G$_\mu$-Closed Sets and G$_m$-Closed Sets in GTMS Spaces

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\textbf{Abstract :} The main purpose of this article is to introduce the concepts of G$_\mu$-closed sets and G$_m$-closed sets, which are a weak forms of closed sets in a generalized topology and minimal structure space. Some of their properties are studied. In particular, the characterizations of $\mu^G$-closed sets and $\mu m^G$-closed sets are obtained using G$_\mu$-closed and G$_m$-closed. Moreover, the notions of GT$_1$-GTMS spaces and GT$_2$-GTMS spaces are introduced.

\textbf{Keywords :} GTMS space; G$_\mu$-closed set; G$_m$-closed set; GT$_1$-GTMS space; GT$_2$-GTMS space.

\textbf{2010 Mathematics Subject Classification :} 54A05; 54C10.

1 Introduction

Generalized topology and minimal structure, which were the generalizations of topology, were first studied by Császár [1] and Popa and Noiri [2], respectively. After that, Buadong et al. [3] introduced the concept of generalized topology and minimal structure space (briefly GTMS-space), which was a non-empty set with generalized topology and minimal structure on its. They studied closed sets,

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This research has been funded by Mahasarakham University.

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open sets and weak separation axioms, which were $T_1$-GTMS and $T_2$-GTMS, in a GTMS space. Later, Zakari [4] proposed the notions of $\mu m$-closed sets, $\mu mG$-closed sets, and $mG$-closed sets in GTMS spaces. Moreover, lower separation axioms, which is $T_{00}$-GTMS and $R_0$-GTMS, were studied in a GTMS space. Also, $\mu m$-continuity on GTMS spaces was introduced by Zakari [5].

In this paper, we introduce the concepts of $G_{\mu}$-closed sets and $G_{m}$-closed sets in a GTMS space and study some properties of such sets. Moreover, we study some separation axioms in the GTMS space using $G_{\mu}$-open and $G_{m}$-open.

2 Preliminaries

In this section, we shall begin by repeating the concepts of minimal structure, see in [1] or [6]. A subcollection $m$ of subsets of a non-empty set $X$ is called a minimal structure (briefly, $m$-structure) on $X$ if $\emptyset \in m$ and $X \in m$. Each member of $m$ is said to be $m$-open and the complement of an $m$-open set is said to be $m$-closed. For a minimal structure $m$ on $X$ and $A \subset X$, $c_m(A) = \bigcap \{F : A \subset F$ and $X \setminus F \in m\}$ and $i_m(A) = \bigcup \{U : U \subset A$ and $U \in m\}$. Clearly, $i_m(A) \subset A \subset c_m(A)$. If $A, B \subset X$ and $m$ is a minimal structure on $X$, the following properties hold:

1. $c_m(X \setminus A) = X \setminus i_m(A)$ and $i_m(X \setminus A) = X \setminus c_m(A)$.
2. If $X \setminus A \in m$, then $c_m(A) = A$ and if $A \in m$, then $i_m(A) = A$.
3. If $A \subset B$, then $c_m(A) \subset c_m(B)$ and $i_m(A) \subset i_m(B)$.
4. $c_m(c_m(A)) = c_m(A)$ and $i_m(i_m(A)) = i_m(A)$.

Moreover, if $x \in X$ and $A \subset X$, then $x \in c_m(A)$ if and only if $U \cap A \neq \emptyset$ for every $U \in m$ containing $x$.

Next, we will recall the notions of generalized topology, see in [1] or [6]. A subfamily $\mu$ of subsets of a non-empty set $X$ is called a generalized topology (briefly, GT) on $X$ if $\emptyset \in \mu$ and any union of elements of $\mu$ belongs to $\mu$. A subset $A$ of $X$ is called $\mu$-open if $A \in \mu$. The complement of a $\mu$-open set is called a $\mu$-closed set. For a GT $\mu$ on $X$ and $A \subset X$, $c_\mu(A)$ is the intersection of all $\mu$-closed sets containing $A$, i.e., the smallest $\mu$-closed set containing $A$, and $i_\mu(A)$ is the union of all $\mu$-open sets contained in $A$, i.e., the largest $\mu$-open set contained in $A$. Obviously, $i_\mu(A) \subset A \subset c_\mu(A)$. If $A, B \subset X$ and $\mu$ is a GT on $X$, then the following statements hold:

1. $c_\mu(X \setminus A) = X \setminus i_\mu(A)$ and $i_\mu(X \setminus A) = X \setminus c_\mu(A)$.
2. If $X \setminus A \in \mu$, then $c_\mu(A) = A$ and if $A \in \mu$, then $i_\mu(A) = A$.
3. If $A \subset B$, then $c_\mu(A) \subset c_\mu(B)$ and $i_\mu(A) \subset i_\mu(B)$.
4. $c_\mu(c_\mu(A)) = c_\mu(A)$ and $i_\mu(i_\mu(A)) = i_\mu(A)$. 
In [7], \( x \in c_{\mu}(A) \) if and only if \( x \in V \in \mu \) implies \( V \cap A \neq \emptyset \).

Next, we will recall some concepts of GTMS spaces in [3] and [4]. A non-empty set \( X \) equipped with a GT \( \mu \) and a minimal structure \( m \) on it is called a generalized topology and minimal structure space or simply a GTMS space, is denoted by \((X, \mu, m)\). For a GTMS space \((X, \mu, m)\), a subset \( A \) of \( X \) is said to be \( \mu m \)-closed (resp. \( \mu m \)-open) \[3\] if \( c_{\mu}(c_{m}(A)) = A \) (resp. \( c_{m}(c_{\mu}(A)) = A \)). The complement of a \( \mu m \)-closed (resp. \( \mu m \)-open) set is said to be \( \mu m \)-open (resp. \( \mu m \)-closed). Then the following are equivalent:

1. \( A \) is \( \mu m \)-closed.
2. \( c_{\mu}(A) = A \) and \( c_{m}(A) = A \).
3. \( A \) is \( \mu m \)-closed.

In a GTMS space \((X, \mu, m)\), a subset \( A \) of \( X \) is said to be \( \text{closed} \) (resp. \( \text{open} \), \( \text{c-closed} \)) \[3\] if \( A \) is \( \mu m \)-closed (resp. \( \mu m \)-open, \( \mu m \)-closed). The complement of a \( \text{closed} \) (resp. \( \text{open} \), \( \text{c-closed} \)) set is said to be \( \text{open} \) (resp. \( \text{closed} \), \( \text{c-closed} \)). Clearly, \( A \) is open in a GTMS space \((X, \mu, m)\) if and only if \( i_{\mu}(A) = A \) and \( i_{m}(A) = A \). A subset \( A \) of \( X \) is said to be \( \mu m G \)-closed (resp. \( \mu m G \)-open, \( \mu m G \)-closed, \( mG \)-closed, \( \mu G \)-closed) \[4\] in a GTMS space \((X, \mu, m)\) if \( c_{\mu}(c_{m}(A)) \subseteq U \) (resp. \( c_{m}(c_{\mu}(A)) \subseteq U \), \( c_{m}(c_{\mu}(A)) \subseteq U \), \( c_{\mu}(c_{m}(A)) \subseteq U \)) whenever \( A \subseteq U \) and \( U \) is open (resp. \( U \) is open, \( U \) is \( \mu m \)-open, \( U \) is \( mG \)-closed). Also, a subset \( A \) of \( X \) is said to be \( G \)-closed (resp. \( G^* \)-closed) \[4\] in a GTMS space \((X, \mu, m)\) if \( A \) is \( \mu m G \)-closed and \( \mu m G \)-closed (resp. \( \mu G \)-closed and \( mG \)-closed).

Now, we recall some separation axioms in a GTMS space.

\textbf{Definition 2.1} \[3\]. A GTMS space \((X, \mu, m)\) is called a \textit{\( T_{0} \)-GTMS space} if for any pair of distinct points \( x \) and \( y \) in \( X \), there exist a subset \( U \) which is either \( \mu \)-open or \( m \)-open such that \( x \in U \), \( y \notin U \) or \( y \in U \), \( x \notin U \).

\textbf{Definition 2.2} \[3\]. A GTMS space \((X, \mu, m)\) is called a \textit{\( T_{1} \)-GTMS space} if for any pair of distinct points \( x \) and \( y \) in \( X \), there exist a \( \mu \)-open set \( U \) and a \( m \)-open set \( V \) such that \( x \in U \), \( y \notin U \) and \( y \in V \), \( x \notin V \).

\textbf{Definition 2.3} \[3\]. A GTMS space \((X, \mu, m)\) is called a \textit{\( T_{2} \)-GTMS space} if for any pair of distinct points \( x \) and \( y \) in \( X \), there exist a \( \mu \)-open set \( U \) and a \( m \)-open set \( V \) such that \( x \in U \), \( y \in V \) and \( U \cap V = \emptyset \).

\textbf{Definition 2.4} \[3\]. A GTMS space \((X, \mu, m)\) is called a \textit{\( R_{0} \)-GTMS space} if \( \{x\} \) is \( G^* \)-closed set for each \( x \in X \).

\textbf{Theorem 2.5} \[4\]. Let \((X, \mu, m)\) be a GTMS space. Then the following are equivalent:

1. \( X \) is a \( T_{1} \)-GTMS space.
2. \( X \) is a \( T_{0} \)-GTMS space and \( R_{0} \)-GTMS space.
3 \( G_\mu \)-Closed Sets and \( G_m \)-Closed Sets

In this section, we shall start by introducing the notion of \( G_\mu \)-closed sets and investigate some of their properties.

**Definition 3.1.** A subset \( A \) of a GTMS space \((X, \mu, m)\) is said to be a \( G_\mu \)-closed set if \( c_\mu(A) \subset U \) whenever \( A \subset U \) and \( U \) is open. The complement of a \( G_\mu \)-closed set is called a \( G_\mu \)-open set.

**Proposition 3.2.** Let \((X, \mu, m)\) be a GTMS space and \( A \subset X \). If \( A \) is a \( \mu mG \)-closed set, then \( A \) is a \( G_\mu \)-closed set.

**Proof.** Assume that \( A \) is \( \mu mG \)-closed and let \( U \) be open such that \( A \subset U \). Then \( c_\mu(c_m(A)) \subset U \). From \( c_\mu(A) \subset c_\mu(c_m(A)) \), we have \( c_\mu(A) \subset U \). Therefore, \( A \) is \( G_\mu \)-closed.

**Proposition 3.3.** Let \((X, \mu, m)\) be a GTMS space and \( A \subset X \). If \( A \) is a \( m\mu G \)-closed set, then \( A \) is a \( G_\mu \)-closed set.

**Proof.** The proof is similar to the proof of Proposition 3.2.

**Proposition 3.4.** Let \((X, \mu, m)\) be a GTMS space and \( A \subset X \). If \( A \) is a \( \mu \)-closed set, then \( A \) is a \( G_\mu \)-closed set.

**Proof.** It follows from the fact that if \( A \) is a \( \mu \)-closed set, then \( c_\mu(A) = A \).

**Remark 3.5.** The converse of Proposition 3.2, 3.3 and 3.4 may not be true as the following example.

**Example 3.6.** Consider the GTMS space \((X, \mu, m)\), where \( X = \{1, 2, 3, 4\} \),
\[ \mu = \{\emptyset, \{3, 4\}, \{1, 2, 3\}, X\} \) and \( m = \{\emptyset, \{2, 4\}, \{1, 2, 3\}, X\} \).
Then \( \{2\} \) is \( G_\mu \)-closed but it is not \( \mu mG \)-closed, \( m\mu G \)-closed and \( \mu \)-closed. Moreover, \( \{1, 3, 4\} \) is \( G_\mu \)-open.

**Proposition 3.7.** Let \((X, \mu, m)\) be a GTMS space and \( A \subset X \). If \( A \) is open and \( G_\mu \)-closed, then \( A \) is \( \mu \)-closed.

**Proof.** Assume that \( A \) is open and \( G_\mu \)-closed. Then \( c_\mu(A) \subset A \). Thus \( A \) is \( \mu \)-closed.

**Proposition 3.8.** Let \((X, \mu, m)\) be a GTMS space and \( A \subset X \). If \( A \) is \( G_\mu \)-closed, then \( c_\mu(A) \setminus A \) does not contain any nonempty closed set.

**Proof.** Assume that \( A \) is \( G_\mu \)-closed. Suppose to the contrary that \( c_\mu(A) \setminus A \) contains a nonempty closed set, say \( F \). Then \( F \subset c_\mu(A) \setminus A = c_\mu(A) \cap (X \setminus A) \). Thus \( A \subset X \setminus F \). Since \( A \) is \( G_\mu \)-closed and \( X \setminus F \) is open, \( c_\mu(A) \subset X \setminus F \). This implies \( F \subset X \setminus c_\mu(A) \). From \( F \subset c_\mu(A), F = \emptyset \) which contradicts with \( F \neq \emptyset \).
Proposition 3.8. Hence $c$ and $F$

By assumption, $X \subseteq X$, then $A$ is $G_\mu$-closed and $A \subset B$, then $B$ is $G_\mu$-closed.

Proof. Assume that $A$ is $G_\mu$-closed and $A \subset B$. Suppose $B$ is not $G_\mu$-closed. Thus there exists an open set $U$ such that $B \subseteq U$ and $c_\mu(B) \nsubseteq U$. Since $A$ is $G_\mu$-closed, $c_\mu(A) \subseteq U$, and so $X \setminus U \subseteq X \setminus c_\mu(A)$. This implies $X = (X \setminus c_\mu(A)) \cup U$ is $\mu$-open which contradicts with $X \not\subseteq \mu$. Thus $B$ is $G_\mu$-closed.

Theorem 3.9. Let $(X, \mu, m)$ be a GTMS space and $A \subset X$. Then $A$ is $G_\mu$-open if and only if $F \subset c_\mu(A)$ whenever $F$ is closed and $F \subset A$.

Proof. $(\Rightarrow)$ Let $F$ be closed such that $F \subset A$. Then $X \setminus F$ is open and $X \setminus A \subset X \setminus F$. By assumption, we obtain that $X \setminus A$ is $G_\mu$-closed, and so $c_\mu(X \setminus A) \subset X \setminus F$. Since $X \setminus i_\mu(A) = c_\mu(X \setminus A)$, $F \subset i_\mu(A)$.

$(\Leftarrow)$ Let $U$ be open such that $X \setminus A \subset U$. Then $X \setminus U$ is closed and $X \setminus U \subset A$. By assumption, $X \setminus U \subset i_\mu(A)$. Thus $X \setminus i_\mu(A) \subset U$, and so $c_\mu(X \setminus A) \subset U$. Hence $X \setminus A$ is $G_\mu$-closed, and so $A$ is $G_\mu$-open.

Proposition 3.11. Let $(X, \mu, m)$ be a GTMS space and $A \subset X$. If $A$ is $G_\mu$-closed, then $c_\mu(A) \setminus A$ is $G_\mu$-open.

Proof. Assume that $A$ is $G_\mu$-closed. Suppose to the contrary that $c_\mu(A) \setminus A$ is not $G_\mu$-open. By Theorem 3.10, there exists a closed set $F$ such that $F \subset c_\mu(A) \setminus A$ and $F \nsubseteq i_\mu(c_\mu(A) \setminus A)$. This implies $\emptyset \neq F \subset c_\mu(A) \setminus A$. It is a contradiction with Proposition 3.8. Hence $c_\mu(A) \setminus A$ is $G_\mu$-open.

Proposition 3.12. Let $(X, \mu, m)$ be a GTMS space and $A, B \subset X$. If $A$ is $G_\mu$-open and $i_\mu(A) \subset B \subset A$, then $B$ is $G_\mu$-open.

Proof. It follows from Theorem 3.9 and the fact that if $B \subset A \subset X$, then $i_\mu(B) \subset i_\mu(A)$ and $i_\mu(i_\mu(A)) \subset i_\mu(A)$.

Next, we will introduce the concept of $G_m$-closed sets and investigate some of their properties.

Definition 3.13. A subset $A$ of a GTMS space $(X, \mu, m)$ is said to a $G_m$-closed set if $c_m(A) \subset U$ whenever $A \subset U$ and $U$ is open. The complement of a $G_m$-closed set is called a $G_m$-open set.

Proposition 3.14. Let $(X, \mu, m)$ be a GTMS space and $A \subset X$. If $A$ is a $\mu mG$-closed set, then $A$ is a $G_m$-closed set.

Proof. The proof is similar to the proof of Proposition 3.12.

Proposition 3.15. Let $(X, \mu, m)$ be a GTMS space and $A \subset X$. If $A$ is a $m\mu G$-closed set, then $A$ is a $G_m$-closed set.

Proof. The proof is similar to the proof of Proposition 3.12.
Proposition 3.16. Let \((X, \mu, m)\) be a GTMS space and \(A \subset X\). If \(A\) is a \(m\)-closed set, then \(A\) is a \(G_m\)-closed set.

Proof. It follows from the fact that if \(A\) is a \(m\)-closed set, then \(c_m(A) = A\).

Remark 3.17. The converse of Propositions 3.14, 3.15 and 3.16 may not be true as the following example.

Example 3.18. In Example 3.6 we see that \(\{3\}\) is \(G_m\)-closed but it is not \(\mu m\)-closed, \(\mu \mu \mu G\)-closed and \(m\)-closed. Moreover, \(\{1, 2, 4\}\) is \(G_m\)-open.

Proposition 3.19. Let \((X, \mu, m)\) be a GTMS space and \(A \subset X\). If \(A\) is open and \(G_m\)-closed, then \(c_m(A) = A\).

Proof. Assume that \(A\) is open and \(G_m\)-closed. Then \(c_m(A) \subset A\). This implies \(c_m(A) = A\).

Proposition 3.20. Let \((X, \mu, m)\) be a GTMS space and \(A \subset X\). If \(A\) is \(G_m\)-closed, then \(c_m(A) \setminus A\) does not contain any nonempty closed set.

Proof. The proof is similar to the proof of Proposition 3.8.

Theorem 3.21. Let \((X, \mu, m)\) be a GTMS space and \(A \subset X\). Then \(A\) is \(G_m\)-open if and only if \(F \subset i_m(A)\) whenever \(F\) is closed and \(F \subset A\).

Proof. The proof is similar to the proof of Theorem 3.10.

Theorem 3.22. Let \((X, \mu, m)\) be a GTMS space and \(A \subset X\). If \(A\) is \(G_m\)-closed, then \(c_m(A) \setminus A\) is \(G_m\)-open.

Proof. It follows from Theorem 3.21 and Proposition 3.20.

Proposition 3.23. Let \((X, \mu, m)\) be a GTMS space and \(A, B \subset X\). If \(A\) is \(G_m\)-open and \(i_m(A) \subset B \subset A\), then \(B\) is \(G_m\)-open.

Proof. It follows from Theorem 3.21 and the fact that if \(B \subset A \subset X\), then \(i_m(B) \subset i_m(A)\) and \(i_m(i_m(A)) \subset i_m(A)\).

Now, we will give a characterization of \(\mu \mu G\)-closed sets and \(\mu m G\)-closed sets using \(G_{\mu}\)-closed sets and \(G_m\)-closed sets.

Theorem 3.24. Let \((X, \mu, m)\) be a GTMS space and \(A \subset X\). Then \(A\) is \(\mu \mu G\)-closed if and only if \(A\) is \(G_{\mu}\)-closed and \(c_{\mu}(A)\) is \(G_m\)-closed.

Proof. \((\Rightarrow)\) Assume that \(A\) is \(\mu \mu G\)-closed. By Proposition 3.3, we have \(A\) is \(G_{\mu}\)-closed. Next, we shall prove that \(c_{\mu}(A)\) is \(G_m\)-closed. Let \(U\) be an open set such that \(c_{\mu}(A) \subset U\). Then \(A \subset U\). Since \(A\) is \(\mu \mu G\)-closed, \(c_m(c_{\mu}(A)) \subset U\). Then \(c_m(A)\) is \(G_m\)-closed.

\((\Leftarrow)\) Assume that \(A\) is \(G_{\mu}\)-closed and \(c_{\mu}(A)\) is \(G_m\)-closed. To show that \(A\) is \(\mu \mu G\)-closed, let \(U\) be an open set such that \(A \subset U\). Since \(A\) is \(G_{\mu}\)-closed, \(c_{\mu}(A) \subset U\). Since \(c_{\mu}(A)\) is \(G_m\)-closed, \(c_m(c_{\mu}(A)) \subset U\). Then \(A\) is \(\mu \mu G\)-closed.
Theorem 3.25. Let \((X, \mu, m)\) be a GTMS space and \(A \subset X\). Then \(A\) is \(\mu m\)-closed if and only if \(A\) is \(G_m\)-closed and \(cm(A)\) is \(G_\mu\)-closed.

Proof. The proof is similar to the proof of Theorem 3.24.

Finally, we will discuss a relation of \(G_\mu\)-closed sets and \(G_m\)-closed sets under some conditions.

Theorem 3.26. Let \((X, \mu, m)\) be a GTMS space such that \(X \notin \mu\) and \(A \subset X\). If \(A\) is \(G_\mu\)-closed, then \(A\) is \(G_m\)-closed.

Proof. Assume that \(A\) is \(G_\mu\)-closed. Suppose to the contrary that \(A\) is not \(G_m\)-closed. Then there exists an open set \(U\) such that \(A \subset U\) and \(cm(A) \notin U\). Since \(A\) is \(G_\mu\)-closed, \(c_\mu(A) \subset U\). From \(c_\mu(A)\) is \(\mu\)-closed, \(X \setminus c_\mu(A)\) is \(\mu\)-open. Since \(U\) is open, \(U\) is \(\mu\)-open. This implies \(X = (X \setminus c_\mu(A)) \cup U\) is \(\mu\)-open. Thus \(X \in \mu\) which contradicts with \(X \notin \mu\). Thus \(A\) is \(G_m\)-closed.

Corollary 3.27. Let \((X, \mu, m)\) be a GTMS space such that \(X \notin \mu\) and \(A \subset X\). If \(A\) is \(G_\mu\)-open, then \(A\) is \(G_m\)-open.

Proof. It follows from Theorem 3.26.

4 GT\(_1\)-GTMS Spaces and GT\(_2\)-GTMS Spaces

In this section, we shall introduce the notions of GT\(_1\)-GTMS spaces and GT\(_2\)-GTMS spaces and investigate some of their characterization. We start by defining the \(G_\mu\)-closure and \(G_m\)-closure of a set in GTMS spaces.

Definition 4.1. Let \((X, \mu, m)\) be a GTMS space and \(A \subset X\). Defined the \(G_\mu\)-closure and \(G_m\)-closure of \(A\) as follows:

\[
c_{G_\mu}(A) = \bigcap \{K : K \text{ is } G_\mu\text{-closed and } A \subset K\}
\]

and

\[
c_{G_m}(A) = \bigcap \{K : K \text{ is } G_m\text{-closed and } A \subset K\},
\]

respectively.

Lemma 4.2. Let \((X, \mu, m)\) be a GTMS space and \(A \subset X\). Then \(x \in c_{G_\mu}(A)\) if and only if \(A \cap U \neq \emptyset\) for all \(G_\mu\)-open \(U\) containing \(x\).

Proof. \((\Rightarrow)\) Assume that there exists a \(G_\mu\)-open set \(U\) containing \(x\) such that \(A \cap U = \emptyset\). Then \(X \setminus U\) is \(G_\mu\)-closed and \(A \subset X \setminus U\). Since \(x \notin X \setminus U\), \(x \not\in c_{G_\mu}(A)\).

\((\Leftarrow)\) Assume that \(x \notin c_{G_\mu}(A)\). Then there exists a \(G_\mu\)-closed set \(K\) such that \(A \subset K\) and \(x \notin K\). Thus \(X \setminus K\) is \(G_\mu\)-open and \(x \in X \setminus K\). Moreover, \(A \cap (X \setminus K) = \emptyset\).
Lemma 4.3. Let $(X, \mu, m)$ be a GTMS space and $A \subset X$. Then $x \in c_{G_m}(A)$ if and only if $A \cap U \neq \emptyset$ for all $G_m$-open $U$ containing $x$.

Proof. The proof is similar to the proof of Lemma 4.2.

Now, we shall give definition of $GT_1$-GTMS spaces.

Definition 4.4. A GTMS space $(X, \mu, m)$ is said to be $GT_1$-GTMS if for pair of distinct points $x$ and $y$ in $X$, there exist a $G_\mu$-open set $U$ and a $G_m$-open set $V$ such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$.

Proposition 4.5. If $(X, \mu, m)$ is $T_1$-GTMS, then $(X, \mu, m)$ is $GT_1$-GTMS.

Proof. It follows from the fact that every $\mu$-open set is $G_\mu$-open and every $m$-open set is $G_m$-open.

Remark 4.6. The converse of Proposition 4.5 may not be true as the following example.

Example 4.7. Consider the GTMS space $(X, \mu, m)$, where $X = \{1, 2, 3\}$,

$$
\mu = \emptyset, \{1, 2\}, \{1, 3\}, X \text{ and } m = \emptyset, X.
$$

Then $(X, \mu, m)$ is $GT_1$-GTMS but it is not $T_1$-GTMS.

Now, we will give a characterization of $GT_1$-GTMS spaces.

Theorem 4.8. Let $(X, \mu, m)$ be a GTMS space. Then the following are equivalent:

1. $(X, \mu, m)$ is $GT_1$-GTMS.
2. $c_{G_\mu}(\{x\}) = \{x\}$ and $c_{G_m}(\{x\}) = \{x\}$ for all $x \in X$.

Proof. (1) $\Rightarrow$ (2) Assume that $X$ is $GT_1$-GTMS. We will show that $c_{G_\mu}(\{x\}) = \{x\}$ and $c_{G_m}(\{x\}) = \{x\}$ for all $x \in X$. Let $x \in X$. It is clear that $\{x\} \subset c_{G_\mu}(\{x\})$. Let $y \in X$ be such that $y \neq x$. By assumption, there exists a $G_\mu$-open set $U$ such that $y \in U$ but $x \notin U$. Then $U \cap \{x\} = \emptyset$. Thus $y \notin c_{G_\mu}(\{x\})$. Hence $c_{G_\mu}(\{x\}) \subset \{x\}$. Then $c_{G_\mu}(\{x\}) = \{x\}$. Similarly, we can prove that $c_{G_m}(\{x\}) = \{x\}$.

(2) $\Rightarrow$ (1) Assume that $c_{G_\mu}(\{x\}) = \{x\}$ and $c_{G_m}(\{x\}) = \{x\}$ for all $x \in X$. To show that $X$ is $GT_1$-GTMS, let $x, y \in X$ with $x \neq y$. By assumption, $c_{G_\mu}(\{x\}) = \{x\}$ and $c_{G_\mu}(\{y\}) = \{y\}$. Thus $x \notin c_{G_\mu}(\{y\})$ and $y \notin c_{G_\mu}(\{x\})$. Then there exist a $G_\mu$-open set $U$ and $G_m$-open set $V$ such that $x \in U$, $\{y\} \cap U = \emptyset$ and $y \in V$, $\{x\} \cap V = \emptyset$. Hence $X$ is $GT_1$-GTMS.

Theorem 4.9. Let $(X, \mu, m)$ be a GTMS space such that $X$ has at least two elements and $X \notin \mu$. Then $X$ is $GT_1$-GTMS if and only if $\{a\}$ is $G_\mu$-open in $X$ for all $a \in X$. 
Proof. \((\Rightarrow)\) Assume that \(X\) is GT1-GTMS. To show that \(\{a\}\) is \(G_\mu\)-open in \(X\) for all \(a \in X\), let \(a \in X\). Suppose \(\{a\}\) is not \(G_\mu\)-open. Then there exists a closed set \(F\) such that \(F \subseteq \{a\}\) and \(F \not\subseteq i_\mu(\{a\})\). This implies \(\{a\} = F\) is closed. Thus \(X \setminus \{a\}\) is open. Since \(X\) has at least two elements, \(X \setminus \{a\} \neq \emptyset\), say \(b \in X \setminus \{a\}\).

By assumption, there exist a \(G_\mu\)-open set \(U\) and a \(G_m\)-open set \(V\) such that 
\[
a \in U, b \notin U \text{ and } b \in V, a \notin V.
\]
Since \(U\) is \(G_\mu\)-open and \(\{a\}\) is closed such that \(\{a\} \subset U\), \(\{a\} \subset i_\mu(U)\). Then \(X = (X \setminus \{a\}) \cup i_\mu(U) \in \mu\) which contradicts with \(X \notin \mu\). Hence \(\{a\}\) is \(G_\mu\)-open.

\((\Leftarrow)\) Assume that \(\{a\}\) is \(G_\mu\)-open in \(X\) for all \(a \in X\). To show that \(X\) is GT1-GTMS, let \(x, y \in X\) be such that \(x \neq y\). By assumption, \(\{x\}\) and \(\{y\}\) is \(G_\mu\)-open. Since \(X \notin \mu\), by Corollary 5.27 \(\{y\}\) is \(G_m\)-open. Set \(U = \{x\}\) and \(V = \{y\}\). Then \(U\) is \(G_\mu\)-open and \(V\) is \(G_m\)-open. Moreover, \(x \in U, y \notin U\) and \(y \in V, x \notin V\). Hence \(X\) is GT1-GTMS.

Next, we will introduce the concepts of GT0-GTMS spaces and GR0-GTMS spaces.

**Definition 4.10.** A GTMS space \((X, \mu, m)\) is called GT0-GTMS if for any pair of distinct points \(x\) and \(y\) in \(X\), there exists a subset \(U\) of \(X\) which is \(G_\mu\)-open or \(G_m\)-open such that \(x \in U, y \notin U\) or \(y \in U, x \notin U\).

**Lemma 4.11.** If \((X, \mu, m)\) is GT1-GTMS, then \((X, \mu, m)\) is GT0-GTMS.

**Proof.** Assume that \((X, \mu, m)\) is GT1-GTMS. To show that \((X, \mu, m)\) is GT0-GTMS, let \(x, y \in X\) with \(x \neq y\). Since \((X, \mu, m)\) is GT1-GTMS, there exist a \(G_\mu\)-open set \(U\) and a \(G_m\)-open set \(V\) such that \(x \in U, y \notin U\) and \(y \in V, x \notin V\). Hence \((X, \mu, m)\) is GT0-GTMS.

**Remark 4.12.** The converse of the previous Lemma 4.11 need not be true as the following example.

**Example 4.13.** Consider the GTMS space \((X, \mu, m)\), where \(X = \{1, 2, 3\}, \mu = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, X\}\), and \(m = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, X\}\).

Then \((X, \mu, m)\) is GT0-GTMS but it is not GT1-GTMS.

**Definition 4.14.** A GTMS space \((X, \mu, m)\) is said to be GR0-GTMS if for each \(x, y \in X\) if \(x \in c_{G_\mu}(c_{G_m}(\{y\}))\), then \(y \in c_{G_\mu}(c_{G_m}(\{x\}))\) and if \(x \in c_{G_m}(c_{G_\mu}(\{y\}))\), then \(y \in c_{G_m}(c_{G_\mu}(\{x\}))\).

**Example 4.15.** Consider the GTMS space \((X, \mu, m)\), where \(X = \{1, 2, 3\}, \mu = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, X\}\).

Then \((X, \mu, m)\) is GR0-GTMS.

**Lemma 4.16.** If \((X, \mu, m)\) is GT1-GTMS, then \((X, \mu, m)\) is GR0-GTMS.
Proof. Assume that \((X, \mu, m)\) is GT\(_1\)-GTMS. To show that \((X, \mu, m)\) is GR\(_0\)-GTMS, let \(x, y \in X\) with \(x \in c_{G_m}(c_{G_m}({\{y\}}))\). Since \(X\) is GT\(_1\)-GTMS, we obtain that \(c_{G_m}(c_{G_m}({\{y\}})) = c_{G_m}({\{y\}}) = \{y\}\). Then \(x \in \{y\}\), and so \(x = y\). Hence \(y \in c_{G_m}(\{x\})\). Similarly, we can prove that if \(x \in c_{G_m}(c_{G_m}({\{y\}}))\), then \(y \in c_{G_m}(\{x\})\). Therefore, \((X, \mu, m)\) is GR\(_0\)-GTMS.

**Theorem 4.17.** \((X, \mu, m)\) is GT\(_1\)-GTMS if and only if \((X, \mu, m)\) is GT\(_0\)-GTMS and GR\(_0\)-GTMS.

Proof. \((\Rightarrow)\) It follows from Lemma 4.11 and 4.16.

\((\Leftarrow)\) Assume that \((X, \mu, m)\) is GT\(_0\)-GTMS and GR\(_0\)-GTMS. To show that \((X, \mu, m)\) is GT\(_1\)-GTMS, fix \(x \in X\). Let \(y \in X\) with \(y \neq x\). Since \(X\) is GT\(_0\)-GTMS, there exists a subset \(U\) of \(X\) which is \(G_m\)-open or \(G_m\)-open such that \(x \in U\), \(y \notin U\) or \(y \in U, x \notin U\). Without loss of generality, we assume that \(U\) is \(G_m\)-open. If \(x \in U\) and \(y \notin U\), then \(x \notin c_{G_m}(\{y\})\). Since \(X\) is GR\(_0\)-GTMS, \(y \notin c_{G_m}(c_{G_m}(\{x\}))\), and so \(y \notin c_{G_m}(\{x\})\). On the other hand, if \(y \in U\) and \(x \notin U\), then \(y \notin c_{G_m}(\{x\})\). Since \(X\) is GR\(_0\)-GTMS, \(x \notin c_{G_m}(c_{G_m}(\{y\}))\). Then \(x \notin c_{G_m}(\{y\})\), and so \(y \notin c_{G_m}(\{x\})\) and \(y \notin c_{G_m}(\{x\})\). This implies \(c_{G_m}(\{x\}) = \{x\}\) and \(c_{G_m}(\{x\}) = \{x\}\). Therefore, \((X, \mu, m)\) is GT\(_1\)-GTMS.

Now, we will introduce the notion of GT\(_2\)-GTMS spaces.

**Definition 4.18.** A GTMS space \((X, \mu, m)\) is is said to be GT\(_2\)-GTMS if for any pair of distinct points \(x\) and \(y\) in \(X\), there exist a \(G_m\)-open set \(U\) and a \(G_m\)-open set \(V\) such that \(x \in U\), \(y \in V\) and \(U \cap V = \emptyset\).

**Proposition 4.19.** If \((X, \mu, m)\) is T\(_2\)-GTMS, then \((X, \mu, m)\) is GT\(_2\)-GTMS.

Proof. Assume that \((X, \mu, m)\) is T\(_2\)-GTMS. To show that \((X, \mu, m)\) is GT\(_2\)-GTMS, let \(x, y \in X\) be such that \(x \neq y\). Since \(X\) is T\(_2\)-GTMS, there exist \(G_m\)-open \(U\) and \(G_m\)-open \(V\) such that \(x \in U\), \(y \in V\) and \(U \cap V = \emptyset\). By Proposition 3.4 and 3.16, \(U\) is \(G_m\)-open and \(V\) is \(G_m\)-open. Therefore, \((X, \mu, m)\) is GT\(_2\)-GTMS.

**Example 4.20.** Consider the GTMS space \((X, \mu, m)\), where \(X = \{1, 2, 3\}\),

\[
\mu = \{\emptyset\} \text{ and } m = \{\emptyset, X\}.
\]

Then \((X, \mu, m)\) is GT\(_2\)-GTMS but it is not T\(_2\)-GTMS.

**Lemma 4.21.** If \((X, \mu, m)\) is GT\(_2\)-GTMS, then \((X, \mu, m)\) is GT\(_1\)-GTMS.

Proof. Assume that \((X, \mu, m)\) is GT\(_2\)-GTMS. To show that \((X, \mu, m)\) is GT\(_1\)-GTMS, let \(x, y \in X\) be such that \(x \neq y\). Since \(X\) is GT\(_2\)-GTMS, there exist \(G_m\)-open \(U\) and \(G_m\)-open \(V\) such that \(x \in U\), \(y \in V\) and \(U \cap V = \emptyset\). Then \(x \in U\), \(y \notin U\) and \(y \in V\), \(x \notin V\). Hence \(X\) is GT\(_1\)-GTMS.

**Remark 4.22.** The converse of Lemma 4.21 need not be true as the following example.
Example 4.23. In Example 4.19 we see that \((X, \mu, m)\) is GT$_1$-GTMS but it is not GT$_2$-GTMS.

Next, we will introduce the concept of GR$_1$-GTMS spaces.

Definition 4.24. A GTMS space \((X, \mu, m)\) is said to be GR$_1$-GTMS if for all \(x, y \in X\) with \(x \neq y\) if \(c_{G\mu}(\{x\}) \neq c_{Gm}(\{y\})\), then there exist disjoint a \(G\mu\)-open set \(U\) and a \(Gm\)-open set \(V\) such that \(c_{G\mu}(\{x\}) \subset U\) and \(c_{Gm}(\{y\}) \subset V\).

Example 4.25. Consider the GTMS space \((X, \mu, m)\), where \(X = \{1, 2\}\),

\[\mu = \{\emptyset, \{1\}, \{2\}, X\} = m.\]

Then \((X, \mu, m)\) is GR$_1$-GTMS.

Lemma 4.26. If \((X, \mu, m)\) is GR$_1$-GTMS, then \((X, \mu, m)\) is GR$_0$-GTMS.

Proof. Assume that \((X, \mu, m)\) is GR$_1$-GTMS. To show that \((X, \mu, m)\) is GR$_0$-GTMS, let \(x, y \in X\) with \(x \neq y\) and \(c_{G\mu}(\{x\}) \neq c_{Gm}(\{y\})\). By assumption, there exist disjoint \(G\mu\)-open set \(U\) and \(Gm\)-open set \(V\) such that \(c_{G\mu}(\{x\}) \subset U\) and \(c_{Gm}(\{y\}) \subset V\). Thus \(x \notin c_{G\mu}(c_{Gm}(\{y\}))\). Similarly, we can prove that if \(y \notin c_{G\mu}(\{x\})\), then \(x \notin c_{Gm}(c_{G\mu}(\{y\}))\). Therefore, \((X, \mu, m)\) is GR$_0$-GTMS.

Remark 4.27. The converse of Lemma 4.30 may not be true as the following example.

Example 4.28. In Example 4.15 we see that \((X, \mu, m)\) is GR$_0$-GTMS but it is not GR$_1$-GTMS.

Lemma 4.29. If \((X, \mu, m)\) is GT$_2$-GTMS, then \((X, \mu, m)\) is GR$_1$-GTMS.

Proof. Assume that \((X, \mu, m)\) is GT$_2$-GTMS. To show that \((X, \mu, m)\) is GR$_1$-GTMS, let \(x, y \in X\) with \(x \neq y\) and \(c_{G\mu}(\{x\}) \neq c_{Gm}(\{y\})\). By assumption and Lemma 4.21 \((X, \mu, m)\) is GT$_1$-GTMS. Then \(c_{G\mu}(\{x\}) = \{x\}\) and \(c_{Gm}(\{y\}) = \{y\}\). Since \((X, \mu, m)\) is GT$_2$-GTMS, there exist disjoint \(G\mu\)-open \(U\) and \(Gm\)-open \(V\) such that \(c_{G\mu}(\{x\}) = \{x\} \subset U\) and \(c_{Gm}(\{y\}) = \{y\} \subset V\). Therefore, \((X, \mu, m)\) is GR$_1$-GTMS.

Theorem 4.30. \((X, \mu, m)\) is GT$_2$-GTMS if and only if \((X, \mu, m)\) is GT$_0$-GTMS and GR$_1$-GTMS.

Proof. \((\Rightarrow)\) It follows from Lemma 4.21, 4.11 and 4.29.

\((\Leftarrow)\) Assume that \((X, \mu, m)\) is GT$_0$-GTMS and GR$_1$-GTMS. By Lemma 4.26 and Theorem 4.17 \((X, \mu, m)\) is GT$_1$-GTMS. To show that \((X, \mu, m)\) is GT$_2$-GTMS, let \(x, y \in X\) with \(x \neq y\). Since \(X\) is GT$_1$-GTMS, thus \(c_{G\mu}(\{x\}) = \{x\} \neq \{y\} = c_{Gm}(\{y\})\). Since \(X\) is GR$_1$-GTMS, there exist disjoint a \(G\mu\)-open set \(U\) and a \(Gm\)-open set \(V\) such that \(c_{G\mu}(\{x\}) = \{x\} \subset U\) and \(c_{Gm}(\{y\}) = \{y\} \subset V\). Therefore, \((X, \mu, m)\) is GT$_2$-GTMS.
Acknowledgements: The authors would like to thank the referees for helpful comments and suggestions on the manuscript. The authors also would like to thank Mahasarakham University for the financial support.

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(Received 5 February 2017)
(Accepted 25 September 2017)