j \) and \( i, j \in \{1, 2, 3, 4, 5\} \). By \( K_{i,j}^{-K_{i,j}}(n_1, n_2, n_3, n_4, n_5) \), we denote the graph obtained from \( K(n_1, n_2, n_3, n_4, n_5) \) by deleting a set of \( s \) edges that induce a \( K_{1,s} \) with its center in \( A_i \) and all its end vertices are in \( A_j \). Note that \( K_{i,j}^{-K_{i,j}}(n_1, n_2, n_3, n_4, n_5) = K_{i,j}^{-K_{i,j}}(n_1, n_2, n_3, n_4, n_5) \) and \( K_{i,j}^{-K_{i,j}}(n_1, n_2, n_3, n_4, n_5) = K_{i,j}^{-K_{i,j}}(n_1, n_2, n_3, n_4, n_5) \) for \( n_i = n_j \) and \( l \neq i, j \).

Lemma 2.7 (Zhao et al. [6]). Suppose that \( \min \{n_1, n_2, n_3, n_4, n_5\} \geq s + 2 \) and \( n_i \neq n_j \) for \( i \neq j, i, j = 1, 2, 3, 4, 5 \), then \( P(K_{i,j}^{-K_{i,j}}(n_1, n_2, n_3, n_4, n_5), \lambda) \neq P(K_{i,j}^{-K_{i,j}}(n_1, n_2, n_3, n_4, n_5), \lambda) \).

3 Classification

In this section, we shall characterize certain complete 5-partite graph \( G = K(n_1, n_2, n_3, n_4, n_5) \) according to the number of 6-independent partitions of \( G \) where \( n_1 + n_2 + n_3 + n_4 + n_5 = 5n + 1, n \geq 1 \).

Theorem 3.1. Let \( G = K(n_1, n_2, n_3, n_4, n_5) \) be a complete 5-partite graph such that \( n_1 + n_2 + n_3 + n_4 + n_5 = 5n + 1, n \geq 1 \). Define \( \theta(G) = [\alpha(G, 6) - 2^{n+1} - 2^n + 5]/2^{n-2} \). Then

(i) \( \theta(G) \geq 0 \);
(ii) \( \theta(G) = 0 \) if and only if \( G = K(n, n, n, n, n + 1) \);
(iii) \( \theta(G) = 1 \) if and only if \( G = K(n - 1, n, n, n + 1, n + 1) \);
(iv) \( \theta(G) = 2 \) if and only if \( G = K(n - 1, n - 1, n + 1, n + 1, n + 1) \);
(v) \( \theta(G) = 5/2 \) if and only if \( G = K(n - 2, n, n + 1, n + 1, n + 1) \);
(vi) \( \theta(G) = 3 \) if and only if \( G = K(n - 1, n, n, n + 2) \);
(vii) \( \theta(G) \geq 4 \) if and only if \( G \) is not a graph appeared in (ii)–(vi);

Proof. For a complete 5-partite graph \( H_t \) with \( 5n + 1 \) vertices, we can construct a sequence of complete 5-partite graphs with \( 5n + 1 \) vertices, say \( H_1, H_2, \ldots, H_t \), such that \( H_t \) is an improvement of \( H_{t-1} \) for each \( i = 2, \ldots, t \), and \( H_t = K(n, n, n, n, n + 1) \). By Lemma 2.3, \( \alpha(H_{t-1}; 6) - \alpha(H_t; 6) > 0 \). So \( \theta(H_{t-1}) - \theta(H_t) > 0 \), which implies \( \theta(G) \geq \theta(H_t) = \theta(K(n, n, n, n, n + 1)) \). From Lemma 2.2 and by a simple calculation, we have \( \theta(K(n, n, n, n, n + 1)) = 0 \). Thus, (ii) is true.

Since \( H_1 = K(n, n, n, n, n + 1) \) and \( H_t \) is an improvement of \( H_{t-1} \), it is not hard to see that \( H_{t-1} \in \{M_0, M_3\} \), where \( M_0 = K(n - 1, n, n, n + 1, n + 1) \) and \( M_3 = K(n - 1, n, n, n, n + 2) \). Hence, by Lemma 2.2, we have \( \theta(M_0) = 1 \), \( \theta(M_3) = 3 \). Note that \( H_{t-1} \) is an improvement of \( H_{t-2} \), one can see that \( H_{t-2} \in \{M_i | i = 1, 2, \ldots, 7\} \), where \( M_i \) and \( \theta(M_i) \) are shown in Table 1.
To complete the proof of the theorem, we need only determine all complete 5-partite graphs $G$ with $5n + 1$ vertices such that $\theta(G) < 4$. By Lemma 2.3, $\theta(H_{t-2}) > 4$ for each $H_{t-2}$ if $H_{t-2} \in \{M_i | i = 4, 5, 6, 7\}$. All graphs $H_{t-2}$ and its $\theta$-values are listed in Table 2 when $H_{t-2} \in \{M_i | i = 1, 2, 3\}$.

It is easy to obtain the following: If $H_{t-2} = M_1$, then $H_{t-3} \in \{M_2, M_4, R_2\}$; $H_{t-3} \in \{M_3, R_1, R_2, R_3\}$ if $H_{t-2} = M_2$ and $H_{t-3} \in \{M_i | i = 4, 5, 6, 7\}$ if $H_{t-2} = M_3$. Thus, from Lemma 2.2, Table 1, Table 2 and the above arguments, we conclude that the theorem holds.

\section{Chromatically Closed 5-Partite Graphs}

In this section, we obtained several $\chi$-closed families of graphs in $K^{-s}(n_1, n_2, n_3, n_4, n_5)$.

\textbf{Theorem 4.1.}

(i) If $n \geq s + 2$, then the family of graphs $K^{-s}(n, n, n, n, n + 1)$ is $\chi$-closed;

(ii) If $n \geq s + 3$, then the family of graphs $K^{-s}(n-1, n, n+1, n+1)$ is $\chi$-closed;

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
$M_i$ & Graphs $H_{t-2}$ & $\theta(M_i)$ \\
\hline
$M_1$ & $K(n-1, n-1, n+1, n+1, n+1)$ & 2 \\
$M_2$ & $K(n-2, n, n+1, n+1, n+1)$ & 5/2 \\
$M_3$ & $K(n-1, n, n, n+2)$ & 3 \\
$M_4$ & $K(n-1, n-1, n, n+1, n+2)$ & 4 \\
$M_5$ & $K(n-2, n, n, n+1, n+2)$ & 9/2 \\
$M_6$ & $K(n-1, n-1, n, n+3)$ & 10 \\
$M_7$ & $K(n-2, n, n, n+3)$ & 21/2 \\
\hline
\end{tabular}
\caption{$H_{t-2}$ and its $\theta$-values}
\end{table}

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
$R_i$ & Graphs $H_{t-3}$ & $\theta(R_i)$ \\
\hline
$R_1$ & $K(n-3, n+1, n+1, n+1, n+1)$ & 17/4 \\
$R_2$ & $K(n-2, n-1, n+1, n+1, n+2)$ & 11/2 \\
$R_3$ & $K(n-3, n, n+1, n+1, n+2)$ & 25/4 \\
\hline
\end{tabular}
\caption{$H_{t-3}$ and its $\theta$-values}
\end{table}
(iii) If \( n \geq s + 3 \), then the family of graphs \( K^{-s}(n - 1, n - 1, n + 1, n + 1) \) is \( \chi \)-closed;

(iv) If \( n \geq s + 4 \), then the family of graphs \( K^{-s}(n - 2, n + 1, n + 1, n + 1) \) is \( \chi \)-closed;

(v) If \( n \geq s + 3 \), then the family of graphs \( K^{-s}(n - 1, n, n, n, n + 2) \) is \( \chi \)-closed.

**Proof.** The proof of each statement of the theorem is similar. So, we only give a proof for (iii) and omit the proofs of the others. For convenience, let \( G_1 = K(n, n, n, n, n + 1) \), \( G_2 = K(n - 1, n, n, n + 1, n + 1) \) and \( G_3 = K(n - 1, n - 1, n + 1, n + 1, n + 1) \). Suppose that \( H \sim G_3 - S \). Then it suffices to show that \( H \in K^{-s}(n - 1, n - 1, n + 1, n + 1, n + 1) \). By Lemma 2.4, there is a complete 5-partite graph \( F = K(y_1, y_2, y_3, y_4, y_5) \) and a set \( S' \) for some \( s' \) edges in \( F \) such that \( H = F - S' \) and \( |S'| = s' = q(F) - q(G_3) + s \geq 0 \). Clearly, \( \alpha(F - S', 6) = \alpha(G_3 - S, 6) \).

By definition, we have

\[
\alpha(G_3 - S, 6) = \alpha(G_3, 6) + \alpha'(G_3 - S) \quad \text{with} \quad s \leq \alpha'(G_3 - S) \leq 2^s - 1,
\]

and

\[
\alpha(F - S', 6) = \alpha(F, 6) + \alpha'(F - S').
\]

So

\[
\alpha(F - S', 6) - \alpha(G_3 - S, 6) = \alpha(F, 6) - \alpha(G_3, 6) + \alpha'(F - S') - \alpha'(G_3 - S) \tag{4.1}
\]

By Theorem 3.1, \( \alpha(F, 6) - \alpha(G_3, 6) = 2^{n-2} \theta(F) - \theta(G_3) \). We distinguish the following two cases.

*Case 1:* \( \alpha(F, 6) < \alpha(G_3, 6) \). By Theorem 3.1, then \( F \in \{G_1, G_2\} \). If \( F = G_1 \), we have \( \alpha(G_1, 6) - \alpha(G_3, 6) = -2^{n-1} \), and \( q(G_1) - q(G_3) = 2 \). From Equation (4.1) above, we have

\[
\alpha(G_1 - S', 6) - \alpha(G_3 - S, 6) = -2^{n-1} + \alpha'(F - S') - \alpha'(G_3 - S).
\]

Note that \( n \geq s + 3 \) and \( s' = q(G_1) - q(G_3) + s = s + 2 \leq n - 1 \). By Lemma 2.5, \( 0 \leq s' \leq \alpha'(F - S') \leq 2^s - 1 \leq 2^{n-1} - 1 \). Since \( 0 \leq s \leq \alpha'(G_3 - S) \leq 2^s - 1 \), we have

\[
\alpha(G_1 - S', 6) - \alpha(G_3 - S, 6) \leq -2^{n-1} + \alpha'(F - S') - \alpha'(G_3 - S) \leq -1,
\]

which contradicts \( \alpha(F - S', 6) = \alpha(G_3 - S, 6) \).

If \( F = G_2 \), by Theorem 3.1, we have \( \alpha(G_2, 6) - \alpha(G_3, 6) = -2^{n-2} \), and \( q(G_2) - q(G_3) = 1 \). From Equation (4.1) above, we have

\[
\alpha(G_2 - S', 6) - \alpha(G_3 - S, 6) = -2^{n-2} + \alpha'(F - S') - \alpha'(G_3 - S).
\]
Note that \( n \geq s + 3 \) and \( s' = q(G_2) - q(G_3) + s = s + 1 \leq n - 2 \). By Lemma 2.5, \( 0 \leq s' \leq \alpha'(F - S') \leq 2^{s' - 1} - 2^{n-2} - 1 \), since \( 0 \leq s' \leq \alpha'(G_3 - S) \leq 2^{s} - 1 \), we have
\[
\alpha(G_2 - S', 6) - \alpha(G_3 - S, 6) \leq -2^{n-2} + \alpha'(F - S') - \alpha'(G_3 - S) \leq -1,
\]
which contradicts \( \alpha(F - S', 6) = \alpha(G_3 - S, 6) \).

**Case 2:** \( \alpha(F, 6) > \alpha(G_3, 6) \). By Theorem 3.1, \( F \neq G_1 \), where \( i = 1, 2, 3 \) and we have \( \alpha(F, 6) - \alpha(G_3, 6) \geq 2^{n-3} \). Hence we have \( \alpha(F - S', 6) - \alpha(G_3 - S, 6) \geq 2^{n-3} + \alpha'(F - S') - \alpha'(G_3 - S) \).

Since \( n - 3 \geq s \), \( 0 \leq \alpha'(F - S') \) and \( 0 \leq s \leq \alpha'(G_3 - S) \leq 2^s - 1 \), we have \( \alpha(F - S', 6) - \alpha(G_3 - S, 6) \geq 1 \), contradicting the fact that \( \alpha(F - S', 6) = \alpha(G_3 - S, 6) \). So, from the above two cases, we conclude that \( \theta(F) - \theta(G_3) = 0 \). Thus \( F = G_3 \) and \( S = S' \). Therefore, \( H \in \mathcal{K}^{-s}(n-1, n-1, n+1, n+1, n+1) \).

\( \square \)

## 5 Chromatically Unique 5-Partite Graphs

In this section, we first study the chromatically unique 5-partite graphs with \( 5n + 1 \) vertices and a set \( S \) of \( s \) edges deleted where the deleted edges induce a star \( K_{1,s} \).

**Theorem 5.1.** If \( n \geq s + 2 \), then the graphs \( K_{i,j}^{-K_{1,s}}(n, n, n, n, n+1) \) are \( \chi \)-unique for each \( (i, j) \in \{(1, 2), (1, 5), (5, 1)\} \).

**Proof.** By Lemma 2.5 and Theorem 4.1(i), we know that \( K_{i,j}^{-K_{1,s}}(n, n, n, n+1) = \{K_{i,j}^{-K_{1,s}}(n, n, n, n, n+1)|(i, j) \in \{(1, 2), (1, 5), (5, 1)\}\) is \( \chi \)-closed for \( n \geq s + 2 \). Note that
\[
t(K_{i,j}^{-K_{1,s}}(n, n, n, n, n+1)) = t(K(n, n, n, n, n+1)) - 3sn \text{ for } (i, j) \in \{(1, 5), (5, 1)\},
\]
\[
t(K_{1,2}^{-K_{1,s}}(n, n, n, n, n+1)) = t(K(n, n, n, n, n+1)) - (3n + 1).
\]
By Lemma 2.1, we have \( K_{1,2}^{-K_{1,s}}(n, n, n, n, n+1) \) is chromatically unique. From Lemma 2.7, we find that \( P(K_{1,5}^{-K_{1,s}}(n, n, n, n, n+1), \lambda) \neq P(K_{2,1}^{-K_{1,s}}(n, n, n, n, n+1), \lambda) \). Hence, the graphs \( K_{i,j}^{-K_{1,s}}(n, n, n, n, n+1) \) is \( \chi \)-unique where \( n \geq s + 2 \) for each \( (i, j) \in \{(1, 2), (1, 5), (5, 1)\} \).

**Theorem 5.2.** If \( n \geq s + 3 \), then the graphs \( K_{i,j}^{-K_{1,s}}(n-1, n, n+1, n+1, n+1) \) are \( \chi \)-unique for each \( (i, j) \in \{(1, 2), (2, 1), (2, 4), (4, 2), (4, 5)\} \).

**Proof.** Let \( F \in \{K_{i,j}^{-K_{1,s}}(n-1, n, n+1, n+1) \mid (i, j) = \{(1, 2), (2, 1), (2, 4), (4, 2), (4, 5)\}\} \) and \( H \sim F \). By Theorem 4.1(ii), \( H \in \mathcal{K}^{-s}(n-1, n, n+1, n+1, n+1) \).
Without loss of generality, we assume $H \sim K_{1,2}^{−K_{1,2}}(n−1, n, n+1, n+1)$, where $(i, j) = (1, 2)$. Since

$$\alpha(H, 6) = \alpha(K_{1,2}^{−K_{1,2}}(n−1, n, n+1, n+1), 6) = \alpha(K(n−1, n, n+1, n+1), 6) + 2^s − 1,$$

from Lemma 2.5, we know that $H \in \{K_{i,j}^{−K_{i,j}}(n−1, n, n+1, n+1) \mid i \neq j, i, j = 1, 2, 3, 4, 5\}$. It easy to see that $H \in \{K_{i,j}^{−K_{i,j}}(n−1, n, n+1, n+1) \mid i \neq j, i, j = 1, 2, 3, 4, 5\} = \{K_{i,j}^{−K_{i,j}}(n−1, n, n+1, n+1) \mid (i, j) \in \{(1, 2), (2, 1), (1, 4), (4, 1), (2, 3), (2, 4), (4, 2), (4, 5)\}\}$.

Now let’s determine the numbers of triangles in $K_{i,j}^{−K_{i,j}}(n−1, n, n+1, n+1)$. Then we obtain that

$$t_{1,2} = t_{2,1} = t(K(n−1, n, n+1, n+1)) − s(3n + 2),$$
$$t_{1,4} = t_{4,1} = t_{2,3} = t_{2,3} = t(K(n−1, n, n+1, n+1)) − s(3n + 1),$$
$$t_{2,4} = t_{4,2} = t(K(n−1, n, n+1, n+1)) − 3ns,$$
$$t_{4,5} = t(K(n−1, n, n+1, n+1)) − s(3n − 1).$$

Recalling $F \in \{K_{i,j}^{−K_{i,j}}(n−1, n, n+1, n+1) \mid (i, j) \in \{(1, 2), (2, 1), (2, 4), (4, 2), (4, 5)\}\}$ and $t(H) = t(F)$, we have

$$H, F \in \{K_{i,j}^{−K_{i,j}}(n−1, n, n+1, n+1) \mid (i, j) \in \{(1, 2), (2, 1)\}\}$$
or

$$H, F \in \{K_{i,j}^{−K_{i,j}}(n−1, n, n+1, n+1) \mid (i, j) \in \{(2, 4), (4, 2)\}\}.$$  

It follows from Lemma 2.7 that

$$P(K_{1,2}^{−K_{1,2}}(n−1, n, n+1, n+1), \lambda) \neq P(K_{2,1}^{−K_{2,1}}(n−1, n, n+1, n+1), \lambda);$$
$$P(K_{2,4}^{−K_{2,4}}(n−1, n, n+1, n+1), \lambda) \neq P(K_{4,2}^{−K_{4,2}}(n−1, n, n+1, n+1), \lambda).$$

Hence, the graphs $K_{i,j}^{−K_{i,j}}(n−1, n, n+1, n+1)$ are $\chi$-unique where $n \geq s + 3$ for each $(i, j) \in \{(1, 2), (2, 1), (2, 4), (4, 2), (4, 5)\}$.

Similarly to the proofs of Theorems 5.1 and 5.2, we can prove Theorems 5.3, 5.4 and 5.5.

**Theorem 5.3.** If $n \geq s + 3$, then the graphs $K_{i,j}^{−K_{i,j}}(n−1, n−1, n+1, n+1)$ are $\chi$-unique for each $(i, j) \in \{(1, 2), (1, 3), (3, 1), (3, 4)\}$.

**Theorem 5.4.** If $n \geq s + 4$, then the graphs $K_{i,j}^{−K_{i,j}}(n−2, n+1, n+1, n+1)$ are $\chi$-unique for each $(i, j) \in \{(1, 2), (2, 1), (1, 3), (3, 1), (2, 3), (3, 2), (3, 4)\}$.

**Theorem 5.5.** If $n \geq s + 3$, then the graphs $K_{i,j}^{−K_{i,j}}(n−1, n, n+2)$ are $\chi$-unique for each $(i, j) \in \{(1, 2), (2, 1), (1, 5), (5, 1), (2, 5), (5, 2), (2, 3)\}$. 
Let $K_{i,j}^{-sK_2}(n_1, n_2, n_3, n_4, n_5)$ denotes the graph obtained from $K(n_1, n_2, n_3, n_4, n_5)$ by deleting a set of $s$ edges that forms a matching in $(A_i \cup A_j)$. We now investigate the chromatically unique 5-partite graphs with $5n + 1$ vertices and a set $S$ of $s$ edges deleted where the deleted edges induce a matching $sK_2$.

**Theorem 5.6.** If $n \geq s + 3$, then the graphs $K_{1,2}^{-sK_2}(n-1, n-1, n+1, n+1, n+1)$ are $\chi$-unique.

**Proof.** Let $F \sim K_{1,2}^{-sK_2}(n-1, n-1, n+1, n+1, n+1)$. It is sufficient to prove that $F = K_{1,2}^{-sK_2}(n-1, n-1, n+1, n+1, n+1)$. By Theorem 4.1(iii) and Lemma 2.5, we have $F \in \mathcal{K}^{-s}(n-1, n-1, n+1, n+1, n+1)$ and $\alpha'(F) = s$. Let $F = G - S$ where $G = K(n-1, n-1, n+1, n+1, n+1)$. Next we consider the number of triangles of $F$. Let $e_i \in S$ and $t(e_i)$ be the number of triangles in $G$ containing the edge $e_i$. Then one can see that $t(e_i) \leq 3n + 3$. As $n - 1 \leq n - 1 < n + 1 \leq n + 1 \leq n + 1$, we know that $t(e_i) = 3n + 3$ if and only if $e_i$ is an edge of the subgraph $(A_1 \cup A_2)$ in $G$. So,

$$t(F) \geq t(G) - s(3n + 3);$$

where the equality holds if and only if each edge $e_i$ in $S$ is an edge of the subgraph $(A_1 \cup A_2)$ in $G$. Note that $t(F) = t(G) - s(3n + 3)$ and $\alpha'(F) = s$. By Lemma 2.5, we know that $F = K_{1,2}^{-sK_2}(n-1, n-1, n+1, n+1, n+1)$. This completes the proof. \hfill \Box

Similarly to the proof of Theorem 5.6, we can prove Theorem 5.7.

**Theorem 5.7.** If $n \geq s + 4$, then the graphs $K_{1,2}^{-sK_2}(n-2, n, n+1, n+1, n+1)$ are $\chi$-unique.

We end this paper with the following two open problems.

1. Study the chromaticity of the graphs $K_{i,j}^{-K_1,r}(n-1, n, n+1, n+1)$ for each $(i, j) \in \{(1, 4), (4, 1), (2, 3)\}$.
2. Study the chromaticity of the graphs $K_{1,2}^{-sK_2}(n, n, n, n, n+1)$, $K_{1,2}^{-sK_2}(n-1, n, n, n+1, n+1)$ and $K_{1,2}^{-sK_2}(n-1, n, n, n, n+2)$.

**Acknowledgements:** We would like to thank the referees for his comments and suggestions on the manuscript. This work was supported by Universiti Sains Malaysia, Penang, Malaysia under Research Incentive Grant 304/JNP/600004.

**References**


(Received 5 May 2011)
(Accepted 9 November 2011)