A New Type of Difference Sequence Spaces

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Abstract: In this paper, we introduce a new type of difference operator \( \triangle_m^n \) for fixed \( m, n \in \mathbb{N} \) and define the sequence spaces

\[
E(\triangle_m^n) = \{ x = (x_k) : (\triangle_m^n x_k) = (\triangle^n x_k - \triangle^n x_{k+m}) \in E, E \in \{ l_\infty, c, c_0 \} \}
\]

and study some topological properties of these spaces. We also obtain some inclusion relations involving these sequence spaces. With different choices of \( m \) and \( n \) it is observed that these spaces include many known spaces as special cases.

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1 Introduction

Throughout the paper, \( \omega, l_\infty, c \) and \( c_0 \) denote the space of all, bounded, convergent and null sequences \( x = (x_k) \) with complex terms respectively, normed by

\[
\|x\| = \sup_{k \geq 1} |x_k|.
\]

The zero sequence is denoted by \( \theta = (0, 0, ...) \).

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Kizmaz [1] defined the difference sequence spaces $Z(\Delta)$ as follows

$$Z(\Delta) = \{x = (x_k) \in \omega : (\Delta x_k) \in Z\}$$

where $Z \in \{l_\infty, c, c_0\}$ and $\Delta x_k = x_k - x_{k+1}$. The above sequence spaces are Banach spaces normed by

$$\|x\|_\Delta = |x_1| + \sup_{k \geq 1} |x_k|.$$

The idea of Kizmaz [1] was applied to introduce the different type of sequence spaces by several authors (see [2–7]) who studied their different properties.

Serigol [8] defined the sequence spaces

$$X(\Delta_q) = \{x = (x_k) : \Delta_q x = k^q(x_k - x_{k+1}) \in X, q < 1\},$$

where $X \in \{l_\infty, c, c_0\}$. Serigol proved that the above spaces are Banach spaces with respect to the norm

$$\|x\|_{\Delta_q} = |x_1| + \sup_{k \geq 1} |k^q(x_k - x_{k+1})|$$

and studied some properties.

Et and Colak [9, 10] defined the sequence spaces

$$X(\Delta_m) = \{x = (x_k) : (\Delta_m x_k) \in X\},$$

where $m \in \mathbb{N}, \Delta_m x_k = \Delta^{m-1} x_k - \Delta^{m-1} x_{k+1}$ and $X \in \{l_\infty, c, c_0\}$ so that

$$\Delta_m x_k = \sum_{\nu=0}^{m} (-1)^\nu \binom{m}{\nu} x_{k+\nu}.$$

They showed that the spaces $l_\infty(\Delta_m^n), c(\Delta_m^n)$ and $c_0(\Delta_m^n)$ are Banach spaces with respect to the norm

$$\|x\|_{\Delta_m} = \sum_{i=1}^{m} |x_i| + \sup_{k \geq 1} |\Delta_m x_k|.$$

Bektas and Colak [3] defined and studied the sequence spaces

$$X(\Delta_m^r) = \{x = (x_k) : (k^r \Delta_m x_k) \in X\},$$

where $m \in \mathbb{N}, r \in \mathbb{R}$ and $X \in \{l_\infty, c, c_0\}$. They showed that the spaces are Banach spaces with respect to the norm

$$\|x\|_{\Delta_m^r} = \sum_{i=1}^{m} |x_i| + \sup_{k} k^r |\Delta_m x_k|.$$
Esi et al. [11] introduced the difference operator $\Delta^q_p$ for fixed $p, q \in \mathbb{N}$ and defined the sequence spaces

$$X(\Delta^q_p) = \{x = (x_k) : (\Delta^q_p x_k) \in X\},$$

where $\Delta^q_p x_k = \Delta^q_p -1 x_k - \Delta^q_p -1 x_{k+p}$ and $X \in \{l_\infty, c, c_0\}$ and proved that the spaces are Banach spaces with respect to the norm

$$\|x\|_{\Delta^q_p} = \sum_{i=1}^{pq} |x_i| + \sup_{k \geq 1} |\Delta^q_p x_k|.$$

### 2 Definitions and Preliminaries

A sequence $X$ is said to be **solid (normal)** if $(x_k) \in X$ implies $(\alpha_k x_k) \in X$ for all sequences of the scalars $(\alpha_k)$ with $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$. A sequence $X$ is said to be **monotonic** if it contains the canonical preimage of all its step spaces. A sequence $X$ is said to be **convergence free** if $(y_k) \in X$ whenever $(x_k) \in X$ and $y_k = 0$ whenever $x_k = 0$. A sequence $X$ is said to be **symmetric**, if $(x_{\pi(k)}) \in X$ whenever $(x_k) \in X$ where $\pi(k)$ is permutation of $\mathbb{N}$, the set of natural numbers.

Let $m, n \geq 1$ be fixed positive integers, then we introduce a new type of difference operators $\Delta^m_n$ where $\Delta^m_n x_k = \Delta^n x_k - \Delta^n x_{k+m}$ and define the sequence spaces $Z(\Delta^m_n)$ as

$$Z(\Delta^m_n) = \{x = (x_k) : (\Delta^m_n x_k) = (\Delta^n x_k - \Delta^n x_{k+m}) \in Z\}$$

where $Z \in \{l_\infty, c, c_0\}$. So that

$$\Delta^m_n x_k = \Delta^n x_k - \Delta^n x_{k+m}$$

$$= \sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} (x_{k+\nu} - x_{k+m+\nu}).$$

### 3 Main Results

**Proposition 3.1.** The spaces $l_\infty(\Delta^m_n), c(\Delta^m_n)$ and $c_0(\Delta^m_n)$ are normed linear spaces normed by

$$\|x\|_{\Delta^m_n} = \sum_{r=1}^{m+n} |x_r| + \sup_{k \geq 1} |\Delta^m_n x_k|.$$  \hspace{1cm} (1.1)

**Proof.** Let $\alpha, \beta$ be scalars and $x, y \in l_\infty(\Delta^m_n)$. Then $\sup_{k \geq 1} |\Delta^m_n x_k| < \infty$ and $\sup_{k \geq 1} |\Delta^m_n y_k| < \infty$. This gives

$$\sup_{k \geq 1} |\Delta^m_n (\alpha x_k + \beta y_k)| \leq |\alpha| \sup_{k \geq 1} |\Delta^m_n x_k| + |\beta| \sup_{k \geq 1} |\Delta^m_n y_k| < \infty.$$
Hence \( l_\infty(\triangle_m^n) \) is a linear space. Similarly, it can be shown that \( c(\triangle_m^n) \) and \( c_0(\triangle_m^n) \) are linear spaces. To show that \( l_\infty(\triangle_m^n) \) is a normed linear space. It is clear that if \( x = \theta \). Then

\[
\|x\|_{\triangle_m^n} = \|\theta\|_{\triangle_m^n} = 0.
\]

Conversely, suppose that \( \|x\|_{\triangle_m^n} = 0 \). This gives

\[
\sum_{r=1}^{m+n} |x_r| + \sup_{k \geq 1} |\triangle_m^n x_k| = 0,
\]

which implies \( x_r = 0 \ \forall r = 1, 2, \ldots, m+n \) and \( \sup_{k \geq 1} |\triangle_m^n x_k| = 0 \), \( \forall k \in \mathbb{N} \), which further implies

\[
\sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} (x_{k+\nu} - x_{k+m+\nu}) = 0.
\]

This gives

\[
\left| \binom{n}{0} (x_k - x_{k+m}) - \binom{n}{1} (x_{k+1} - x_{k+m+1}) + \cdots \right. \\
\left. + (-1)^{n-1} \binom{n}{n-1} (x_{k+n-1} - x_{k+m+n-1}) + (-1)^n \binom{n}{n} (x_{k+n} - x_{k+m+n}) \right| = 0.
\]

Put \( k = 1 \), we get

\[
\left| \binom{n}{0} (x_1 - x_{m+1}) - \binom{n}{1} (x_2 - x_{m+2}) + \cdots \right. \\
\left. + (-1)^{n-1} \binom{n}{n-1} (x_n - x_{m+n}) + (-1)^n \binom{n}{n} (x_{n+1} - x_{m+n+1}) \right| = 0,
\]

which implies

\[
\left| (-1)^n \binom{n}{n} x_{m+n+1} \right| = 0.
\]

This gives \( x_{(m+n)+1} = 0 \). Proceeding in this way, we have \( x_k = 0 \), \( \forall k \in \mathbb{N} \). Thus, \( \|x\|_{\triangle_m^n} = 0 \iff x = \theta \). Further

\[
\|x\|_{\triangle_m^n} = \sum_{r=1}^{m+n} |x_r + y_r| + \sup_{k \geq 1} |\triangle_m^n (x_k + y_k)| \\
\leq \|x\|_{\triangle_m^n} + \|y\|_{\triangle_m^n}.
\]

Finally, we have

\[
\|\lambda x\|_{\triangle_m^n} = \sum_{r=1}^{m+n} |\lambda x_r| + \sup_{k \geq 1} |\triangle_m^n (\lambda x_k)| = |\lambda| \|x\|_{\triangle_m^n}.
\]
Hence $l_\infty(\Delta_m^n)$ is a normed linear space. Similarly, it can be shown that $c(\Delta_m^n)$ and $c_0(\Delta_m^n)$ are normed linear spaces.

The following proposition is easily obtained.

**Proposition 3.2.**

1. $c_0(\Delta_m^n) \subset c(\Delta_m^n) \subset l_\infty(\Delta_m^n)$ and the inclusions are proper.
2. $Z(\Delta_m^n) \subset Z(\Delta_m^n)$ for $Z \in \{l_\infty, c, c_0\}$, $1 \leq i < n$ and the inclusions are strict.

**Theorem 3.3.** The spaces $l_\infty(\Delta_m^n), c(\Delta_m^n)$ and $c_0(\Delta_m^n)$ are Banach spaces under the norm defined in (1.1).

**Proof.** Let $(x^i)$ be a Cauchy sequence in $l_\infty(\Delta_m^n)$ where $x^i = (x^i_k) = (x^i_1, x^i_2, \ldots)$. Then for given $\epsilon > 0$, we can find a positive integer $n_0$ such that

$$\|x^i - x^j\| < \epsilon, \quad \forall i, j \geq n_0.$$

This gives

$$\sum_{r=1}^{m+n} |x^i_r - x^j_r| < \epsilon \quad \text{and} \quad \sup_{k \geq 1} |\Delta_m^n(x^i_k - x^j_k)| < \epsilon, \quad \forall i, j \geq n_0,$$

which gives

$$|x^i_r - y^i_r| < \epsilon, \quad \forall i, j \geq n_0 \text{ and } r = 1, 2, \ldots, m+n.$$

This shows that $(x^i_k)$ is a Cauchy sequence for $1 \leq k \leq m+n$. Let $\lim_{i \to \infty} x^i_k = x_k$ for $1 \leq k \leq m+n$. Also, since $\sup_{k \geq 1} |\Delta_m^n(x^i_k - x^j_k)| < \epsilon, \quad \forall i, j \geq n_0$, and $k \in N$. This shows that $(\Delta_m^n x^i_k)$ is also a Cauchy sequence $\forall k \in N$. Let $\lim_{i \to \infty} \Delta_m^n x^i_k = y_k, \quad \forall k \in N$. This gives

$$\lim_{i \to \infty} \left[ \sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} (x^i_{k+\nu} - x^i_{k+m+\nu}) \right] = y_k.$$

Put $k = 1$, we get

$$\lim_{i \to \infty} \left[ \sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} (x^i_{1+\nu} - x^i_{1+m+\nu}) \right] = y_1.$$

This gives

$$\lim_{i \to \infty} \left[ \binom{n}{0} (x^i_1 - x^i_{m+1}) - \binom{n}{1} (x^i_2 - x^i_{m+2}) + \cdots \right].$$
which implies by using \( \lim_{i \to \infty} x^i_k = x_k \) for \( 1 \leq k \leq m + n \) that
\[
\left[ \binom{n}{0} (x_1 - x_{m+1}) - \binom{n}{1} (x_2 - x_{m+2}) + \cdots \right.
\left. + (-1)^n \binom{n}{n} (x_{1+n} - \lim_{i \to \infty} x^i_{m+n+1}) \right] = y_1.
\]
This gives
\[
\lim_{i \to \infty} x^i_{(m+n)+1} = x_{(m+n)+1},
\]
where
\[
x_{(m+n)+1} = \frac{1}{y_1 - \left\{ \binom{n}{0} (x_1 - x_{m+1}) - \binom{n}{1} (x_2 - x_{m+2}) + \cdots \right.}
\left. + (-1)^n \binom{n}{n} (x_{1+n}) \right\}].
\]
Proceeding similarly, we get
\[
\lim_{i \to \infty} x^i_k = x_k, \quad \forall k \geq 1.
\]
Now \( \sum_{r=1}^{m+n} |x^i_k - x^j_k| < \epsilon, \quad \forall i, j \geq n_0. \) This gives
\[
\lim_{j \to \infty} \sum_{r=1}^{m+n} |x^i_r - x^j_r| < \epsilon, \quad \forall i \geq n_0,
\]
which implies
\[
\sum_{r=1}^{m+n} |x^i_r - x^j_r| < \epsilon, \quad \forall i \geq n_0.
\]
Also, we have
\[
|\Delta^m_{n}x^i_k - \Delta^m_{n}x^j_k| < \epsilon, \quad \forall i, j \geq n_0 \text{ and } k \geq 1.
\]
This gives
\[
\lim_{j \to \infty} |\Delta^m_{n}x^i_k - \Delta^m_{n}x^j_k| < \epsilon, \quad \forall i \geq n_0 \text{ and } k \geq 1,
\]
which gives
\[
|\Delta^m_{n}x^i_k - \lim_{j \to \infty} \sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} (x^j_{k+\nu} - x^j_{k+m+\nu})| < \epsilon \quad \forall i \geq n_0 \text{ and } k \geq 1,
\]
which further gives

\[ |\triangle_m^n x_k - \triangle_m^n x_k| < \epsilon, \forall i \geq n_0 \text{ and } k \geq 1. \]

This gives

\[ |\triangle_m^i x_k - \triangle_m^i x_k| < \epsilon, \forall i \geq n_0 \text{ and } k \geq 1. \]

Hence

\[ \sum_{r=1}^{m+n} |x_i - x_r| + \sup_{k \geq 1} |\triangle_m^n (x_k) - x_k| < 2\epsilon, \forall i \geq n_0. \]

This shows that \( x^i \rightarrow x \) as \( i \rightarrow \infty \). Also since

\[
|\triangle_m^n x_k| = \sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} (x_{k+\nu} - x_{k+m+\nu}) \\
= \sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} \left( x_{k+m+\nu} - x_{k+m} - \left( x_{k+\nu} - x_{k+m+\nu} \right) + \left( x_{k+\nu} - x_{k+m+\nu} \right) \right) \\
\leq \sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} \left( x_{k+m} - x_{k+m+\nu} - x_{k+m+\nu} + x_{k+m+\nu} \right) \\
+ \sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} \left( x_{k+m+\nu} - x_{k+m+\nu} \right) \\
\leq \|x^n - x\|_{\triangle_m^n} + ||\triangle_m^n x^n|| = O(1). 
\]

Hence \( x \in l_\infty(\triangle_m^n) \). This shows that \( l_\infty(\triangle_m^n) \) is a Banach space. Similarly, it can be shown that \( c(\triangle_m^n) \) and \( c_0(\triangle_m^n) \) are Banach spaces.

**Corollary 3.4.** The spaces \( c(\triangle_m^n) \) and \( c_0(\triangle_m^n) \) are nowhere dense subsets of \( l_\infty(\triangle_m^n) \).

**Proof.** From Proposition 3.1, the inclusion \( c(\triangle_m^n) \subset l_\infty(\triangle_m^n) \) and \( c_0(\triangle_m^n) \subset l_\infty(\triangle_m^n) \) are strict. Further from Theorem 3.3, it follows that the spaces \( c(\triangle_m^n) \) and \( c_0(\triangle_m^n) \) are closed. Hence the spaces \( c(\triangle_m^n) \) and \( c_0(\triangle_m^n) \) are nowhere dense subsets of \( l_\infty(\triangle_m^n) \).

**Theorem 3.5.** The spaces \( l_\infty(\triangle_m^n) \), \( c(\triangle_m^n) \) and \( c_0(\triangle_m^n) \) are not solid in general.

**Proof.** To show that the above spaces are not solid in general. Let \( m = n = 2 \) and consider the sequence \((x_k)\) defined as

\[ x_1 = 1 \text{ and } x_{k+1} = x_k + k + 2, \forall k \in N. \]

Then \((x_k) \in c_0(\Delta_2^2) \subset c(\Delta_2^2) \subset l_\infty(\Delta_2^2)\). Now consider the sequence of scalars \((\alpha_k)\) defined by

\[
\alpha_k = \begin{cases} 
1, & \text{if } k = 3i, i \in N, \\
0, & \text{otherwise.}
\end{cases}
\]
Then \((\alpha_k x_k) \notin l_\infty(\Delta^2_2)\). Hence, the space \(l_\infty(\Delta^m_n)\) are not solid in general. Similarly, we can show that \(c(\Delta^m_n)\) and \(c_0(\Delta^m_n)\) are not solid in general.

**Theorem 3.6.** The spaces \(l_\infty(\Delta^m_n), c(\Delta^m_n)\) and \(c_0(\Delta^m_n)\) are not symmetric in general.

**Proof.** To show that the above spaces are not symmetric in general let \(m = n = 2\) and consider the sequence \((x_k)\) defined in Theorem 3.5. Then \((x_k) \in c_0(\Delta^2_2) \subset c(\Delta^2_2) \subset l_\infty(\Delta^2_2)\). Now consider the rearrangement \((y_k)\) of \((x_k)\) as

\[
y_k = \begin{cases} 
1, & \text{if } k = 3n - 2, n \in \mathbb{N}, \\
x_{k+1}, & \text{if } k \text{ is even}, k \neq 3n - 2, n \in \mathbb{N}, \\
x_{k-1}, & \text{if } k \text{ is odd}, k \neq 3n - 2, n \in \mathbb{N}.
\end{cases}
\]

Then \((y_k) \notin l_\infty(\Delta^2_2)\). Hence, the space \(l_\infty(\Delta^2_2)\) is not symmetric in general. Similarly, we can show that \(c(\Delta^m_n)\) and \(c_0(\Delta^m_n)\) are not symmetric in general.

**Theorem 3.7.** The spaces \(l_\infty(\Delta^m_n), c(\Delta^m_n)\) and \(c_0(\Delta^m_n)\) are not convergence free in general.

**Proof.** To show that the above spaces are not convergence free in general let \(m = n = 2\) and \(n = 1\) and consider the sequence \((x_k)\) defined by \(x_k = 1, \forall k \in \mathbb{N}\). Then \((x_k) \in c_0(\Delta^1_1)\). Now consider the sequence \((y_k)\) as \(y_k = k^2, \forall k \in \mathbb{N}\). Then \((y_k) \notin c_0(\Delta^1_1)\). Hence, \(c_0(\Delta^m_n)\) is not convergence free in general. Similarly we can show that \(l_\infty(\Delta^m_n)\) and \(c(\Delta^m_n)\) are not convergence free in general.

**Theorem 3.8. Theorem 3.8.** The spaces \(l_\infty(\Delta^m_n), c(\Delta^m_n)\) and \(c_0(\Delta^m_n)\) are not monotonic in general.

**Proof.** Let \(m = 3\) and \(n = 2\) and consider the sequence \((x_k)\) defined as

\[
x_1 = 1, \text{ and } x_{k+1} = x_k + k + 1, \forall k \in \mathbb{N}.
\]

Then \(x_k \in c_0(\Delta^2_3)\). Now consider the sequence \((y_k)\) in its preimage as

\[
y_k = \begin{cases} 
1, & \text{if } k \text{ odd}, \\
0, & \text{if } k \text{ even}.
\end{cases}
\]

Then \((y_k)\) neither belongs to \(c_0(\Delta^m_n)\) nor \(c(\Delta^2_3)\). Hence \(c(\Delta^2_3)\) and \(c_0(\Delta^m_n)\) are not monotonic in general. Similarly, we can show that \(l_\infty(\Delta^m_n)\) is not monotonic in general.

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