Idempotents and Similar Operators

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Abstract: If $T$ is an idempotent operator on a Hilbert space $H$ such that the spectrum of $T$ contains two different points, then $T$ has the representation $I \oplus \begin{bmatrix} 0 & A \\ 0 & I \end{bmatrix}$, where $A$ is a unique, positive, and similar operator. Furthermore, if $T$ on $\mathbb{Z} = K \oplus H$ has the form $\begin{bmatrix} \alpha I & A \\ 0 & \beta I \end{bmatrix}$, where $\alpha$ and $\beta$ are scalars, then a characterization of the norm of $T$ in terms of the norm of $A$ is proved. Examples of such operators are also given. In particular, genetic operators are defined. Finally, a necessary and a sufficient condition for similar operators is provided.

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1 Introduction

Throughout this article, $H$ and $K$ will denote Hilbert spaces, and $\mathbb{Z} = K \oplus H$. The identity operator is denoted by $I$. We write $\text{Sp}(T)$ and $r(T)$ for the spectrum and the spectral radius of an operator $T$ respectively.

Let $T_1$ and $T_2$ be two operators in $B(H)$ and $B(K)$ respectively. Then, $T_1$ is similar to $T_2$ if there exists an invertible operator $S$ from $H$ onto $K$ such that $T_2S = ST_1$.

An operator $T$ is called a contraction if the norm of $T$ is less than one, [1]. Also, an operator on a Hilbert space is similar to a contraction if and only if its spectrum is contained in the interior of the unit disc; see [1].
Let \( H_1 \) be a subset of \( H \). Let \( T \) be a continuous map from \( H_1 \) to \( H \), then \( T \) is called a \( t \)-contraction if there exists a constant \( t \geq 0 \) such that for any nonempty bounded subset \( H_0 \) of \( H \) we have \( \alpha(T(H_0)) \leq t\alpha(H_0) \), where \( \alpha(T(H_0)) \) is the measure of noncompactness of \( T(H_0) \) and \( \alpha(T) = \inf \{ t > 0 : T \) is a \( t \)-contraction\}, see [2].

We note that if \( \Phi \) is a compact operator and \( \Phi_0 \) is a contraction such that \( \alpha(\varphi_0) < 1 \), then an operator \( \psi = \Phi + \varphi_0 \) is a \( t \)-contraction with \( ||\psi|| < 1 \). We also note that an operator \( A \) is a \( ||A|| \)-contraction and \( \alpha(A) \leq ||A|| \).

## 2 Idempotent Operators

In this section, we prove some results on idempotent operators where the spectrum contains two distinct points.

**Theorem 2.1.** Let \( T \) be an idempotent operator on \( H \) with \( Sp(T) = \{\alpha, \beta\} \), where \( \alpha \) and \( \beta \) are distinct scalars. Then \( T \) is similar to an operator of the form

\[
\begin{bmatrix}
0 & A \\
0 & I
\end{bmatrix},
\]

where \( A \) is a unique, positive (strict), and similar operator.

**Proof.** Let \( T \) be an idempotent operator with \( Sp(T) = \{\alpha, \beta\} \), where \( \alpha \) and \( \beta \) are distinct scalars. Then \( \left( \frac{I - \alpha A}{1 - \beta} \right) \) is idempotent. Let \( N(T) \) and \( N(T^*) \) denote the null spaces of \( T \) and \( T^* \) respectively. If \( K_0 \) is the orthogonal complement of \( N(T) \cap N(T^*) \) in \( H \) and \( T \) is idempotent, then \( T \) can be written as \( 0 \oplus S \) on \( H \), where \( S = \begin{bmatrix} 0 & A_1 \\ 0 & A_2 \end{bmatrix} \) on \( K_0 \), see [3]. If \( R(S^*) \) denotes the range of \( S^* \) then \( K_0 = N(S) \oplus R(S^*) \).

We also note that if \( S \) is idempotent (which is the case here), then \( A_2 = I \) and \( S \) has a better representation of the form \( I \oplus \begin{bmatrix} 0 & D \\ 0 & I \end{bmatrix} \) on \( N(B) \oplus [N(S) \oplus (R(S^*) \oplus N(B))] \) where \( \oplus \) denotes the orthogonal complement of \( N(B) \). The \( R(D) \) is dense (\( D \) is one to one) and \( A = U^*D \), where \( U \) is the unitary operator. This means the operator \( D \) can be decomposed so that \( D = UA \). In this case, a simple matrix multiplication yields:

\[
\begin{bmatrix}
0 & U^*D \\
0 & I
\end{bmatrix}
\begin{bmatrix}
U & 0 \\
0 & I
\end{bmatrix}
= \begin{bmatrix}
0 & A \\
0 & I
\end{bmatrix}.
\]

Thus, by the definition of similar operators, the operator \( S \) is similar to \( I \oplus \begin{bmatrix} 0 & A \\ 0 & I \end{bmatrix} \), and the representation of \( T \) is valid.

For uniqueness, let \( T = 0 \oplus I \oplus \begin{bmatrix} 0 & A \\ 0 & I \end{bmatrix} \) on \( [N(T) \cap N(T^*)] \oplus [N(T - I) \cap N(T - I)^*] \). In fact, the dimensions of these spaces are uniquely determined by \( T \).

For two positive operators \( A_0 \) and \( A \), let \( \begin{bmatrix} 0 & A_0 \\ 0 & I \end{bmatrix} \) and \( \begin{bmatrix} 0 & A \\ 0 & I \end{bmatrix} \) be similar.
U is the unitary operator, then \( U \begin{bmatrix} 0 & A \\ 0 & I \end{bmatrix} = \begin{bmatrix} 0 & A_0 \\ 0 & I \end{bmatrix} U \), and this gives us the following equations:

\[ A_0 U_4 = U_1 A + U_2 \quad \text{and} \quad U_3 = 0 \quad \text{where} \quad U = \begin{bmatrix} U_1 & U_2 \\ U_3 & U_4 \end{bmatrix}. \]

Furthermore,

\[ U^* \begin{bmatrix} 0 & A_0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} 0 & A_0 \\ 0 & I \end{bmatrix} U^* \Rightarrow U_2 = 0 \quad \text{and} \quad A_0 U_4 = U_1 A. \]

Also, \( A_0^2 = (U_1^* A_0)(A_0 U_4) = (U_1^* A_0)(U_1 A) = A^2 \Rightarrow A_0 \) and \( A \) are similar operators.

**Theorem 2.2.** Let \( T = \begin{bmatrix} \alpha I & A \\ 0 & \beta I \end{bmatrix} \) on \( \mathcal{H} = K \oplus H \), where \( A \) is the same as in Theorem 1 above, and \( \alpha \) and \( \beta \) are scalars. Then \( ||T|| = \left\| \begin{bmatrix} \alpha & ||A|| \\ 0 & \beta \end{bmatrix} \right\|. \)

**Proof.** Let \( \mathcal{H} = K \oplus H \) and \( v \) be a unit vector in \( \mathcal{H} \). If \( v_1 \) and \( v_2 \) are the unit vectors in \( K \) and \( H \) respectively, then \( v = pv_1 + qv_2 \), where \( p \) and \( q \) are scalars with \( |p|^2 + |q|^2 = 1 \). Then

\[
Tv = (\alpha pv_1 + qAv_2) \oplus (\beta qv_2)
\]

\[
\Rightarrow ||Tv||^2 = ||\alpha pv_1 + qAv_2||^2
\]

\[
= |\alpha|^2 |p|^2 + |\beta|^2 |q|^2 + |q|^2 ||Av_2||^2 + 2 Re(\overline{\alpha q}(Av, v_1))
\]

\[
\Rightarrow \left\| \begin{bmatrix} \alpha & (Av, v_1) \\ 0 & \beta \end{bmatrix} v \right\| \leq ||Tv||^2.
\]

Let

\[ \eta = \sup \left\{ \left\| \begin{bmatrix} \alpha & (Av, v_1) \\ 0 & \beta \end{bmatrix} \right\| : v_1 \in K, v_2 \in H, \text{ and } ||v_1|| = 1 = ||v_2|| \right\}. \]

Then \( \eta \leq ||T|| \). Let the norm of the matrix \( \begin{bmatrix} \alpha & \gamma \\ 0 & \beta \end{bmatrix} \) for the first fixed \( \alpha \) and \( \beta \) be a monotonically increasing function of \( |\gamma| \). Then \( ||A|| = \sup\{|(Av_2, v_1)| : v_1 \in K, v_2 \in H, \text{ and } ||v_1|| = 1 = ||v_2||\} \) and \( \left\| \begin{bmatrix} \alpha & ||A|| \\ 0 & \beta \end{bmatrix} \right\| \leq ||T|| \). Let \( r \) be a real number such that \( \alpha \beta \overline{q} ||A||(\cos r + i \sin r) \geq 0 \). Then, from above, we have \( ||T|| \leq \left\| \begin{bmatrix} \alpha & ||A|| \\ 0 & \beta \end{bmatrix} \right\|, \) which proves the theorem.

**Corollary 2.3.** \( ||Tv|| = ||T|| \Leftrightarrow ||Av_2|| = ||A||. \)
Proof. Let \( v \in \mathcal{S} \) and \(|Tv| = |T|\). Let \( q = 0 \). Then from Theorem 1, we have \(|Tv| = |T| = |\alpha qv_2| = |\alpha| \geq ||T^*(v_1 + 0)|| = \sqrt{||\alpha|^2 + ||A^*v_1||^2} \Rightarrow A = 0\). Next, let \( q \neq 0 \). Then from Theorem 2.1, it follows that \(|Av_2| = |A|\). The implication is a straightforward application of Theorem 1 and the fact that if \( v_1 \) is a unit vector in \( H \), then \( v_1 = \frac{Av_2}{||Av_2||} \) with \(|A| = (Av_2, v_1)\).

**Corollary 2.4.** \(|T - \alpha I|^2 = |\alpha - \beta|^2 + ||A||^2\).

**Proof.** The corollary is an easy consequence of Theorem 2.1.

**Example 2.5.** It is known that the boundary of the spectrum of an operator is contained in the approximate spectrum of the operator. Since \( W(T)(W(T) \) is the numerical range of \( T \)) is convex, we have \( \text{coSp}(T) \subseteq \overline{W(T)} \). For more on numerical ranges, refer to [4]. The equality does not hold in general. For example, let \( H = C^2 \), and let \( T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \). Then, \( \text{Sp}(T) = \{0\} \), although \( T \) is not idempotent. On the other hand,

\[
W(T) = \{zw^* : z, w \in C, |z|^2 + |w|^2 = 1\} = \left\{ \lambda \in C : |\lambda| \leq \frac{1}{2} \right\}.
\]

**Remark 2.6.** If \( T \) is a normal operator, then \( \text{coSp}(T) = \overline{W(T)} \).

**Example 2.7.** Let \( T = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \in B(C)^2 \) be a hermitian operator such that

\[
||(z, w)|| = \max \left\{ |z|, |w|, \frac{|z - w|}{\sqrt{2}} \right\}.
\]

Then \( \text{Sp}(T) \) contains two distinct points \( 0 \) and \( 1 \). Also, by Theorem 2.2, it follows that \( W(T) \) is either the closed line segment with two points \( \alpha \) and \( \beta \) or the disc with foci at \( \alpha \) and \( \beta \), where the major and minor axes are given by \( ||T - \alpha I|| \) and \( \sqrt{||T - \alpha I||^2 - |\alpha - \beta|^2} \), respectively. In fact, \( W(T) = \text{co}\{0, 1, \frac{1+i}{2}\} \).

**Example 2.8.** Let \( T = x_1 \otimes x_2 \) with rank of \( T = 1 \). Then \( W(T) \) is the closed disc with foci \( (x_1, x_2) \) and zero. Major and minor axes are given by \( ||x_1|| ||x_2|| \) and \( \sqrt{||x_1||^2 ||x_2||^2 - ||(x_1 - x_2)||^2} \), respectively. The numerical radius \( w(T) \) is given by \( \frac{1}{2}(||x_1|| ||x_2|| + ||(x_1, x_2)||) \). Compare this with [5] and [6].

**Remark 2.9.** If \( T = \begin{bmatrix} \alpha & (Av_2, v_1) \\ 0 & \beta \end{bmatrix} \), then \( W(T) \) is the closed disc with foci at \( \alpha \) and \( \beta \).

The major and minor axes are given by \( \sqrt{(Av_2, v_1)^2 + |\alpha - \beta|^2} \) and \( |(Av_2, v_1)| \) respectively. The numerical range \( W(T) \) is closed by Corollary 2.4. In fact, \( W(T) \) is closed if and only if \(|A| = |(Av_2, v_1)|\).
Example 2.10. This example has its roots in the biological sciences. We note that genetic operators are idempotent. Let \( \Gamma = \{ x = (x_0, \ldots, x_{n-1}) : h(x) = 1, x_j \geq 0 \} \) be the set of all possible populations, where \( h(x) = \sum_{j=0}^{n-1} x_j, x \in \mathbb{R}^n \). Let \( P \) be a population distribution in \( \Gamma \) from which the first parent is drawn and \( q \in \Gamma \) for the second parent. Let \( L(p, q) \in \Gamma \) be the population distribution resulting from crossing other random parents from \( p \) and \( q \). Let \( e_i \) be the vector with a 1 at position \( i \) and zeroes elsewhere. Then \( e_i \) represents a population comprised entirely of copies of \( i \). In this case, the other component (\( k \)th component) of \( L(e_i, e_j) \) equals the probability that crossing \( i \) and \( j \) will produce \( k \). Therefore,

\[
L(p, q)_k = \sum p_i q_j L(e_i, e_j)_k
\]

\[
\Rightarrow L(p, q) = \sum_k \left( \sum_{i,j} p_i q_j L(e_i, e_j)_k \right) e_k
\]

\[
= \sum_{i,j} p_i q_j L(e_i, e_j).
\]

The sum \( L(p, q) \) is completely determined by vectors \( L(e_i, e_j) \), and this can be extended to \( \mathbb{R}^n \). We define the following operator:

\[
\Psi : \mathbb{R}^n \to \mathbb{R}^n \text{ by } \Psi(x_1, \ldots, x_n) = \sum_{i,j} x_i x_j L(e_i, e_j).
\]

Then, \( \Psi \) defines an idempotent operator. These operators are nonlinear. For more information on this topic, refer to [7].

Remark 2.11. It is easy to see that \( \| \Psi \| = 1 \). If \( A \) is a linear operator, then the compositions \( A \circ \Psi \) and \( \Psi \circ A \) are idempotent. This operator represents the effect of applying crossover to a population vector.

3 Similar Operators

Definition 3.1. The essential spectrum of an operator \( A \), denoted by \( \text{Sp}_e(A) \), is the set of scalars \( \lambda \) such that at least one of the following conditions holds:

(a) Range of \( (\lambda I - A) \) is not closed.

(b) \( \bigcup_{n=1}^{\infty} N((\lambda I - A)^n) \) is infinite dimensional.

The essential spectral radius is defined by \( r_e(A) = \sup \{ |\lambda| : \lambda \in \text{Sp}_e(A) \} \).

Theorem 3.2. The essential spectra of two similar operators on a Hilbert space \( H \) are the same.

Proof. Let \( P \) and \( Q \) be two similar operators on \( H \). Suppose that \( S \) is an invertible operator on \( H \) such that \( Q = S^{-1} PS \). Then the essential spectra of \( P \) and \( Q \) are equal since their spectra are the same. Furthermore, \( S^{-1}[(\lambda I - P)]^n = (\lambda I - Q)^n \) and \( [R((\lambda I - P))]^n = S^{-1}[R(\lambda I - P)]^n \). Also, for each positive integer \( n \), \( N(\lambda I - Q)^n = S^{-1}[N(\lambda I - P)^n] \).

\( \square \)
**Theorem 3.3.** An operator $T$ on $H$ is similar to a contraction if and only if $r_e(T) < 1$.

**Proof.** Let $S$ be a bounded and invertible operator on $H$. Then by Theorem 3.2, we have the following inequality: $r_e(T) = r_e(S^{-1}TS) \leq \alpha(S^{-1}TS)$, where $\alpha(S^{-1}TS)$ is a measure of noncompactness of $(S^{-1}TS)$. If $S$ is linear and bijective, then $r_e(T) \leq \inf\{\alpha(S^{-1}TS)\}$. Now, let $\delta > 0$. Then, for a finite dimensional operator $\psi$, we have from Theorem 3.2, $r(T + \psi) \leq r_e(T) + \delta/m, m = 2, 3, 4, \ldots$. Hence, $||S^{-1}(T + \psi)S|| \leq r_e(T) + \delta$. Furthermore, $\alpha$ is a seminorm and therefore the following is true:

$$\alpha(S^{-1}TS) = \alpha(S^{-1}TS + S^{-1}\psi S) = \alpha(S^{-1}(T + \psi)S) \leq ||S^{-1}(T + \psi)S||,$$

which implies that $\alpha(S^{-1}TS) \leq r_e(T) + \delta$. Since $\delta$ is arbitrarily chosen, the theorem follows from Theorem 3.2. 

**Remark 3.4.** We denote by $X'$ the set of invertible elements of a unital $C^*$ algebra $X$. Let $x$ be an element of $X$ such that the spectral radius $r(x)$, of $x$ is strictly less than one. Then, the series $\sum_{n=0}^{\infty} ||x^n||^2$ is convergent and belongs to $X$. Also, if $x_0 = \sum_{n=0}^{\infty} ||x^n||^2 x^n$, then $x_0 \geq 1$. Let $y$ be an element of $X$ such that $y = \sqrt{x_0}$ and $y \geq 1$, then $y \in X'$. Hence,

$$||yx^{-1}|| = \left|\left| y^{-1} \sum_{n=1}^{\infty} (x^*)^n y^{-1} \right|\right| = r(1 - y^{-2}) < 1 \Rightarrow ||yx^{-1}|| < 1.$$

Let $M$ be a closed two sided ideal of $X$. If $x$ is in $X$ and $X/M$ is a $C^*$-algebra, then $r(x + M) = \inf_{y \in M} r(x + y), y \in X'$. 

**Remark 3.5.** If $\kappa(H)$ denotes the set of compact operators on $H$ and if $T \in \kappa(H)$, then $||\kappa(H) + T|| = \alpha(T)$.

**References**


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