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A New Type of Difference Sequence Spaces

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Abstract : In this paper, we introduce a new type of difference operator $\triangle_m^n$ for fixed $m, n \in N$ and define the sequence spaces

$$E(\triangle_m^n) = \{x = (x_k) : (\triangle_m^n x_k) = (\triangle^n x_k - \triangle^n x_{k+m}) \in E,\ E \in \{l_\infty, c, c_0\}\}$$

and study some topological properties of these spaces. We also obtain some inclusion relations involving these sequence spaces. With different choices of m and n it is observed that these spaces include many known spaces as special cases.

Keywords : Difference sequence Space; Banach space; Solid space; Symmetric space; Completeness.
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1 Introduction

Throughout the paper, ω, l_∞, c and c_0 denote the space of all, bounded, convergent and null sequences $x = (x_k)$ with complex terms respectively, normed by

$$\|x\| = \sup_{k \geq 1} |\ x_k\ | \ .$$

The zero sequence is denoted by $\theta = (0, 0, ...)$.

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Kizmaz [1] defined the difference sequence spaces $Z(\triangle)$ as follows

$$Z(\triangle) = \{x = (x_k) \in \omega : (\triangle x_k) \in Z\}$$

where $Z \in \{l_\infty, c, c_0\}$ and $\triangle x_k = x_k - x_{k+1}$. The above sequence spaces are Banach spaces normed by

$$\|x\|_\triangle =| x_1 | + \sup_{k\geq 1} | x_k | .$$

The idea of Kizmaz [1] was applied to introduce the different type of sequence spaces by several authors (see [2–7]) who studied their different properties.

Serigol [8] defined the sequence spaces

$$X(\triangle_q) = \{x = (x_k) : \triangle_q x = k^q(x_k - x_{k+1}) \in X, q < 1\},$$

where $X \in \{l_\infty, c, c_0\}$. Serigol proved that the above spaces are Banach spaces with respect to the norm

$$\|x\|_{\triangle_q} =| x_1 | + \sup_{k\geq 1} | k^q(x_k - x_{k+1}) |$$

and studied some properties.

Et and Colak [9, 10] defined the sequence spaces

$$X(\triangle^m) = \{x = (x_k) : (\triangle^m x_k) \in X\},$$

where $m \in N, \triangle^m x_k = \triangle^{m-1}x_k - \triangle^{m-1}x_{k+1}$ and $X \in \{l_\infty, c, c_0\}$ so that

$$\triangle^m x_k = \sum_{\nu=0}^{m} (-1)^\nu \binom{m}{\nu} x_{k+\nu}.$$

They showed that the spaces $l_\infty(\triangle^n_m), c(\triangle^n_m)$ and $c_0(\triangle^n_m)$ are Banach spaces with respect to the norm

$$\|x\|_{\triangle^m} = \sum_{i=1}^{m} | x_i | + \sup_{k\geq 1} | \triangle^m x_k | .$$

Bektas and Colak [3] defined and studied the sequence spaces

$$X(\triangle^m_r) = \{x = (x_k) : (k^r \triangle^m x_k) \in X\},$$

where $m \in N, r \in R$ and $X \in \{l_\infty, c, c_0\}$. They showed that the spaces are Banach spaces with respect to the norm

$$\|x\|_{\triangle^m_r} = \sum_{i=1}^{m} | x_i | + \sup_{k} k^r | \triangle^m x_k | .$$

Esi et al. [11] introduced the difference operator $\triangle_p^q$ for fixed $p, q \in N$ and defined the sequence spaces

$$X(\triangle_p^q) = \{x = (x_k) : (\triangle_p^q x_k) \in X\},$$

where $\triangle_p^q x_k = \triangle_p^{q-1} x_k - \triangle_p^{q-1} x_{k+p}$ and $X \in \{l_\infty, c, c_0\}$ and proved that the spaces are Banach spaces with respect to the norm

$$\|x\|_{\triangle_p^q} = \sum_{i=1}^{pq} \mid x_i \mid + \sup_{k \geq 1} \mid \triangle_p^q x_k \mid .$$

2 Definitions and Preliminaries

A sequence X is said to be *solid (normal)* if $(x_k) \in X$ implies $(\alpha_k x_k) \in X$ for all sequences of the scalars (α_k) with $|\alpha_k| \leq 1$ for all $k \in N$. A sequence X is said to be *monotonic* if it contains the canonical preimage of all its step spaces. A sequence X is said to be *convergence free* if $(y_k) \in X$ whenever $(x_k) \in X$ and $y_k = 0$ whenever $x_k = 0$. A sequence X is said to be *symmetric*, if $(x_{\pi(k)}) \in X$ whenever $(x_k) \in X$ where $\pi(k)$ is permutation of N, the set of natural numbers.

Let $m, n \geq 1$ be fixed positive integers, then we introduce a new type of difference operators $\triangle_m^n$ where $\triangle_m^n x_k = \triangle^n x_k - \triangle^n x_{k+m}$ and define the sequence spaces $Z(\triangle_m^n)$ as

$$Z(\triangle_m^n) = \{x = (x_k) : (\triangle_m^n x_k) = (\triangle^n x_k - \triangle^n x_{k+m}) \in Z\}$$

where $Z \in \{l_\infty, c, c_0\}$. So that

$$\begin{aligned} \triangle_m^n x_k &= \triangle^n x_k - \triangle^n x_{k+m} \\ &= \sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} (x_{k+\nu} - x_{k+m+\nu}). \end{aligned}$$

3 Main Results

Proposition 3.1. *The spaces $l_\infty(\triangle_m^n), c(\triangle_m^n)$ and $c_0(\triangle_m^n)$ are normed linear spaces normed by*

$$\|x\|_{\triangle_m^n} = \sum_{r=1}^{m+n} \mid x_r \mid + \sup_{k \geq 1} \mid \triangle_m^n x_k \mid . \tag{1.1}$$

Proof. Let α, β be scalars and $x, y \in l_\infty(\triangle_m^n)$. Then $\sup_{k \geq 1} \mid \triangle_m^n x_k \mid < \infty$ and $\sup_{k \geq 1} \mid \triangle_m^n y_k \mid < \infty$. This gives

$$\sup_{k \geq 1} \mid \triangle_m^n (\alpha x_k + \beta y_k) \mid \leq \mid \alpha \mid \sup_{k \geq 1} \mid \triangle_m^n x_k \mid + \mid \beta \mid \sup_{k \geq 1} \mid \triangle_m^n y_k \mid < \infty.$$

Hence $l_\infty(\triangle_m^n)$ is a linear space. Similarly, it can be shown that $c(\triangle_m^n)$ and $c_0(\triangle_m^n)$ are linear spaces. To show that $l_\infty(\triangle_m^n)$ is a normed linear space. It is clear that if $x=\theta$. Then

$$\|x\|_{\triangle_m^n} = \|\theta\|_{\triangle_m^n} = 0.$$

Conversely, suppose that$\|x\|_{\triangle_m^n} = 0$. This gives

$$\sum_{r=1}^{m+n} \mid x_r \mid + \sup_{k\geq 1} \mid \triangle_m^n x_k \mid = 0,$$

which implies $x_r = 0 \ \forall r = 1,2,...,m+n$ and $\sup_{k\geq 1} \mid \triangle_m^n x_k \mid = 0, \ \forall k \in N$, which further implies

$$\sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} (x_{k+\nu} - x_{k+m+\nu}) = 0.$$

This gives

$$\left| \binom{n}{0}(x_k - x_{k+m}) - \binom{n}{1}(x_{k+1} - x_{k+m+1}) + \cdots \right.$$
$$\left. +(-1)^{n-1}\binom{n}{n-1}(x_{k+n-1} - x_{k+m+n-1}) + (-1)^n \binom{n}{n}(x_{k+n} - x_{k+m+n}) \right| = 0.$$

Put $k=1$, we get

$$\left| \binom{n}{0}(x_1 - x_{m+1}) - \binom{n}{1}(x_2 - x_{m+2}) + \cdots \right.$$
$$\left. +(-1)^{n-1}\binom{n}{n-1}(x_n - x_{m+n}) + (-1)^n \binom{n}{n}(x_{n+1} - x_{m+n+1}) \right| = 0,$$

which implies

$$\left| (-1)^n \binom{n}{n} x_{m+n+1} \right| = 0.$$

This gives $x_{(m+n)+1} = 0$. Proceeding in this way, we have $x_k = 0, \ \forall k \in N$. Thus, $\|x\|_{\triangle_m^n} = 0 \iff x = \theta$. Further

$$\begin{aligned} \|x\|_{\triangle_m^n} &= \sum_{r=1}^{m+n} \mid x_r + y_r \mid + \sup_{k\geq 1} \mid \triangle_m^n (x_k + y_k) \mid \\ &\leq \|x\|_{\triangle_m^n} + \|y\|_{\triangle_m^n}. \end{aligned}$$

Finally, we have

$$\|\lambda x\|_{\triangle_m^n} = \sum_{r=1}^{m+n} \mid \lambda x_r \mid + \sup_{k\geq 1} \mid \triangle_m^n (\lambda x_k) \mid = \mid \lambda \mid \|x\|_{\triangle_m^n}.$$

Hence $l_\infty(\triangle_m^n)$ is a normed linear space. Similarly, it can be shown that $c(\triangle_m^n)$ and $c_0(\triangle_m^n)$ are normed linear spaces. □

The following proposition is easily obtained.

Proposition 3.2.

(1) $c_0(\triangle_m^n) \subset c(\triangle_m^n) \subset l_\infty(\triangle_m^n)$ *and the inclusions are proper.*

(2) $Z(\triangle_m^i) \subset Z(\triangle_m^n)$ *for* $Z \in \{l_\infty, c, c_0\},\ 1 \le i < n$ *and the inclusions are strict.*

Theorem 3.3. *The spaces* $l_\infty(\triangle_m^n), c(\triangle_m^n)$ *and* $c_0(\triangle_m^n)$ *are Banach spaces under the norm defined in (1.1).*

Proof. Let (x^i) be a Cauchy sequence in $l_\infty(\triangle_m^n)$ where $x^i = (x_k^i) = (x_1^i, x_2^i, ...)$. Then for given $\epsilon > 0$, we can find a positive integer n_0 such that

$$\|x^i - x^j\| < \epsilon, \quad \forall i, j \ge n_0.$$

This gives

$$\sum_{r=1}^{m+n} \mid x_r^i - x_r^j \mid < \epsilon \text{ and } \sup_{k\ge 1} \mid \triangle_m^n(x_k^i - x_k^j) \mid < \epsilon,\ \forall i, j \ge n_0,$$

which gives

$$\mid x_r^i - y_r^j \mid < \epsilon,\ \forall i, j \ge n_0 \text{ and } r = 1, 2, ..., m+n.$$

This shows that (x_k^i) is a Cauchy sequence for $1 \le k \le m+n$. Let $\lim_{i\to\infty} x_k^i = x_k$ for $1 \le k \le m+n$. Also, since $\sup_{k\ge 1} \mid \triangle_m^n(x_k^i - x_k^j) \mid < \epsilon,\ \forall i, j \ge n_0,$ and $k \in N$. This shows that $(\triangle_m^n x_k^i)$ is also a Cauchy sequence $\forall k \in N$. Let $\lim_{i\to\infty} \triangle_m^n x_k^i = y_k,\ \forall k \in N$. This gives

$$\lim_{i\to\infty} \left[\sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} (x_{k+\nu}^i - x_{k+m+\nu}^i) \right] = y_k.$$

Put $k = 1$, we get

$$\lim_{i\to\infty} \left[\sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} (x_{1+\nu}^i - x_{1+m+\nu}^i) \right] = y_1.$$

This gives

$$\lim_{i\to\infty} \left[\binom{n}{0} (x_1^i - x_{m+1}^i) - \binom{n}{1} (x_2^i - x_{m+2}^i) + \cdots \right.$$

$$+(-1)^n \binom{n}{n} (x^i_{1+n} - x^i_{m+n+1})\Bigg] = y_1,$$

which implies by using $\lim_{i\to\infty} x^i_k = x_k$ for $1 \le k \le m+n$ that

$$\Bigg[\binom{n}{0}(x_1 - x_{m+1}) - \binom{n}{1}(x_2 - x_{m+2}) + \cdots$$
$$+(-1)^n \binom{n}{n} \left(x_{1+n} - \lim_{i\to\infty} x^i_{m+n+1}\right)\Bigg] = y_1.$$

This gives

$$\lim_{i\to\infty} x^i_{(m+n)+1} = x_{(m+n)+1},$$

where

$$x_{(m+n)+1} = \pm \Bigg[y_1 - \Bigg\{\binom{n}{0}(x_1 - x_{m+1}) - \binom{n}{1}(x_2 - x_{m+2}) + \cdots$$
$$+(-1)^n \binom{n}{n}(x_{1+n})\Bigg\}\Bigg].$$

Proceeding similarly, we get

$$\lim_{i\to\infty} x^i_k = x_k,\ \forall k \ge 1.$$

Now $\sum_{r=1}^{m+n} \mid x^i_k - x^j_r \mid < \epsilon,\ \forall i,j \ge n_0$. This gives

$$\lim_{j\to\infty} \sum_{r=1}^{m+n} \mid x^i_r - x^j_r \mid < \epsilon,\ \forall i \ge n_0,$$

which implies

$$\sum_{r=1}^{m+n} \mid x^i_r - x_r \mid < \epsilon,\ \forall i \ge n_0.$$

Also, we have

$$\mid \triangle^n_m x^i_k - \triangle^n_m x^j_k \mid < \epsilon,\ \forall i,j \ge n_0 \text{ and } k \ge 1.$$

This gives

$$\lim_{j\to\infty} \mid \triangle^n_m x^i_k - \triangle^n_m x^j_k \mid < \epsilon,\ \forall i \ge n_0 \text{ and } k \ge 1,$$

which gives

$$\left| \triangle^n_m x^i_k - \lim_{j\to\infty} \sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} (x^j_{k+\nu} - x^j_{k+m+\nu}) \right| < \epsilon\ \forall i \ge n_0 \text{ and } k \ge 1,$$

which further gives

$$\left|\triangle_m^n x_k^i - \sum_{\nu=0}^{n}(-1)^\nu \binom{n}{\nu}(x_{k+\nu} - x_{k+m+\nu})\right| < \epsilon \ \forall i \geq n_0 \text{ and } k \geq 1.$$

This gives

$$| \triangle_m^n x_k^i - \triangle_m^n x_k |< \epsilon, \ \forall i \geq n_0 \text{ and } k \geq 1.$$

Hence

$$\sum_{r=1}^{m+n} | x_r^i - x_r | + \sup_{k\geq 1} | \triangle_m^n (x_k^i - x_k) |< 2\epsilon, \ \forall i \geq n_0.$$

This shows that $x^i \to x$ as $i \to \infty$. Also since

$$\begin{aligned}|\triangle_m^n x_k| &= \left|\sum_{\nu=0}^{n}(-1)^\nu \binom{n}{\nu}(x_{k+\nu} - x_{k+m+\nu})\right| \\ &= \left|\sum_{\nu=0}^{n}(-1)^\nu \binom{n}{\nu}\left[x_{k+\nu} - x_{k+m+\nu} - (x_{k+\nu}^{n_0} - x_{k+m+\nu}^{n_0}) + (x_{k+\nu}^{n_0} - x_{k+m+\nu}^{n_0})\right]\right| \\ &\leq \left|\sum_{\nu=0}^{n}(-1)^\nu \binom{n}{\nu}\left[(x_{k+\nu}^{n_0} - x_{k+m+\nu}^{n_0}) - (x_{k+\nu} - x_{k+m+\nu})\right]\right| \\ &+ \left|\sum_{\nu=0}^{n}(-1)^\nu \binom{n}{\nu}(x_{k+\nu}^{n_0} - x_{k+m+\nu}^{n_0})\right| \\ &\leq \|x^{n_0} - x\|_{\triangle_m^n} + \|\triangle_m^n x^{n_0}\| = O(1).\end{aligned}$$

Hence $x \in l_\infty(\triangle_m^n)$. This shows that $l_\infty(\triangle_m^n)$ is a Banach space. Similarly, it can be shown that $c(\triangle_m^n)$ and $c_0(\triangle_m^n)$ are Banach spaces. □

Corollary 3.4. *The spaces $c(\triangle_m^n)$ and $c_0(\triangle_m^n)$ are nowhere dense subsets of $l_\infty(\triangle_m^n)$.*

Proof. From Proposition 3.1, the inclusion $c(\triangle_m^n) \subset l_\infty(\triangle_m^n)$ and $c_0(\triangle_m^n) \subset l_\infty(\triangle_m^n)$ are strict. Further from Theorem 3.3, it follows that the spaces $c(\triangle_m^n)$ and $c_0(\triangle_m^n)$ are closed. Hence the spaces $c(\triangle_m^n)$ and $c_0(\triangle_m^n)$ are nowhere dense subsets of $l_\infty(\triangle_m^n)$. □

Theorem 3.5. *The spaces $l_\infty(\triangle_m^n), c(\triangle_m^n)$ and $c_0(\triangle_m^n)$ are not solid in general.*

Proof. To show that the above spaces are not solid in general. Let $m = n = 2$ and consider the sequence (x_k) defined as

$$x_1 = 1 \text{ and } x_{k+1} = x_k + k + 2, \ \forall k \in N.$$

Then $(x_k) \in c_0(\triangle_2^2) \subset c(\triangle_2^2) \subset l_\infty(\triangle_2^2)$. Now consider the sequence of scalars (α_k) defined by

$$\alpha_k = \begin{cases} 1, \text{ if } k = 3i, i \in N, \\ \\ 0, \text{ otherwise.} \end{cases}$$

Then $(\alpha_k x_k) \notin l_\infty(\triangle_2^2)$. Hence, the space $l_\infty(\triangle_m^n)$ are not solid in general. Similarly, we can show that $c(\triangle_m^n)$ and $c_0(\triangle_m^n)$ are not solid in general. □

Theorem 3.6. *The spaces $l_\infty(\triangle_m^n), c(\triangle_m^n)$ and $c_0(\triangle_m^n)$ are not symmetric in general.*

Proof. To show that the above spaces are not symmetric in general let $m = n = 2$ and consider the sequence (x_k) defined in Theorem 3.5. Then $(x_k) \in c_0(\triangle_2^2) \subset c(\triangle_2^2) \subset l_\infty(\triangle_2^2)$. Now consider the rearrangement (y_k) of (x_k) as

$$y_k = \begin{cases} 1, & \text{if } k = 3n-2, n \in N, \\ x_{k+1}, & \text{if } k \text{ is even, } k \neq 3n-2,\ n \in N, \\ x_{k-1}, & \text{if } k \text{ is odd, } k \neq 3n-2,\ n \in N. \end{cases}$$

Then $(y_k) \notin l_\infty(\triangle_2^2)$. Hence, the space $l_\infty(\triangle_2^2)$ is not symmetric in general. Similarly, we can show that $c(\triangle_m^n)$ and $c_0(\triangle_m^n)$ are not symmetric in general. □

Theorem 3.7. *The spaces $l_\infty(\triangle_m^n), c(\triangle_m^n)$ and $c_0(\triangle_m^n)$ are not convergence free in general.*

Proof. To show that the above spaces are not convergence free in general let $m = 2$ and $n = 1$ and consider the sequence (x_k) defined by $x_k = 1,\ \forall k \in N$. Then $(x_k) \in c_0(\triangle_2^1)$. Now consider the sequence (y_k) as $y_k = k^2,\ \forall k \in N$. Then $y_k \notin c_0(\triangle_2^1)$. Hence, $c_0(\triangle_m^n)$ is not convergence free in general. Similarly we can show that $l_\infty(\triangle_m^n)$ and $c(\triangle_m^n)$ are not convergence free in general. □

Theorem 3.8. Theorem 3.8. *The spaces $l_\infty(\triangle_m^n), c(\triangle_m^n)$ and $c_0(\triangle_m^n)$ are not monotonic in general.*

Proof. Let $m = 3$ and $n = 2$ and consider the sequence (x_k) defined as

$$x_1 = 1, \text{ and } x_{k+1} = x_k + k + 1,\ \forall k \in N.$$

Then $x_k \in c_0(\triangle_3^2)$. Now consider the sequence (y_k) in its preimage as

$$y_k = \begin{cases} 1, \text{ if } k \text{ odd}, \\ 0, \text{ if } k \text{ even}. \end{cases}$$

Then (y_k) neither belongs to $c_0(\triangle_m^n)$ nor $c(\triangle_3^2)$. Hence $c(\triangle_3^2)$ and $c_0(\triangle_m^n)$ are not monotonic in general. Similarly, we can show that $l_\infty(\triangle_m^n)$ is not monotonic in general. □

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