Best Proximity Point Theorems for $G$-Proximal Generalized Contraction in Complete Metric Spaces endowed with Graphs

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Abstract: In this work, we present the notion of a $G$-proximal generalized contraction which is a development of well known mappings by Banach, Kannan, Chatterjea from self mappings to non-self mappings and we prove best proximity point theorems for this mapping in a complete metric space endowed with a directed graph. Moreover, we apply our main theorems to prove an existence of coupled best proximity points in a complete metric space endowed with a directed graph.

Keywords: best proximity point; proximally $G$-edge preserving; $G$-proximal generalized contraction.

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1 Introduction and Preliminaries

In 1922, Stefan Banach [1] presented the notion of contractions and established the famous theorem which is called a Banach contraction principle or a Banach fixed point theorem.

Theorem 1.1. [1] Let $(X,d)$ be a complete metric space and a self mapping $S : X \to X$ be a contraction, that is, there exists a nonnegative real number $k < 1$
such that  
\[ d(Sx, Sy) \leq kd(x, y), \quad \text{for all } x, y \in X. \]  
(1.1)

Then \( S \) has a unique fixed point in \( X \), i.e., there exists \( x \in X \) such that \( Sx = x \).

The Banach contraction principle has been applied for solving the existence of solutions of various equations in many fields of analysis such as Applied Mathematics, Applied Sciences, Physics, Economics, etc. In fact, if \( S \) satisfies (1.1), then it is always forced continuity.

In 1968, Kannan [2] introduced the concept of a Kannan mapping which is another notion of contraction that need not be continuous as follows:

A mapping \( S : X \to X \) is called a Kannan mapping if there exists a nonnegative real number \( a < \frac{1}{2} \) such that  
\[ d(Sx, Sy) \leq a[d(x, Sx) + d(y, Sy)], \quad \text{for all } x, y \in X. \]  
(1.2)

He proved the existence of a fixed point of the Kannan mapping in a complete metric space. Based on the condition (1.2), Chatterjea [3] introduced the concept of a \( C \)-contraction mapping as follows:

\[ d(Sx, Sy) \leq a[d(x, Sy) + d(y, Sx)], \quad \text{for all } x, y \in X. \]  
(1.3)

He proved that every mappings satisfy the condition (1.3) in a complete metric space have a unique fixed point. It can be seen in [4] that the conditions (1.1) and (1.2) are independent. Similarly, (1.1) and (1.3) are also independent. Some generalizations of Banach, Kannan, \( C \)-contractions were studied in [4–7].

The study and inspiration of literatures mentioned above, the purpose of this article is to study the best proximity point theorems of non-self mappings which is general than the mappings above. Let \( W \) and \( V \) be two nonempty subsets of a metric space \( (X, d) \) and let \( S : W \to V \) a non-self mapping. Observe that the equation \( Sx = x \) may not have a solution, if \( W \cap V \) is nonempty. So, it is natural to ask that how far is the distance between \( x \) and \( Sx \)? Therefore, the study of a best proximity point has played an important role and it is a problem of global optimization for determining the minimum valued of the distance  
\[ d(x, Sx) = \min \{d(x, y) : x \in W \text{ and } y \in V \}. \]

In 1969, Fan [8] presented the first result concerning best proximity point theorems. He proved that if \( S : W \to X \) is a continuous non-self mapping, where \( W \) is a nonempty compact convex subset in a normed vector space \( X \), then there exists \( w \in W \) such that \( \|w - Sw\| = d(Sw, W) \) where \( d(Sw, W) := \min\{\|Sw - a\| : a \in W\} \). Following the Fan’s Theorem, best proximity point theorems of non-self mappings get a lot of attention and have been studied by many researchers. For more details about best proximity point theorems, see Kirk et al. [9], Reich [10], Polla [11], Sehgal and Singh [12, 13], Vetrivel et al. [14], Amuradha and Veeramani [15], Basha [16, 17], Basha and Veeramani [18], Eldred et al. [19], Eldred and Veeramani [20], Raj [21], Abkar and Gabeleh [22], and Gabeleh [23].
Throughout this article, we denote $W$ and $V$ are nonempty subsets of a metric space $(X, d)$ and we also need the following notions:

$$d(W, V) := \inf \{ d(x, y) : x \in W \text{ and } y \in V \},$$

$$W_0 := \{ x \in W : d(x, y) = d(W, V) \text{ for some } y \in V \},$$

$$V_0 := \{ y \in V : d(x, y) = d(W, V) \text{ for some } x \in W \}.$$

In 2011, Basha [16] gave the following definition of a proximal contraction for non-self mappings in a metric space:

**Definition 1.2.** [16] Let $S : W \to V$ be a non-self mapping. Then $S$ is called a proximal contraction if there exists $k \in [0, 1)$ and for every $u_1, u_2, x, y \in W$,

$$d(u_1, Sx) = d(W, V) \quad d(u_2, Sy) = d(W, V)$$

$$\implies d(u_1, u_2) \leq kd(x, y). \quad (1.4)$$

Inspired and motivated by the above works, in this article, we introduce the new concept of a $G$-proximal generalized contraction for non-self mappings and establish best proximity point theorems for a $G$-proximal generalized contraction in a complete metric space endowed with a directed graph. Moreover, we can apply our main results to prove an existence of coupled best proximity point in a complete metric space endowed with a directed graph. An example to support and explain our main result is also presented.

Next, we recall some mappings and notions regarding a graph.

Let $(X, d)$ be a metric space and $G = (V(G), E(G))$ a directed graph which has no parallel edges such that the set $V(G)$ of its vertices coincides with $X$ and the set $E(G)$ of its edges is a subset of $X \times X$. The conversion of a graph $G$ denoted by $G^{-1}$ i.e.,

$$E(G^{-1}) = \{(x, y) \in X \times X : (y, x) \in E(G)\}.$$

We start with the following definition:

**Definition 1.3.** Let $(X, d)$ be a metric space and $G = (V(G), E(G))$ a directed graph such that $V(G) = X$. A non-self mapping $S : W \to V$ is called a $G$-proximal Kannan mapping if there exists $b \in [0, \frac{1}{2})$ such that

$$\begin{cases}
(x, y) \in E(G) \\
\quad d(u, Sx) = d(W, V) \\
\quad d(v, Sy) = d(W, V)
\end{cases} \implies d(u, v) \leq b[d(x, v) + d(y, u)], \quad (1.5)$$

where $x, y, u, v \in W$.

**Definition 1.4.** Let $(X, d)$ be a metric space and $G = (V(G), E(G))$ a directed graph such that $V(G) = X$. A non-self mapping $S : W \to V$ is said to be
(i) proximally $G$-edge-preserving if for each $x, y, u, v \in W$, 
\[
\begin{align*}
(x, y) \in E(G) \\
d(u, Sx) = d(W, V) \\
d(v, Sy) = d(W, V)
\end{align*}
\] 
\[\Rightarrow (u, v) \in E(G);\]

(ii) $G$-proximal generalized contraction if there exists $k \in [0, 1)$ and each $x, y, u, v \in W$ such that 
\[
\begin{align*}
(x, y) \in E(G) \\
d(u, Sx) = d(W, V) \\
d(v, Sy) = d(W, V)
\end{align*}
\] 
\[\Rightarrow d(u, v) \leq kM(x, y), \quad (1.6)\]
where $M(x, y) = \max \left\{d(x, y), d(x, u), d(y, v), \frac{d(x, v) + d(y, u)}{2}\right\}$.

From the Definition 1.4(ii), we observe that (1) $S$ is said to be a $G$-proximal contraction, if $M(x, y) = d(x, y)$, and 
(2) $S$ is said to be a $G$-proximal $C$-contraction, if $M(x, y) = \frac{d(x, v) + d(y, u)}{2}$.

2 Main Results

In this section, we will prove best proximity point theorems for a $G$-proximal generalized contraction in a complete metric space endowed with a directed graph.

**Theorem 2.1.** Let $(X, d)$ be a complete metric space, $G = (V(G), E(G))$ a directed graph such that $V(G) = X$. Let $W$ and $V$ be nonempty closed subsets of $X$ with $W_0$ is nonempty and let $S : W \to V$ be a non-self mapping which satisfies the following properties:

(i) $S$ is proximally $G$-edge-preserving, continuous and $G$-proximal generalized contraction such that $S(W_0) \subseteq V_0$;

(ii) there exist $x_0, x_1 \in W_0$ such that 
\[d(x_1, Sx_0) = d(W, V) \text{ and } (x_0, x_1) \in E(G).\]

Then $S$ has a best proximity point in $W$, that is, there exists an element $w \in W$ such that 
\[d(w, Sw) = d(W, V).\]

Further, the sequence $\{x_n\}$, defined by 
\[d(x_n, Sx_{n-1}) = d(W, V), \text{ for all } n \in \mathbb{N},\]
converges to the element $w$. 
Proof. From the condition (ii), there exist \( x_0, x_1 \in W_0 \) such that
\[
d(x_1, Sx_0) = d(W, V) \quad \text{and} \quad (x_0, x_1) \in E(G). \tag{2.1}
\]
Since \( S(W_0) \subseteq V_0 \), we have \( Sx_1 \in V_0 \) and hence there exits \( x_2 \in W_0 \) such that
\[
d(x_2, Sx_1) = d(W, V). \tag{2.2}
\]
By the proximally \( G \)-edge preserving of \( S \) and using both (2.1) and (2.2), we get
\[
(x_1, x_2) \in E(G). \tag{2.3}
\]
Next, we will show that \( S \) has a best proximity point in \( W_0 \). Suppose that there exists \( n_0 \in \mathbb{N}, \) such that \( x_{n_0} = x_{n_0+1} \). By using (2.3), we obtain that
\[
d(x_{n_0}, Sx_{n_0}) = d(W, V) \quad \text{and} \quad (x_{n_0}, x_{n_0+1}) \in E(G). \tag{2.4}
\]
Thus we have
\[
d(x_n, x_{n+1}) \leq kM(x_{n-1}, x_n), \tag{2.5}
\]
where
\[
M(x_{n-1}, x_n) = \max \left\{ d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_{n+1}) + d(x_n, x_n)}{2} \right\}
\]
\[
= \max \left\{ d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_{n+1})}{2} \right\}
\]
Case 1. If \( M(x_{n-1}, x_n) = d(x_{n-1}, x_n) \), for all \( n \in \mathbb{N} \), then by (2.4) we obtain
\[
d(x_n, x_{n+1}) \leq kd(x_{n-1}, x_n), \quad \text{for all} \quad n \in \mathbb{N}.
\]
By above inequality, we have
\[
d(x_1, x_2) \leq kd(x_0, x_1),
\]
and hence
\[
d(x_2, x_3) \leq k^2d(x_0, x_1).
\]
By induction, we can conclude that
\[
d(x_n, x_{n+1}) \leq k^n d(x_0, x_1), \quad \text{for all} \quad n \in \mathbb{N}. \tag{2.6}
\]
From (2.6), for each \( m, n \in \mathbb{N} \) with \( m > n \),
\[
d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{m-1}, x_m)
\leq k^n d(x_0, x_1) + k^{n+1} d(x_0, x_1) + \ldots + k^{m-1} d(x_0, x_1)
\]
\[
= d(x_0, x_1) \sum_{i=n}^{m-1} k^i
\leq \frac{k^n}{1-k} d(x_0, x_1).
\]
Since \( 0 \leq k < 1 \), it follows that \( \{x_n\} \) is a Cauchy sequence in \( W \).

**Case 2.** If \( M(x_{n-1}, x_n) = d(x_n, x_{n+1}) \), for all \( n \in \mathbb{N} \), then by using (2.4), we have
\[
d(x_n, x_{n+1}) \leq k d(x_{n-1}, x_{n+1})
\leq \frac{k}{2} [d(x_{n-1}, x_n) + d(x_n, x_{n+1})].
\]
It implies that
\[
d(x_n, x_{n+1}) \leq \frac{k}{2} d(x_{n-1}, x_n). \quad (2.7)
\]
By using the same method as in the Case 1 and \( 0 < \frac{k}{2-k} < 1 \), we obtain that \( \{x_n\} \) is a Cauchy sequence in \( W \). Therefore, \( \{x_n\} \) is a Cauchy sequence in \( W \). Since \( W \) is closed, there exists \( w \in W \) such that \( x_n \to w \). By the continuing of \( S \), we have \( Sx_n \to Sw \) as \( n \to \infty \). As the metric function is continuous, we obtain
\[
d(x_{n+1}, Sx_n) \to d(w, Sw) \text{ as } n \to \infty.
\]
Similarly, By (2.3) we can conclude that
\[
d(w, Sw) = d(W, V).
\]
This implies that \( w \in W \) is a best proximity point of \( S \).
Indeed, the sequence \( \{x_n\} \) defined by
\[
d(x_{n+1}, Sx_n) = d(W, V), \ n \in \mathbb{N},
\]
converges to an element \( w \). The proof is completed.

**Example 2.2.** Let \( X = \mathbb{R}^2 \) equipped with the metric \( d \) given by
\[
d((x, y), (u, v)) = \sqrt{(x-u)^2 + (y-v)^2}.
\]
Let $W = \{(x, 1) : 0 \leq x \leq 1\}$ and $V = \{(x, -1) : 0 \leq x \leq 1\} \cup \{(0, y) : -2 \leq y \leq -1\}$. It is easy to see that $d(W, V) = 2$, $W_0 = W$, $V_0 = \{(x, -1) : 0 \leq x \leq 1\}$, and $W, V$ are closed subsets of $X$. Define a directed graph $G = (V(G), E(G))$ by $V(G) = X$ and

$$E(G) = \{((x, y), (u, v)) \in \mathbb{R}^2 \times \mathbb{R}^2 : x \leq u \text{ and } |y - v| \leq \frac{1}{2}\}.$$

Let $S : W \to V$ be a mapping defined by

$$S(x, 1) = \left(\frac{x}{2}, -1\right), \text{ for all } (x, 1) \in W.$$

Then $S$ is continuous and $S(W_0) \subseteq V_0$.

We will show that $S$ is both proximally $G$-edge preserving and $G$-proximal generalized contraction. Let $(x, 1), (y, 1) \in W$ such that

$$((x, 1), (y, 1)) \in E(G), d((u, 1), S(x, 1)) = d(W, V) = d((v, 1), S(y, 1)),$$

where $(u, 1), (v, 1) \in W$. Then

$$x \leq y, d((u, 1), (\frac{x}{2}, -1)) = 2 = d((v, 1), (\frac{y}{2}, -1)).$$

This implies that $u = \frac{x}{2}$ and $v = \frac{y}{2}$. Since $x \leq y$, it follows that. Thus

$$(u, 1), (v, 1) \in W.$$
closed. Then there exists an element \( w \in W \) such that \( d(w, Sw) = d(W, V) \).

Further, the sequence \( \{x_n\} \), defined by

\[
d(x_{n+1}, Sx_n) = d(W, V), \quad \text{for all } n \in \mathbb{N},
\]

converges to the element \( w \).

**Proof.** Following the proof of Theorem 2.1, there exists a sequence \( \{x_n\} \) in \( W_0 \) satisfying

\[
d(x_{n+1}, Sx_n) = d(W, V) \quad \text{with } (x_{n-1}, x_n) \in E(G), \quad \text{for all } n \in \mathbb{N}, \tag{2.8}
\]

and \( x_n \to u \in W \). Since \( W_0 \) is closed, we get \( u \in W_0 \). Again, by using (i) of Theorem 2.1 we have \( S(W_0) \subseteq V_0 \), so \( Su \in V_0 \). Then there exists \( w \in W \) such that

\[
d(w, Su) = d(W, V). \tag{2.9}
\]

Since \( X \) has Property (A) and \( (x_{n-1}, x_n) \in E(G) \), and \( x_n \to u \) as \( n \to \infty \), there exists a subsequence \( \{x_{n_r}\} \) of \( \{x_n\} \) such that \( (x_{n_r}, u) \in E(G), \) for all \( r \in \mathbb{N} \).

Indeed, by using (2.8), (2.9), and \( S \) is a \( G \)-proximal generalized contraction, we get

\[
d(x_{n_r+1}, w) \leq kM(x_{n_r}, u), \tag{2.10}
\]

where

\[
M(x_{n_r}, u) = \max \left\{ d(x_{n_r}, u), d(x_{n_r}, x_{n_r+1}), d(u, w), \frac{d(x_{n_r+1}, w) + d(x_{n_r}, u)}{2} \right\}.
\]

By taking the limit in the above inequality, we get

\[
\lim_{r \to \infty} M(x_{n_r}, u) = d(u, w).
\]

Suppose that \( d(u, w) > 0 \). From (2.10), we have

\[
\lim_{r \to \infty} d(x_{n_r+1}, w) \leq kd(u, w).
\]

Since \( x_n \to u \) and \( k \in [0, 1) \), we get

\[
0 = \lim_{r \to \infty} [d(x_{n_r+1}, w) - d(u, w)] \leq (k - 1)d(u, w) < 0,
\]

which is a contradiction. Hence \( u = w \). Therefore there exists \( w \in W \) such that \( d(w, Sw) = d(W, V) \). The proof is completed. \( \square \)

The following corollaries are obtained directly from Theorems 2.1 and 2.3.

**Corollary 2.4.** Let \((X, d)\) be a complete metric space, \( G = (V(G), E(G)) \) a directed graph such that \( V(G) = X \). Let \( W \) and \( V \) be nonempty closed subsets of \( X \) with \( W_0 \) is nonempty and let \( S : W \to V \) be proximally \( G \)-edge-preserving and \( G \)-proximal contraction such that \( S(W_0) \subseteq V_0 \). Assume that there exist \( x_0, x_1 \in W_0 \) such that

\[
d(x_1, Sx_0) = d(W, V) \quad \text{and } (x_0, x_1) \in E(G).
\]

Suppose that either
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(i) $S$ is continuous or 

(ii) $X$ has the Property (A) and $W_0$ is closed.

Then there exists an element $w \in W$ such that $d(w, Sw) = d(W, V)$.

Further, the sequence $\{x_n\}$, defined by

$$d(x_n, Sx_{n-1}) = d(W, V), \text{ for all } n \in \mathbb{N},$$

converges to the element $w$.

**Corollary 2.5.** Let $(X, d)$ be a complete metric space, $G = (V(G), E(G))$ a directed graph such that $V(G) = X$. Let $W$ and $V$ be nonempty closed subsets of $X$ with $W_0$ nonempty and let $S : W \to V$ be proximally $G$-edge-preserving and a $G$-proximal Kannan mapping such that $S(W_0) \subseteq V_0$. Assume that there exist $x_0, x_1 \in W_0$ such that

$$d(x_1, Sx_0) = d(W, V) \text{ and } (x_0, x_1) \in E(G).$$

Suppose that either

(i) $S$ is continuous or

(ii) $X$ has the Property (A) and $W_0$ is closed.

Then there exists an element $w \in W$ such that $d(w, Sw) = d(W, V)$.

Further, the sequence $\{x_n\}$, defined by

$$d(x_n, Sx_{n-1}) = d(W, V), \text{ for all } n \in \mathbb{N},$$

converges to the element $w$.

**Corollary 2.6.** Let $(X, d)$ be a complete metric space, $G = (V(G), E(G))$ a directed graph such that $V(G) = X$. Let $W$ and $V$ be nonempty closed subsets of $X$ with $W_0$ nonempty and let $S : W \to V$ be proximally $G$-edge-preserving and a $G$-proximal C-contraction such that $S(W_0) \subseteq V_0$. Assume that there exist $x_0, x_1 \in W_0$ such that

$$d(x_1, Sx_0) = d(W, V) \text{ and } (x_0, x_1) \in E(G).$$

Suppose that either

(i) $S$ is continuous or

(ii) $X$ has the Property (A) and $W_0$ is closed.

Then there exists an element $w \in W$ such that $d(w, Sw) = d(W, V)$.

Further, the sequence $\{x_n\}$, defined by

$$d(x_n, Sx_{n-1}) = d(W, V), \text{ for all } n \in \mathbb{N},$$

converges to the element $w$. 
3 Applications to Coupled Best Proximity Point Theorems

In this section, we prove the existence of a coupled best proximity point by applications of Theorems 2.1 and 2.3 in Section 2. Now, we recall some definitions and notions regarding coupled best proximity points in a complete metric space endowed with a directed graph.

Let $W$ and $V$ be nonempty subsets of any set $X$, $F : W \times W \rightarrow V$ a non-self mapping. An element $(x, y) \in W \times W$ is called a coupled best proximity point of $F$ if $d(x, F(x, y)) = d(W, V)$ and $d(y, F(y, x)) = d(W, V)$. Recently, the coupled best proximity point theorems were investigated by many authors (see [25-27] and the references therein). We assume throughout this section that $W$ and $V$ are nonempty subsets of a metric space $(X, d)$. We define a new mapping $\eta : Y \times Y \rightarrow [0, \infty)$ by

$$
\eta((x, y), (u, v)) = d(x, u) + d(y, v), \text{ for all } (x, y), (u, v) \in Y,
$$

where $Y = X \times X$.

It is easy to show that $(X, d)$ is a metric space if and only if $(Y, \eta)$ is a metric space. Moreover, we can prove that $(X, d)$ is a complete metric space if and only if $(Y, \eta)$ is a complete metric space. We set $W^* = W \times W$, $V^* = V \times V$, $W_0^* = W_0 \times W_0$, $V_0^* = V_0 \times V_0$ and the following notions are used in this section:

$$
\eta(W^*, V^*) := \inf \{ \eta(x, y) : x = (x_1, y_1) \in W^* \text{ and } y = (x_2, y_2) \in V^* \},
$$

$$
W_0^* := \{ x = (x_1, y_1) \in W^* : \eta(x, y) = \eta(W^*, V^*) \text{ for some } y = (x_2, y_2) \in V^* \},
$$

$$
V_0^* := \{ y = (x_2, y_2) \in V^* : \eta(x, y) = \eta(W^*, V^*) \text{ for some } x = (x_1, y_1) \in W^* \}.
$$

Remark 3.1. We have the following facts:

1. $\eta(W^*, V^*) = 2d(W, V)$.

2. If $x = (x_1, y_1) \in W^*$ and $y = (x_2, y_2) \in V^*$ such that $\eta(x, y) = \eta(W^*, V^*)$, then $d(x_1, x_2) = d(y_1, y_2) = d(W, V)$.

For a non-self mapping $F : W \times W \rightarrow V$, we define the non-self mapping $S_F : W^* \rightarrow V^*$ by

$$
S_F(x, y) = (F(x, y), F(y, x)) \text{ for all } (x, y) \in Y.
$$

We note that an element $(x, y) \in W \times W$ is a coupled best proximity point of $F$ if and only if $(x, y)$ is a best proximity point of $S_F$.

Let $(X, d)$ be a metric space and $G = (V(G), E(G))$ a directed graph which has no parallel edges such that the set $V(G)$ of its vertices coincides with $X$ and the set $E(G)$ of its edges is a subset of $X \times X$. So, we define $G_Y = (V(G_Y), E(G_Y))$ such that $V(G_Y) = Y$ and $E(G_Y) = \{(x, y), (u, v)) \in Y \times Y :
(x, u) ∈ E(G) and (y, v) ∈ E(G^{-1})} where \( Y = X \times X \). Hence \( G_Y \) is also a directed graph which has no parallel edges.

In 2014, Chifu and Petrusel [28] presented the concept of edge preserving as the following.

**Definition 3.2.** [28] We say that \( F : X \times X \to X \) is edge preserving if \((x, u) \in E(G), (y, v) \in E(G^{-1}) \) implies \((F(x, y), F(u, v)) \in E(G) \) and \((F(y, x), F(v, u)) \in E(G^{-1}) \).

Now, we give definition of a proximal mixed \( G \)-edge preserving for non-self mapping from the product space \( W^* \) into \( V \) as follows:

**Definition 3.3.** We say that \( F : W^* \to V \) is a proximally mixed \( G \)-edge preserving if for each \( x, y, u, v \in W \),

\[
\begin{align*}
(x, u) \in E(G) \text{ and } (y, v) \in E(G^{-1}) & \implies (u_1, v_2) \in E(G), \\
\begin{align*}
& d(u_1, F(x, y)) = d(W, V) \\
& d(u_2, F(u, v)) = d(W, V)
\end{align*}
\end{align*}
\]

and
\[
\begin{align*}
(x, u) \in E(G) \text{ and } (y, v) \in E(G^{-1}) & \implies (v_1, v_2) \in E(G^{-1}). \\
& d(v_1, F(y, x)) = d(W, V) \\
& d(v_2, F(v, u)) = d(W, V)
\end{align*}
\]

Indeed, by taking \( A = B = X \) in the above definition, the proximal mixed \( G \)-edge preserving reduces to edge preserving of Definition 3.2.

**Theorem 3.4.** Let \( (X, d) \) be a complete metric space, \( G = (V(G), E(G)) \) a directed graph such that \( V(G) = X \), and let \( W \) and \( V \) be two nonempty closed subsets of \( X \) such that \( W_0 \) is a nonempty subset of \( W \). Let \( F : W^* \to V \) be a mapping satisfying the following properties:

(i) \( F \) is proximally mixed \( G \)-edge preserving, continuous and \( F(W_0^*) \subseteq V_0 \);

(ii) there exist \((x_0, y_0), (x_1, y_1) \in W_0^* \) such that \( d(x_1, F(x_0, y_0)) = d(W, V) \), \( d(y_1, F(y_0, x_0)) = d(W, V) \), and \((x_0, x_1) \in E(G), (y_0, y_1) \in E(G^{-1}) \);

(iii) there exists \( k \in [0, 1) \), for each \( x, y, u, v, w_1, w_2, z_1, z_2 \in W_0 \)

\[
\begin{align*}
(x, u) \in E(G) \text{ and } (y, v) \in E(G^{-1}) & \implies d(w_1, F(x, y)) + d(z_1, F(y, x)) = 2d(W, V) \\
& d(w_2, F(u, v)) + d(z_2, F(v, u)) = 2d(W, V)
\end{align*}
\]

implies
\[
\begin{align*}
& d(w_1, w_2) + d(z_1, z_2) \leq k \max \{d(x, u) + d(y, v), d(x, w_1) + d(y, z_1), \\
& d(u, w_2) + d(v, z_2), \frac{d(x, w_2) + d(y, z_2) + d(u, w_1) + d(v, z_1)}{2}\}.
\end{align*}
\]

Then \( F \) has a coupled best proximity point in \( W^* \), i.e., there exists an element \((x^*, y^*) \in W^* \) such that \( d(x^*, F(x^*, y^*)) = d(W, V) \) and \( d(y^*, F(y^*, x^*)) = d(W, V) \).
Proof. Let $Y = X \times X$. By using the mapping $\eta : Y \times Y \to [0, \infty)$ according the equation (3.1), we get $(Y, \eta)$ is a complete metric space. Set $G_Y = (V(G_Y), E(G_Y))$ such that $V(G_Y) = Y$ and 

$$E(G_Y) = \{((x, y), (u, v)) \in Y \times Y : (x, u) \in E(G) \text{ and } (y, v) \in E(G^{-1})\}.$$ 

Hence $G_Y$ is a directed graph which has no parallel edges. Let $S_F : W^* \to V^*$ be a non-self mapping defined by (3.2). Since $F(W_0^*) \subseteq V_0$, we get $S_F(W_0^*) \subseteq V_0^*$. Now, we will show that $S_F$ satisfies all conditions of Theorem 2.1. We start by the proving that $S_F$ is a proximally $G$-edge preserving as follows: Let $(x, y), (u, v) \in Y$ such that 

$$(x, y), (u, v) \in E(G_Y),$$

$$\eta((u_1, v_1), S_F(x, y)) = \eta(W^*, V^*),$$

$$\eta((u_2, v_2), S_F(u, v)) = \eta(W^*, V^*).$$

By using the definition of $S_F$ and $E(G_Y)$, we have 

$$(x, u) \in E(G) \text{ and } (y, v) \in E(G^{-1}),$$

$$d(u_1, F(x, y)) + d(v_1, F(y, x)) = 2d(W, V),$$

$$d(u_2, F(u, v)) + d(v_2, F(v, u)) = 2d(W, V).$$

Using the Remark 3.1(2), we obtain 

$$(x, u) \in E(G) \text{ and } (y, v) \in E(G^{-1}),$$

$$d(u_1, F(x, y)) = 2d(W, V),$$

$$d(u_2, F(u, v)) = 2d(W, V),$$

and 

$$(x, u) \in E(G) \text{ and } (y, v) \in E(G^{-1}),$$

$$d(v_1, F(y, x)) = 2d(W, V),$$

$$d(v_2, F(v, u)) = 2d(W, V).$$

Since $F$ is a proximally mixed $G$-edge preserving, we have $(u_1, u_2) \in E(G)$ and $(v_1, v_2) \in E(G^{-1})$. Again, by the definition of $E(G_Y)$ it follows that $((u_1, v_1), (u_2, v_2)) \in E(G_Y)$. Hence $S_F$ is a proximally $G$-edge preserving. From the continuity of $F$, it is easy to show that $S_F$ is also continuous. Next, from (iii) there exist $(x_0, y_0), (x_1, y_1) \in W_0^*$ such that $d(x_1, F(x_0, y_0)) = d(y_1, F(y_0, x_0)) = d(W, V)$ and $(x_0, x_1) \in E(G), (y_0, y_1) \in E(G^{-1})$. It means that $((x_0, y_0), (x_1, y_1)) \in E(G_Y)$ and 

$$d(x_1, F(x_0, y_0)) + d(y_1, F(y_0, x_0)) = 2d(W, V).$$

It implies that 

$$\eta((x_1, y_1), (F(x_0, y_0), F(y_0, x_0))) = \eta(W, V),$$

$$E(G_Y) = \{((x, y), (u, v)) \in Y \times Y : (x, u) \in E(G) \text{ and } (y, v) \in E(G^{-1})\}.$$
that is,
\[
\eta((x_1, y_1), S_F(x_0, y_0)) = \eta(W, V) \text{ and } ((x_0, y_0), (x_1, y_1)) \in E(G_Y).
\]

Thus \(S_F\) satisfies the condition \((ii)\) of Theorem 2.1 Finally, we will show that \(S_F\) is a G-proximal generalized contraction: Let \((x_0, y_0), (x_1, y_1) \in W^*_0\) such that
\[
\eta((x_1, y_1), S_F(x_0, y_0)) = \eta(W^*, V^*),
\]
where \((u_1, v_1), (u_2, v_2) \in W^*_0\). Hence
\[
(x_0, x_1) \in E(G) \text{ and } (y_0, y_1) \in E(G^{-1}),
\]
\[
d(u_1, F(x_0, y_0)) + d(v_1, F(y_0, x_0)) = 2d(W, V),
\]
\[
d(u_2, F(x_1, y_1)) + d(v_2, F(y_1, x_1)) = 2d(W, V).
\]

By \((iii)\), we have
\[
d(u_1, u_2) + d(v_1, v_2) \leq k \max\{d(x_0, x_1) + d(y_0, y_1), d(x_0, u_1) + d(y_0, v_1),
\]
\[
d(x_1, u_2) + d(y_1, v_2), \frac{d(x_0, u_2) + d(y_0, v_2) + d(x_1, u_1) + d(y_1, v_1)}{2}\}.
\]

It means that
\[
\eta((u_1, v_1), (u_2, v_2)) \leq k \max\{\eta((x_0, y_0), (x_1, y_1)), \eta((x_0, y_0), (u_1, v_1)),
\]
\[
\eta((x_1, y_1), (u_2, v_2)), \frac{\eta((x_0, y_0), (u_2, v_2)) + \eta((x_1, y_1), (u_1, v_1))}{2}\}.
\]

Therefore all conditions of Theorem 3.4 are satisfied. Hence \(S_F\) has a best proximity point in \(W^*\), that is, there exists \(w^* = (x^*, y^*) \in W^*\) such that \(\eta(w^*, S_F(w^*)) = \eta(W^*, V^*)\). Obviously, \(w^* = (x^*, y^*)\) is a coupled best proximity point of \(F\). The proof is completed.

Next, we will use the following property instead of continuity of \(F\) in Theorem 3.4 for proving the existence of a coupled best proximity point.

**Property (B).** Let \(\{x_n\}\) and \(\{y_n\}\) be sequences in \(X\) such that \(\{x_n\}\) and \(\{y_n\}\) have the following properties:

- if \(x_n \to x\), for some \(x \in X\) and \((x_n, x_{n+1}) \in E(G)\), for all \(n \in \mathbb{N}\), then \((x_n, x) \in E(G)\) for all \(n \in \mathbb{N}\) and
- if \(y_n \to y\), for some \(y \in X\) and \((y_n, y_{n+1}) \in E(G^{-1})\), for all \(n \in \mathbb{N}\), then \((y_n, y) \in E(G^{-1})\) for all \(n \in \mathbb{N}\).

**Theorem 3.5.** Suppose that all assumptions of Theorem 3.4 hold, except the continuity of \(F\). In addition, suppose that \(X\) has the Property (B) and \(W_0\) is closed. Then \(F\) has a coupled best proximity point in \(W^*\), i.e., there exists an element \((x^*, y^*) \in W^*\) such that \(d(x^*, F(x^*)) = d(W, V)\) and \(d(y^*, F(y^*, x^*)) = d(W, V)\).
Proof. Let $Y$, $G_Y$, and both mappings $\eta$ and $S_F$ be as in the proof of Theorem 3.4. Then it is sufficient to prove that $Y$ has the Property (A). Let $(x_n, y_n)$ be a sequence in $Y$ with $(x_n, y_n) \to (x, y)$, for some $(x, y) \in Y$ as $n \to \infty$ and $((x_n, y_n), (x_{n+1}, y_{n+1})) \in E(G_Y)$, for all $n \in \mathbb{N}$. Then we get $x_n \to x$, $y_n \to y$ as $n \to \infty$, and $(x_n, x_{n+1}) \in E(G)$, $(y_n, y_{n+1}) \in E(G^{-1})$ for all $n \in \mathbb{N}$. Since $X$ has the Property (B), we have $(x_n, x) \in E(G)$ and $(y_n, y) \in E(G^{-1})$. It implies that $((x_n, y_n), (x, y)) \in E(G_Y)$, for all $n \in \mathbb{N}$. Therefore $Y$ has Property (A). By using Theorem 2.3 we obtain that $S_F$ has a best proximity point in $W^*$, that is, there exists $w^* = (x^*, y^*) \in W^*$ such that $\eta(w^*, S_F(w^*)) = \eta(W^*, X^*)$ or $w^* = (x^*, y^*)$ is coupled best proximity point of $F$. \hfill \Box

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