Tripartite Ramsey Number $r_t(K_{2,4}, K_{2,4})$

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Abstract: A graph $G$ is $n$-partite, $n \geq 1$, if it is possible to partition the set of points $V(G)$ into $n$ subsets $V_1, V_2, \ldots, V_n$ (called partite sets) such that every element of the set of lines $E(G)$ joins a point of $V_i$ to a point of $V_j, i \neq j$. For $n = 2$, and $n = 3$ such graphs are called bipartite graph, and tripartite graph respectively. A complete $n$-partite graph $G$ is an $n$-partite graph with the added property that if $u \in V_i$ and $v \in V_j, i \neq j$, then the line $uv \in E(G)$. If $|V_i| = p_i$, then this graph is denoted by $K_{p_1,p_2,\ldots,p_n}$.

For the complete tripartite graph $K_{s,s,s}$ with the number of points $p = 3s$, let each line of the graph has either red or blue colour. The smallest number $s$ such that $K_{s,s,s}$ always contains $K_{m,n}$ with all lines of $K_{m,n}$ have one colour (red or blue) is called tripartite Ramsey number and denoted by $r_t(K_{m,n}, K_{m,n})$. In this paper, we show that

$$r_t(K_{2,4}, K_{2,4}) = 7.$$ 

Keywords: Tripartite Ramsey numbers; Bipartite Ramsey numbers; Ramsey numbers; Tripartite graphs; Bipartite graphs.

2010 Mathematics Subject Classification: 05C55; 05D10.

1 Introduction

A graph $G$ is $n$-partite, $n \geq 1$, if it is possible to partition the set of points $V(G)$ into $n$ subsets $V_1, V_2, \ldots, V_n$ (called partite sets) such that every element of the set of lines $E(G)$ joins a point of $V_i$ to a point of $V_j, i \neq j$, see [1].
2-coloured

The Value of $r_t(K_{2,4}, K_{2,4})$

2-coloured.

Consider a subgraph $K_{m,n}$ of 2-coloured $K_{s,s}$. If all lines of $K_{m,n}$ have red (blue) colour, we shall say that the $K_{s,s}$ contains a red (blue) $K_{m,n}$. The smallest number $s$ of points such that $K_{s,s}$ always contains red $K_{m,n}$ or blue $K_{m,n}$ is called bipartite Ramsey number and denoted by $r_b(K_{m,n}, K_{m,n})$.

According to the definition of bipartite Ramsey number in this paper, Longani [4], has found that $r_b(K_{1,n}, K_{1,n}) = 2n - 1(n = 1, 2, 3, \ldots)$, $r_b(K_{2,2}, K_{2,2}) = 5$, and $r_b(K_{2,3}, K_{2,3}) = 9$.

Beineke and Schwenk [2] have also found that $r_b(K_{2,2}, K_{2,2}) = 5$ and $r_b(K_{3,3}, K_{3,3}) = 17$.

In this paper, instead of considering a complete bipartite graph $K_{s,s}$, we shall consider a complete tripartite graph $K_{s,s,s}$ of order $p = 3s$. Let each line of $K_{s,s,s}$ be coloured by using either red or blue colour. The smallest number $s$ of points such that $K_{s,s,s}$ always contains red $K_{m,n}$ or blue $K_{m,n}$ is called tripartite Ramsey number and denoted by $r_t(K_{m,n}, K_{m,n})$.

In [3], Leamyoo have found that $r_t(K_{2,2}, K_{2,2}) = 4$.

2 The Value of $r_t(K_{2,4}, K_{2,4})$

We find the value of $r_t(K_{2,4}, K_{2,4})$ by considering a particular 2-coloured $K_{6,6,6}$ and 2-coloured $K_{7,7,7}$. For a $K_{7,7,7}$, consider all ninety eight lines that are adjacent to all points of a $V_i$. We call such lines as the lines of the $V_i$.

Lemma 2.1. Let $K_{7,7,7}$ be a 2-coloured complete tripartite graph with $p = 21$ and each $V_1$, $V_2$, and $V_3$ be the set of seven non-adjacent points of the $K_{7,7,7}$. There exists at least one $V_i$ of which the number of red lines and blue lines of the $V_i$ are not equal.

Proof. Consider the three $V_i$'s. Suppose there are forty nine red lines and forty nine blue lines of each $V_i$.

Since there are totally forty nine red lines of $V_1$, consider when there are $n (n \geq 0)$ red lines which join points of $V_1$ and $V_2$, and so there are $49 - n$ red lines which join points of $V_1$ and $V_3$. Since for $V_2$ there are also exactly forty nine red lines of $V_2$, therefore there are $49 - n$ red lines which join points of $V_2$ and $V_3$.

Now we can see that there are $(49 - n) + (49 - n)$ red lines of $V_3$. Since there
are exactly forty nine red lines of $V_3$, therefore

\[(49 - n) + (49 - n) = 49 \]

\[n = 24.5.\]

This is not possible. Therefore, there exist some $V_i$'s of which the number of red lines and blue lines of the $V_i$'s are not equal. \qed

In order to prove Theorem 2.2 we need to represent 2-colored $K_{m,n}$ with with an $m \times n$ matrix as follows:

Given a 2-colored $K_{m,n}$ with $V_1$ and $V_2$ as its partite sets size $m$ and $n$, respectively. Put $V_1 = \{r_1, r_2, \ldots, r_m\}$ and $V_2 = \{c_1, c_2, \ldots, c_n\}$. Let $B = [b_{ij}]$ be an $m \times n$ matrix where $b_{ij} = 1$ if the line $r_i c_j$ is red, otherwise $b_{ij} = 0$. The following example is to illustrate 2-colored $K_{5,4}$. As in Figure 2.1 (a), we use the dark lines to indicate red lines while dash lines for blue lines.

\[
\begin{bmatrix}
1 & 0 & 1 & 0 & 1 \\
0 & 1 & 1 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 & 0 \\
\end{bmatrix}
\]

Figure 2.1.

**Theorem 2.2.** $r_t(K_{2,4}, K_{2,4}) = 7$.

**Proof.** Consider the 2-coloured $K_{6,6,6}$ graph illustrated in Figure 2.2.
It can be verified that the $K_{6,6,6}$ contains neither red $K_{2,4}$ nor blue $K_{2,4}$. Therefore $r_t(K_{2,4}, K_{2,4}) > 6$. That is
\[ r_t(K_{2,4}, K_{2,4}) \geq 7. \] (2.1)

Let $K_{7,7,7}$ be a 2-coloured complete tripartite graph. Consider the set $V_1$, $V_2$ and $V_3$ of seven non-adjacent points of the $K_{7,7,7}$:

\[
\begin{align*}
V_1 &= \{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}, \\
V_2 &= \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}, \\
V_3 &= \{w_1, w_2, w_3, w_4, w_5, w_6, w_7\}.
\end{align*}
\]

From Lemma 2.1, we can assume that from $V_1$, the number of red lines are greater than the number of blue lines, that is the number of red lines are equal to fifty or greater. We only need to consider the case when the number of red lines from $V_1$ is fifty and show that in such case the $K_{7,7,7}$ always contains red $K_{2,4}$. For the cases when the number of red lines is greater than fifty, the results follow immediately.

Let $V(G_1) = V_1$ and $V(G_2) = V_2 \cup V_3$. For $V(G_1)$, let $u_1, u_2, u_3, u_4, u_5, u_6, u_7$ be respectively replaced by $r_1, r_2, r_3, r_4, r_5, r_6, r_7$. Also for $V(G_2)$, let $v_1, v_2, v_3, v_4, v_5, v_6, v_7, w_1, w_2, w_3, w_4, w_5, w_6, w_7$ be respectively replaced by $c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9, c_{10}, c_{11}, c_{12}, c_{13}, c_{14}$. That is,

\[
\begin{align*}
V(G_1) &= \{r_1, r_2, r_3, r_4, r_5, r_6, r_7\}, \\
V(G_2) &= \{c_1, c_2, \ldots, c_{14}\}.
\end{align*}
\]
By ignoring the lines between \( V_2 \) and \( V_3 \) and consider the defined \( V(G_1) \) and \( V(G_2) \) the \( K_{7,7,7} \) is now reduced to 2-coloured \( K_{7,14} \). In order to prove the theorem we only need to show that this \( K_{7,14} \) always contains red \( K_{2,4} \).

We find the value of \( r_t(K_{2,4}, K_{2,4}) \) by considering the 2-coloured \( K_{7,14} \).

If there are \( m, n, s, t, u, v \) (\( 1 \leq m, n, s, t, u, v, v \leq 14 \)) such that some submatrices

\[
\begin{bmatrix}
  b_{ms} & b_{mt} & b_{mu} & b_{mv} \\
  b_{ns} & b_{nt} & b_{nu} & b_{nv}
\end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\
  1 & 1 & 1 & 1
\end{bmatrix}
\]

(2.2)

then the \( K_{7,14} \) contains red \( K_{2,4} \).

Let \( d_1, d_2, d_3, d_4, d_5, d_6, d_7 \) be degrees of red lines of \( r_1, r_2, r_3, r_4, r_5, r_6, r_7 \) respectively. We can choose \( r_i \)’s such that \( d_1 \geq d_2 \geq d_3 \geq d_4 \geq d_5 \geq d_6 \geq d_7 \).

Here we have the conditions that

\[ d_1 + d_2 + d_3 + d_4 + d_5 + d_6 + d_7 = 50 \]

and \( 0 \leq d_i \leq 14, \forall i = 1, 2, 3, 4, 5, 6, 7 \).

Next, we consider two main cases.

**Case 1.** \( d_1 + d_2 \geq 18 \).

Here, the possible \( d_1 \geq d_2 \) are \( 9 \geq 9, 10 \geq 8, 10 \geq 9, 10 \geq 10, 11 \geq 7, 11 \geq 8, 11 \geq 9, 11 \geq 10, 11 \geq 11, 12 \geq 7, 12 \geq 8, 12 \geq 9, 12 \geq 10, 12 \geq 11, 12 \geq 12, 13 \geq 7, 13 \geq 8, 13 \geq 9, 13 \geq 10, 13 \geq 11, 13 \geq 12, 13 \geq 13, 14 \geq 7, 14 \geq 8, 14 \geq 9, 14 \geq 10, 14 \geq 11, 14 \geq 12, 14 \geq 13, 14 \geq 14. \)

It is easy to show that for all of these cases the \( K_{7,14} \) always contains red \( K_{2,4} \).

For example, consider cases \( d_1 = 9 \), and \( d_2 = 9 \). For a case in Table 2.1, parts of the matrix involving \( r_1 \) and \( r_2 \) would be

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Table 2.1.

from which submatrix of the form (2.2) appears, that is the \( K_{7,7,7} \) contains red \( K_{2,4} \).

**Case 2.** \( d_1 + d_2 < 18 \).

With the conditions for \( d_i \), there are five subcases to consider.

**Subcase 2.1.** \( d_1 = 10, d_2 = 7, d_3 = 7, d_4 = 7, d_5 = 7, d_6 = 6, d_7 = 6. \)

In this case we consider three points \( r_1, r_2, r_3 \) of \( V(G_1) \). When there are four or more points \( c_i \)’s each of which is joined to both of \( r_1 \) and \( r_2 \) by red lines, then we can see that the \( K_{7,14} \) contains red \( K_{2,4} \).

For other cases, suppose that \( r_1 \) is joined by ten red lines to \( c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9, c_{10} \) and \( r_2 \) is joined by seven red lines to \( c_8, c_9, c_{10}, c_{11}, c_{12}, c_{13}, c_{14} \), we consider seven red lines joining to \( r_3 \). Either at least four of seven red lines are joined from \( r_3 \) to some points among \( c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8 \) or at least four of these seven red lines are joined to some points among \( c_9, c_{10}, c_{11}, c_{12}, c_{13}, c_{14} \). In
either case, we can see that red \( K_{2,4} \) is contained in the \( K_{7,14} \), see Table 2.2 for example.

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Table 2.2.

**Subcase 2.2.** \( d_1 = 9, d_2 = 8, d_3 = 7, d_4 = 7, d_5 = 7, d_6 = 6, d_7 = 6 \).

In this case, as in subcase 2.1, we also need to consider three points \( r_1, r_2, r_3 \) of \( V(G_1) \). The method in showing that the \( K_{7,14} \) always contain red \( K_{2,4} \) is almost exactly the same as in the subcase 2.1 above.

**Subcase 2.3.** \( d_1 = 9, d_2 = 7, d_3 = 7, d_4 = 7, d_5 = 7, d_6 = 7, d_7 = 6 \).

In this case we consider five points \( r_1, r_2, r_3, r_4, r_5 \) of \( V(G_1) \). When there are four or more points \( c_i \)'s each of which is joined to both of \( r_1 \) and \( r_2 \) by red lines, then we can see that the \( K_{7,14} \) contains red \( K_{2,4} \).

**2.3.1.** Two \( c_i \)'s are joined by red lines to both of \( r_1 \) and \( r_2 \).

Suppose that \( r_1 \) is joined by nine red lines to \( c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9 \) and \( r_2 \) is joined by seven red lines to \( c_8, c_9, c_{10}, c_{11}, c_{12}, c_{13}, c_{14} \).

Consider seven red lines joining to \( r_3 \). Either at least four of seven red lines are joined from \( r_3 \) to some points among \( c_1, c_2, c_3, c_4, c_5, c_6, c_7 \) or at least four of these seven red lines are joined to some points among \( c_8, c_9, c_{10}, c_{11}, c_{12}, c_{13}, c_{14} \). In either case, we see that red \( K_{2,4} \) is contained in the \( K_{7,14} \), see Table 2.3 for example.

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Table 2.3.

**2.3.2.** Three \( c_i \)'s are joined by red lines to both of \( r_1 \) and \( r_2 \).

Suppose that \( r_1 \) is joined by nine red lines to \( c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9 \) and \( r_2 \) is joined by seven red lines to \( c_7, c_8, c_9, c_{10}, c_{11}, c_{12}, c_{13} \).

Consider when \( c_{14} \) is not joined to some \( r_i \)'s (\( i = 3, 4, 5, 6, 7 \)) by red lines.

Suppose \( c_{14} \) is not joined to \( r_3 \) for example, then either at least four red lines from \( r_3 \) are joined to points among \( c_1, c_2, c_3, c_4, c_5, c_6, c_7 \) or at least four red lines from \( r_3 \) are joined to points among \( c_8, c_9, c_{10}, c_{11}, c_{12}, c_{13} \). In either case, we see
that red $K_{2,4}$ is contained in $K_{7,14}$, see Table 2.4 for example.

\[
\begin{array}{cccccccccccc}
 & c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 & c_8 & c_9 & c_{10} & c_{11} & c_{12} & c_{13} & c_{14} \\
r_1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
r_2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
r_3 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

Table 2.4.

Consider when the point $c_{14}$ is joined to all $r_i$'s ($i = 3, 4, 5, 6, 7$) by red lines. Here we consider two subcases.

(1) Some $r_i, i = 3, 4, 5, 6, 7$ are joined by red lines to some of $c_7, c_8, c_9$.

Suppose $r_3$ is joined to $c_7$ by red line, for example, then there are five other red lines joining $r_3$. Either at least three of the red lines are joined to points among $c_1, c_2, c_3, c_4, c_5, c_6$ or at least three of red lines are joined to points among $c_8, c_9, c_{10}, c_{11}, c_{12}, c_{13}$. In either case, red $K_{2,4}$ is formed. For example, see Table 2.5.

Similarly, if $r_3$ is joined to $c_8$ or $c_9$ by red line, we can show that red $K_{2,4}$ is also formed.

\[
\begin{array}{cccccccccccc}
 & c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 & c_8 & c_9 & c_{10} & c_{11} & c_{12} & c_{13} & c_{14} \\
r_1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
r_2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
r_3 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

Table 2.5

(2) None of $r_i, i = 3, 4, 5, 6, 7$ are joined by red lines to $c_7, c_8$ and $c_9$.

Since each of $r_i, i = 3, 4, 5, 6, 7$ is joined by red lines to $c_{14}$, there are six other red lines joining $r_i$. The six red lines from each $r_3$ and $r_4$ will join to points among $c_1, c_2, c_3, c_4, c_5, c_6, c_{10}, c_{11}, c_{12}, c_{13}$. Therefore, there are at least two points among $c_1, c_2, c_3, c_4, c_5, c_6, c_{10}, c_{11}, c_{12}, c_{13}$ which join $r_3$ and $r_4$. First, consider the case when there are at least three points among $c_1, c_2, c_3, c_4, c_5, c_6, c_{10}, c_{11}, c_{12}, c_{13}$ which join $r_3$ and $r_4$. Also, since $c_{14}$ is joined by red lines to $r_3, r_4$, we can see that red $K_{2,4}$ is contained in the $K_{7,14}$, see Table 2.6 for example.

\[
\begin{array}{cccccccccccc}
 & c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 & c_8 & c_9 & c_{10} & c_{11} & c_{12} & c_{13} & c_{14} \\
r_1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
r_2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
r_3 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
r_4 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

Table 2.6.

Next, we consider the case when there are two points among $c_1, c_2, c_3, c_4, c_5, c_6, c_{10}, c_{11}, c_{12}, c_{13}$ which join $r_3$ and $r_4$. Suppose that these two points are $c_{12}$ and $c_{13}$. When we consider only $r_1, r_2, r_3, r_4$ there are cases when the $K_{7,14}$ does
not contain red $K_{2,4}$, see Table 2.7 for example. For such cases, we shall consider the seven red lines joining $r_5$.

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Table 2.7.

From Table 2.7, $r_3$ is joined to $c_1, c_2, c_3, c_{11}, c_{12}, c_{13}, c_{14}$ and $r_4$ is joined to $c_4, c_5, c_6, c_{10}, c_{12}, c_{13}, c_{14}$ and since $r_5$ is joined by one of the red lines to $c_{14}$, there are six other red lines joining $r_5$. Either at least three red lines are joined to points among $c_1, c_2, c_3, c_{11}$ or at least three red lines are joined to points among $c_4, c_5, c_6, c_{10}, c_{12}, c_{13}$. Assume that there are three red lines which are joined to points among $c_1, c_2, c_3, c_{11}$, say $c_2, c_3, c_{11}$ or there are three red lines which are joined to points among $c_4, c_5, c_6, c_{10}, c_{12}, c_{13}$, say $c_4, c_{10}, c_{12}$, since all $r_3, r_4$, and $r_5$ are joined to $c_{14}$. We can verify that in either case the $K_{7,14}$ contains red $K_{2,4}$, see Table 2.8 for example.

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Table 2.8.

Subcase 2.4. $d_1 = 8, d_2 = 8, d_3 = 7, d_4 = 7, d_5 = 7, d_6 = 7, d_7 = 6$.

In this case, as in Subcase 2.3, we also need to consider five points $r_1, r_2, r_3, r_4, r_5$ of $V(G_1)$. The method in showing that the $K_{7,14}$ always contain red $K_{2,4}$ is almost exactly the same as in the Subcase 2.3 above.

Subcase 2.5. $d_1 = 8, d_2 = 7, d_3 = 7, d_4 = 7, d_5 = 7, d_6 = 7, d_7 = 7$.

When there are four or more $c_i$’s each of which is joined by red lines to both of $r_1$ and $r_2$, then we see that the $K_{7,14}$ contains red $K_{2,4}$, see Table 2.9 for example.

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Table 2.9.

We consider three more cases.

2.5.1. One of $c_i$’s is joined by red lines to both of $r_1$ and $r_2$.

Suppose that $r_1$ is joined by eight red lines to $c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8$ and $r_2$ is joined by seven red lines to $c_8, c_9, c_{10}, c_{11}, c_{12}, c_{13}, c_{14}$. 

Consider the seven red lines joining \( r_3 \). Either at least four of the seven red lines are joined, from \( r_3 \), to points among \( c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8 \) or at least four red lines are joined to points among \( c_9, c_{10}, c_{11}, c_{12}, c_{13}, c_{14} \). In either case, we see that \( K_{2,4} \) is contained in the \( K_{7,14} \), see Table 2.10 for example.

|   | \( c_1 \) | \( c_2 \) | \( c_3 \) | \( c_4 \) | \( c_5 \) | \( c_6 \) | \( c_7 \) | \( c_8 \) | \( c_9 \) | \( c_{10} \) | \( c_{11} \) | \( c_{12} \) | \( c_{13} \) | \( c_{14} \) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| \( r_1 \) | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| \( r_2 \) | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| \( r_3 \) | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

Table 2.10.

**2.5.2.** Two of \( c_i \)'s are joined by red lines to both of \( r_1 \) and \( r_2 \).

Suppose that \( r_1 \) is joined by eight red lines to \( c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8 \) and \( r_2 \) is joined by seven red lines to \( c_7, c_8, c_9, c_{10}, c_{11}, c_{12}, c_{13} \). Consider two subcases (1) and (2).

(1) \( c_{14} \) is not joined to some \( r_i \)'s (\( i = 3, 4, 5, 6, 7 \)) by red lines.

If \( c_{14} \) is not joined to \( r_3 \) for example, then either at least four red lines from \( r_3 \) are joined to points among \( c_1, c_2, c_3, c_4, c_5, c_6, c_7 \) or at least four red lines from \( r_3 \) are joined to points among \( c_8, c_9, c_{10}, c_{11}, c_{12}, c_{13} \). In either case, we see that red \( K_{2,4} \) is contained in \( K_{7,14} \), see Table 2.11 for example.

|   | \( c_1 \) | \( c_2 \) | \( c_3 \) | \( c_4 \) | \( c_5 \) | \( c_6 \) | \( c_7 \) | \( c_8 \) | \( c_9 \) | \( c_{10} \) | \( c_{11} \) | \( c_{12} \) | \( c_{13} \) | \( c_{14} \) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| \( r_1 \) | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| \( r_2 \) | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| \( r_3 \) | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

Table 2.11.

(2) \( c_{14} \) is joined to all \( r_i \)'s (\( i = 3, 4, 5, 6, 7 \)) by red lines.

Here we consider two possibilities (a) and (b).

(a) Some \( r_i \)'s (\( i = 3, 4, 5, 6, 7 \)) are joined to \( c_7 \) or \( c_8 \) or both by red lines.

For example, suppose \( r_3 \) is joined to \( c_7 \) by red line, then there are five other red lines joining \( r_3 \). From these five red lines, either at least three of the red lines are joined to points among \( c_1, c_2, c_3, c_4, c_5, c_6 \) or at least three red lines are joined to points among \( c_8, c_9, c_{10}, c_{11}, c_{12}, c_{13} \). In either case, red \( K_{2,4} \) is formed. For example, see Table 2.12.

Similarly, if \( r_3 \) is joined to \( c_8 \) by red line, we can show that red \( K_{2,4} \) is also formed, and if \( r_3 \) is joined to \( c_7 \) and \( c_8 \) by red lines, then there are four other red lines joining \( r_3 \). For these four red lines, either at least two of the red lines are joined to points among \( c_1, c_2, c_3, c_4, c_5, c_6 \) or at least two red lines are joined to points among \( c_9, c_{10}, c_{11}, c_{12}, c_{13} \). In either case, we can see that red \( K_{2,4} \) is
contained in $K_{7,14}$.

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Table 2.12.

(b) None of $r_i$'s ($i = 3, 4, 5, 6, 7$) are joined to $c_7$ and $c_8$ by red lines.

We consider $r_4$. Since each of $r_i(i = 3, 4, 5, 6, 7)$ is joined by red lines to $c_{14}$, there are six other red lines joining each $r_i$. The six red lines joining each $r_3$ and $r_4$ will join to points among $c_1, c_2, c_3, c_4, c_5, c_6, c_9, c_{10}, c_{11}, c_{12}, c_{13}$. Therefore, there are at least one point among $c_1, c_2, c_3, c_4, c_5, c_6, c_9, c_{10}, c_{11}, c_{12}, c_{13}$ which join $r_3$ and $r_4$. First, consider the case when there are at least three points among $c_1, c_2, c_3, c_4, c_5, c_6, c_9, c_{10}, c_{11}, c_{12}, c_{13}$ each of which joins $r_3$ and $r_4$. Also, since $c_{14}$ is joined by red lines to $r_3, r_4$, we can see that red $K_{2,4}$ is contained in the $K_{7,14}$, see Table 2.13 for example.

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Table 2.13.

Thus we shall consider the case when there are one and two points among $c_1, c_2, c_3, c_4, c_5, c_6, c_9, c_{10}, c_{11}, c_{12}, c_{13}$ which join $r_3$ and $r_4$.

Consider when there is one point among $c_1, c_2, c_3, c_4, c_5, c_6, c_9, c_{10}, c_{11}, c_{12}, c_{13}$ which join $r_3$ and $r_4$. Suppose this point is $c_{13}$, see Table 2.14 for example. When we consider only $r_1, r_2, r_3, r_4$ there are cases when the $K_{7,14}$ does not contain red $K_{2,4}$, see Table 2.14 for example. For such cases, we shall consider the seven red lines joining $r_5$.

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Table 2.14.

From Table 2.14, $r_3$ is joined to $c_1, c_2, c_3, c_{11}, c_{12}, c_{13}$ and $r_4$ is joined to $c_4, c_5, c_6, c_9, c_{10}, c_{13}$ and since $r_5$ is joined by one of the red lines to $c_{14}$, there are six other red lines joining $r_5$. From these six red lines, either at least three red lines are joined to points among $c_1, c_2, c_3, c_{11}, c_{12}, c_{13}$ or at least three red lines are joined to points among $c_4, c_5, c_6, c_9, c_{10}, c_{13}$. Assume that there are three red lines that are joined to points among $c_1, c_2, c_3, c_{11}, c_{12}, c_{13}$, say $c_2, c_3, c_{11}$ or there are
three red lines that are joined to points among $c_4, c_5, c_6, c_9, c_{10}, c_{13}, c_9, c_{10}, c_{13}$. Also, since $c_{14}$ is joined by red lines to $r_3, r_4$ and $r_5$, we can verify that in either case the $K_{7,14}$ contains red $K_{2,4}$, see Table 2.15 for example.

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Table 2.15.

Consider when there are two points among $c_1, c_2, c_3, c_4, c_5, c_6, c_9, c_{10}, c_{11}, c_{12}, c_{13}$ which join $r_3$ and $r_4$.

Suppose these points are $c_{12}$ and $c_{13}$. There are cases when the $K_{7,14}$ does not contain red $K_{2,4}$, see Table 2.16 for example. So, we consider the seven red lines joining $r_5$.

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Table 2.16.

From Table 2.16, $r_3$ is joined to $c_1, c_2, c_3, c_{11}, c_{12}, c_{13}$ and $r_4$ is joined to $c_4, c_5, c_6, c_{10}, c_{12}, c_{13}$ and since $r_5$ is joined by one of the red lines to $c_{14}$. If $r_5$ is not joined to $c_9$, there are six other red lines joining $r_5$. From these six red lines, either at least three red lines are joined to points among $c_1, c_2, c_3, c_{11}, c_{12}, c_{13}$ or at least three red lines are joined to points among $c_4, c_5, c_6, c_{10}, c_{12}, c_{13}$, we can see that the $K_{7,14}$ contains red $K_{2,4}$. If $r_5$ is joined to $c_9$, there are five other red lines joining $r_5$. From these five red lines, either at least three red lines joining to points among $c_1, c_2, c_3, c_{11}, c_{12}, c_{13}$, say $c_1, c_2, c_{11}$ or at least three red lines are joined to points among $c_4, c_5, c_6, c_{10}, c_{12}, c_{13}$, say $c_4, c_{10}$. Also, since $c_{14}$ is joined by red lines to $r_1, r_2$ and $r_5$. We can verify that in either case the $K_{7,14}$ contains red $K_{2,4}$, see Table 2.17 for example.

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Table 2.17.

2.5.3. Three of $c_i$’s are joined by red lines to both of $r_1$ and $r_2$. 

Suppose that $r_1$ is joined by eight red lines to $c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8$ and $r_2$ is joined by seven red lines to $c_6, c_7, c_8, c_9, c_{10}, c_{11}, c_{12}$. We consider two subcases.

(1) Some $r_i$'s ($i = 3, 4, 5, 6, 7$) are not joined to $c_{13}$ and $c_{14}$ by red lines.

If both $c_{13}$ and $c_{14}$ are not joined to $r_3$ for example, then either at least four red lines from $r_3$ are joined to points among $c_1, c_2, \ldots, c_7$ or at least four red lines are joined to points among $c_8, c_9, \ldots, c_{12}$. In either case, we see that red $K_{2,4}$ is contained in $K_{7,14}$, see Table 2.18 for example.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|c|c|c|c|}
\hline
 & $c_1$ & $c_2$ & $c_3$ & $c_4$ & $c_5$ & $c_6$ & $c_7$ & $c_8$ & $c_9$ & $c_{10}$ & $c_{11}$ & $c_{12}$ & $c_{13}$ & $c_{14}$ \\
\hline
$r_1$ & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
$r_2$ & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
$r_3$ & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
\end{tabular}
\caption{Table 2.18.}
\end{table}

(2) Each of all $r_i$'s ($i = 3, 4, 5, 6, 7$) is joined to $c_{13}$ or $c_{14}$ or both by red lines. Here we consider two possibilities (2.1) and (2.2).

(2.1) Three or more $r_i$'s ($i = 3, 4, 5, 6, 7$) are joined to one of $c_{13}$ and $c_{14}$.

Suppose $r_3, r_4, r_5$ are joined to one of $c_{13}$ and $c_{14}$, then at least two of $r_3, r_4, r_5$ are joined to $c_{13}$ or at least two of $r_3, r_4, r_5$ are joined to $c_{14}$. Let $r_3, r_4$ be joined to $c_{13}$. Consider 4 subcases.

(2.1.1) $r_3$ or $r_4$ are joined to $c_6, c_7,$ and $c_8$.

Suppose that $r_3$ is joined to $c_6, c_7, c_8$. There are three other red lines joining $r_3$. From these three red lines joining $r_3$, either at least two of the red lines are joined to points among $c_1, c_2, \ldots, c_5$ or at least two red lines are joined to points among $c_9, c_{10}, \ldots, c_{12}$. In either case, red $K_{2,4}$ is formed. For this subcase, see Table 2.19 for example.

\begin{table}[h]
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\begin{tabular}{|c|c|c|c|c|c|c|c|c|c|c|}
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 & $c_1$ & $c_2$ & $c_3$ & $c_4$ & $c_5$ & $c_6$ & $c_7$ & $c_8$ & $c_9$ & $c_{10}$ & $c_{11}$ & $c_{12}$ & $c_{13}$ & $c_{14}$ \\
\hline
$r_1$ & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
$r_2$ & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
$r_3$ & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
\end{tabular}
\caption{Table 2.19.}
\end{table}

(2.1.2) $r_3$ or $r_4$ are joined to two of $c_6, c_7,$ and $c_8$.

Suppose that $r_3$ is joined to $c_7$ and $c_8$, there are four other red lines joining $r_3$. From these four red lines, either at least two of the red lines are joined to points among $c_1, c_2, \ldots, c_5$ or at least two red lines are joined to points among $c_9, c_{10}, \ldots, c_{12}$. In either case, we see that red $K_{2,4}$ is contained in the $K_{7,14}$, see
Table 2.20 for example.

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Table 2.20.

(2.1.3) r₃ or r₄ are joined to one of c₆, c₇, and c₈.

Suppose that r₃ is joined to c₈, there are five other red lines joining r₃. From these five red lines joining r₃, either at least three of the red lines are joined to points among c₁, c₂, . . . , c₅ or at least three red lines are joined to points among c₉, c₁₀, . . . , c₁₂. In either case, we can see that red $K₂,₄$ is contained in the $K₇,₁₄$, see Table 2.21 for example.

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Table 2.21.

(2.1.4) r₃ and r₄ are not joined to c₆, c₇, and c₈.

Since r₃ is joined to c₁₃ and not joined to c₆, c₇ and c₈. There are six other red lines joining r₃. From these six red lines joining r₃, either at least three of the red lines are joined to points among c₁, c₂, . . . , c₅ or at least three red lines are joined to points among c₉, c₁₀, . . . , c₁₂. Consider the case when there are at least four points among c₁, c₂, . . . , c₅ or at least four points among c₉, c₁₀, . . . , c₁₂ that are joined to r₃. In either case, we can see that red $K₂,₄$ is contained in the $K₇,₁₄$.

Thus we consider the case when there are three points among c₁, c₂, . . . , c₅, say c₁, c₂, c₃ and three points among c₉, c₁₀, . . . , c₁₂, say c₉, c₁₀, c₁₁ that are joined to r₃ by red lines. There are cases, see Table 2.22 for example, when the $K₇,₁₄$ does not contain red $K₂,₄$, so we consider r₄.

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Table 2.22.

Since r₄ is joined to c₁₃ and not joined to c₆, c₇, c₈, there are six other red lines joining r₄. From these six red lines, we can see that there are at least three
points among $c_1, c_2, \ldots, c_5, c_9, c_{10}, \ldots, c_{12}$ that are joined to both $r_3$ and $r_4$. In either case, red $K_{2,4}$ is contained in the $K_{7,14}$, see Table 2.23 for example.

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Table 2.23.

From subcases (2,1,1), (2,1,2), and (2,1,3), if there is at least one of $r_i$'s ($i = 3, 4, 5, 6, 7$) that is joined to some $c_6, c_7, c_8$, then we can see that red $K_{2,4}$ is contained in the $K_{7,14}$. Later, if we consider the cases when $r_i$'s ($i = 3, 4, 5, 6, 7$) are joined to one of $c_{13}$ and $c_{14}$, we shall consider only the case when $r_i$'s ($i = 3, 4, 5, 6, 7$) is not joined to $c_6, c_7, c_8$.

(2.2) Less than three $r_i$'s ($i = 3, 4, 5, 6, 7$) that are joined to one of $c_{13}$ and $c_{14}$. Consider three subcases.

(2.2.1) One of $r_i$'s ($i = 3, 4, 5, 6, 7$), say $r_7$ is joined to one of $c_{13}$ and $c_{14}$. In these cases there are four of $r_i$'s ($i = 3, 4, 5, 6, 7$), say $r_3, r_4, r_5$ and $r_6$ that are joined to both $c_{13}$ and $c_{14}$. Consider two possibilities (a) and (b).

(a) Some $r_i$'s ($i = 3, 4, 5, 6$) are joined to at least two of $c_6, c_7$ and $c_8$. Suppose that $r_3$ is joined to $c_7$ and $c_8$. There are three other red lines joining $r_3$. From these three red lines, either at least two of the red lines are joined to points among $c_1, c_2, \ldots, c_5$ or at least two red lines are joined to points among $c_9, c_{10}, \ldots, c_{12}$. In either case, red $K_{2,4}$ is contained in the $K_{7,14}$, see Table 2.24 for example.

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Table 2.24.

(b) Each of $r_i$'s ($i = 3, 4, 5, 6$) is joined to one or none of $c_6, c_7$ and $c_8$. There are three possibilities.

(i) Each of all $r_i$'s ($i = 3, 4, 5, 6$) is joined to one of $c_6, c_7$ and $c_8$. Since all $r_i$'s ($i = 3, 4, 5, 6$) are joined to one of $c_6, c_7$ and $c_8$, then at least two of $r_i$ are joined to $c_6$ or at least two of $r_i$ are joined to $c_7$ or at least two of $r_i$ are joined to $c_8$. Let $r_3, r_4$ are joined to $c_8$. For this case we consider only three of $r_3, r_4, r_5, r_6$ then we can see that red $K_{2,4}$ is contained in the $K_{7,14}$. Since $r_3$ is joined by red lines to $c_8, c_{13}$, and $c_{14}$. There are four other red lines joining $r_3$. From these four red lines, either at least two of the red lines are
joined to points among \(c_1, c_2, \ldots, c_5\) or at least two red lines are joined to points among \(c_9, c_{10}, \ldots, c_{12}\). For the cases when there are three or more red lines joining \(r_3\) and points among \(c_1, c_2, \ldots, c_5\) or \(c_9, c_{10}, \ldots, c_{12}\), we can see that red \(K_{2,4}\) is contained in the \(K_{7,14}\) see Table 2.25 for example.

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</table>

Table 2.25.

Thus we shall consider when each \(r_i\) \((i = 3, 4, 5, 6)\) has two red lines that are joined to points among \(c_1, c_2, \ldots, c_5\) and two red lines that are joined to points among \(c_9, c_{10}, \ldots, c_{12}\).

Suppose that \(r_3\) is joined by red lines to two points among \(c_1, c_2, \ldots, c_5\), say \(c_1, c_2\), and joined by red lines to two points among \(c_9, c_{10}, \ldots, c_{12}\), say \(c_9, c_{10}\). There are cases when the \(K_{7,14}\) does not contain red \(K_{2,4}\), see Table 2.26 for example. We consider \(r_4\).

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Table 2.26.

Since \(r_4\) is joined to \(c_8, c_{13}, \) and \(c_{14}\), there are four other red lines joining \(r_4\). From these four red lines, either at least two of the red lines are joined to points among \(c_1, c_2, \ldots, c_5\) and at least two red lines are joined to points among \(c_9, c_{10}, \ldots, c_{12}\). For the case when there is at least one of \(c_1, c_2, c_9, c_{10}\) that is joined to both \(r_3\) and \(r_4\), since \(c_8, c_{13}, \) and \(c_{14}\) are joined to both \(r_3\) and \(r_4\), then red \(K_{2,4}\) is contained in the \(K_{7,14}\) see Table 2.27 for example.

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Table 2.27.

Thus we consider the case when none of \(c_1, c_2, c_9, c_{10}\) is joined to both \(r_3\) and \(r_4\), there are cases when the \(K_{7,14}\) does not contain red \(K_{2,4}\), see Table 2.28 for example, so we consider \(r_5\).

Since \(r_5\) is joined to one of \(c_6, c_7\) and \(c_8\), say \(c_7\), if \(r_5\) is not joined to \(c_5\), there are four red lines joining \(r_5\). From these four red lines, either two of the red lines
are joined to points among $c_1, c_2, c_9, c_{10}$ or two of the red lines are joined to points among $c_3, c_4, c_{11}, c_{12}$, and since $c_{13}$ and $c_{14}$ are joined to both $r_3$ and $r_4$, then we can see that red $K_{2,4}$ is contained in the $K_{7,14}$. If $r_5$ is joined to $c_3$, there are three red lines joining $r_5$. From these three red lines, either two of the red lines are joined to points among $c_1, c_2, c_9, c_{10}$ or two red lines are joined to points among $c_3, c_4, c_{11}, c_{12}$. Also, since $c_{13}$ and $c_{14}$ are joined to $r_3$, $r_4$, and $r_5$, so we can see that red $K_{2,4}$ is contained in the $K_{7,14}$, see Table 2.28 for example.

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Table 2.28.

(ii) All $r_i$'s ($i = 3, 4, 5, 6$) are not joined to $c_6, c_7,$ and $c_8$.

For this case we consider only three of $r_3, r_4, r_5, r_6$ and we can see that red $K_{2,4}$ is contained in the $K_{7,14}$. Since $r_3$ is joined to $c_{13}$ and $c_{14}$, there are five other red lines joining $r_3$. For the cases when $r_3$ is joined by red lines to at least four points among $c_1, c_2, \ldots, c_5$ or at least four points among $c_9, c_{10}, c_{11}, c_{12}$, we can see that the $K_{7,14}$ contains red $K_{2,4}$.

Thus we shall consider the cases when each of $r_i$'s ($i = 3, 4, 5, 6$) is joined by red lines to three points among $c_1, c_2, \ldots, c_5$ or three points among $c_9, c_{10}, c_{11}, c_{12}$. Consider when $r_3$ is joined by red lines to three points among $c_1, c_2, \ldots, c_5$ or three points among $c_9, c_{10}, c_{11}, c_{12}$. There are cases when the $K_{7,14}$ does not contain red $K_{2,4}$, see Table 2.29 for example, so we consider $r_4$.

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Table 2.29.

Since $r_4$ is joined to $c_{13}$ and $c_{14}$, there are other five red lines joining $r_4$. We can see that there is at least one of $c_1, c_2, c_3, c_9, c_{10}$ that is joined by red lines to both $r_3$ and $r_4$. For the cases when $r_4$ is joined by red lines to at least two points among $c_1, c_2, c_3, c_9, c_{10}$, we can see that the $K_{7,14}$ contains red $K_{2,4}$. Thus we consider the case when one point among $c_1, c_2, c_3, c_9, c_{10}$, say $c_3$, is joined by red line to both $r_3$ and $r_4$. There are cases when the $K_{7,14}$ does not contain red $K_{2,4}$,
see Table 2.30 for example, so we consider $r_5$.

\begin{tabular}{|c|c|c|c|c|c|c|c|c|c|c|c|}
\hline
 & $c_1$ & $c_2$ & $c_3$ & $c_4$ & $c_5$ & $c_6$ & $c_7$ & $c_8$ & $c_9$ & $c_{10}$ & $c_{11}$ & $c_{12}$ & $c_{13}$ & $c_{14}$ \\
\hline
$r_1$ & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
$r_2$ & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
$r_3$ & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
$r_4$ & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
\end{tabular}

Table 2.30.

Since $r_5$ is joined to $c_{13}$ and $c_{14}$, there are other five red lines joining $r_5$. Either there are at least three red lines that are joined to points among $c_1, c_2, c_3, c_9, c_{10}$ or at least three red lines that are joined to points among $c_3, c_4, c_5, c_{11}, c_{12}$. In either case, the $K_{7,14}$ contains red $K_{2,4}$, see Table 2.31 for example.

\begin{tabular}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}
\hline
 & $c_1$ & $c_2$ & $c_3$ & $c_4$ & $c_5$ & $c_6$ & $c_7$ & $c_8$ & $c_9$ & $c_{10}$ & $c_{11}$ & $c_{12}$ & $c_{13}$ & $c_{14}$ \\
\hline
$r_1$ & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
$r_2$ & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
$r_3$ & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
$r_4$ & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
$r_5$ & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
\end{tabular}

Table 2.31.

From subcases (a) of (2.2.1), if there is at least one of $r_i$'s ($i = 3, 4, 5, 6, 7$) that is joined to at least two of $c_6, c_7, c_8$, then we can see that red $K_{2,4}$ is contained in the $K_{7,14}$. Later, if we consider the cases when $r_i$'s ($i = 3, 4, 5, 6, 7$) are joined to $c_{13}$ and $c_{14}$, we shall consider only the case when $r_i$'s ($i = 3, 4, 5, 6, 7$) is joined to one or none of $c_6, c_7, c_8$.

(iii) Some $r_i$'s ($i = 3, 4, 5, 6$) are joined by red line to one point among $c_6, c_7, c_8$ and some $r_i$'s ($i = 3, 4, 5, 6$) are not joined to $c_6, c_7,$ and $c_8$.

We need to consider only two key cases 1) and 2). It will become clear later that the required results, for all subcases with condition (iii), will follow from the results of 1) or 2).

1) Two of $r_i$'s ($i = 3, 4, 5, 6$) are joined to one of $c_6, c_7, c_8$ and one of the remaining $r_i$'s is joined to none of $c_6, c_7, c_8$.

Suppose two of $r_i$'s ($i = 3, 4, 5, 6$) are joined to one of $c_6, c_7, c_8$ are $r_3$, $r_5$ and one of $r_i$ is joined to none of $c_6, c_7, c_8$ is $r_4$.

Suppose that $r_3$ is joined to one of $c_6, c_7$ and $c_8$, say $c_8$, and since $r_3$ is joined to $c_{13}, c_{14}$, there are four other red lines joining $r_3$. From these four red lines, either at least two of the red lines are joined to points among $c_1, c_2, \ldots, c_5$ or at least two red lines are joined to points among $c_9, c_{10}, \ldots, c_{12}$.

For the cases when three or more red lines joining $r_3$ and points among $c_1, c_2, \ldots, c_5$ or $c_9, c_{10}, \ldots, c_{12}$, we can see that red $K_{2,4}$ is contained in the $K_{7,14}$, see Table 2.25.
Thus we consider when each of $r_3$, $r_5$ has two red lines that are joined to points among $c_1, c_2, \ldots, c_5$, and two red lines that are joined to points among $c_9, c_{10}, \ldots, c_{12}$.

Suppose that $r_3$ is joined by red lines to two points among $c_1, c_2, \ldots, c_5$, say $c_1, c_2$, and joined by red lines to two points among $c_9, c_{10}, \ldots, c_{12}$, say $c_9, c_{10}$. For this case there are cases when the $K_{7,14}$ does not contain red $K_{2,4}$, see Table 2.26, so we consider $r_4$.

Since $r_4$ is joined to $c_{13}$ and $c_{14}$, there are five other red lines joining $r_4$. For the cases when at least four of the red lines are joined to points among $c_1, c_2, \ldots, c_5$, or when at least four of the red lines are joined to points among $c_9, c_{10}, c_{11}, c_{12}$, we can see that the $K_{7,14}$ contains red $K_{2,4}$. Thus we shall consider the case when there are three red lines that are joined to points among $c_1, c_2, \ldots, c_5$ or there are three red lines that are joined to points among $c_9, c_{10}, c_{11}, c_{12}$. When there are at least two points among $c_1, c_2, c_9, c_{10}$ that are joined by red line to both $r_3$ and $r_4$, we can see that the $K_{7,14}$ contains red $K_{2,4}$. Thus we consider when there is at most one point among $c_1, c_2, c_9, c_{10}$ that is joined by red line to both $r_3$ and $r_4$.

Consider the case when there is one point among $c_1, c_2, c_9, c_{10}$ that is joined by red line to both $r_3$ and $r_4$, say $c_{10}$. There are cases when the $K_{7,14}$ does not contain red $K_{2,4}$ see Table 2.32 for example, so we consider $r_5$.

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</tbody>
</table>

Table 2.32.

Since $r_5$ is joined to $c_{13}$ and $c_{14}$ and is joined to one of $c_6, c_7, c_8$, say $c_7$. If $r_5$ is not joined by red lined to $c_5$, there are four other red lines joining $r_5$. Either there are at least two red lines that are joined to points among $c_1, c_2, c_9, c_{10}$ or at least two red lines that are joined to points among $c_3, c_4, c_{10}, c_{11}, c_{12}$, we can see that the $K_{7,14}$ contains red $K_{2,4}$. If $r_5$ is joined to $c_5$, there are three red lines joining $r_5$. Either there are at least two red lines that are joined to points among $c_1, c_2, c_9, c_{10}$ or at least two red lines that are joined to points among $c_3, c_4, c_{10}, c_{11}, c_{12}$, we can see that the $K_{7,14}$ contains red $K_{2,4}$, see Table 2.33 for example.

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</table>

Table 2.33.
Consider the case when all of $c_1, c_2, c_9, c_{10}$ are not joined to $r_4$. There are cases when the $K_{7,14}$ does not contain red $K_{2,4}$, see Table 2.34 for example, so we consider $r_5$.

<table>
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</tbody>
</table>

Table 2.34.

Since $r_5$ is joined to $c_{13}$ and $c_{14}$ and is joined by red line to one point among $c_6, c_7, c_8$, say $c_7$, there are four other red lines joining $r_5$. Either there are at least two red lines that are joined to points among $c_1, c_2, c_9, c_{10}$ or at least two red lines that are joined to points among $c_3, c_4, c_5, c_{11}, c_{12}$, we can see that the $K_{7,14}$ contains red $K_{2,4}$, see Table 2.35 for example.

<table>
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</tbody>
</table>

Table 2.35.

2) One of $r_i$'s ($i = 3, 4, 5, 6$) is joined to one of $c_6, c_7, c_8$ and two of the remaining $r_i$'s are joined to none of $c_6, c_7, c_8$.

Suppose that one of $r_i$'s ($i = 3, 4, 5, 6$) that is joined to one of $c_6, c_7, c_8$ is $r_3$ and two of $r_i$ that are joined to none $c_6, c_7, c_8$ are $r_4, r_5$.

From Table 2.34, we can see that there are cases when the $K_{7,14}$ does not contain red $K_{2,4}$, so we consider $r_5$. Since $r_5$ is joined to $c_{13}$ and $c_{14}$, there are five other red lines joining $r_5$. Either there are at least three red lines that are joined to points among $c_1, c_2, c_9, c_{10}$ or at least three red lines that are joined to points among $c_3, c_4, c_5, c_{11}, c_{12}$, we can see that the $K_{7,14}$ contains red $K_{2,4}$, see Table 2.36 for example.

<table>
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</table>

Table 2.36.

(2.2.2) Two of $r_i$'s ($i = 3, 4, 5, 6, 7$), say $r_3$ and $r_4$, are joined to one of $c_{13}$ and $c_{14}$.
In these cases there are three of \( r_i \)'s \((i = 3, 4, 5, 6, 7)\), say \( r_5, r_6 \) and \( r_7 \), that are joined to both \( c_{13} \) and \( c_{14} \).

From what we have mentioned at the end of (2.1), here we need to consider only the cases when \( r_3 \) and \( r_4 \) are not joined to \( c_6, c_7, \) and \( c_8 \).

(a) \( r_3 \) and \( r_4 \) are joined by red lines to \( c_{13} \).

From Table 2.23, we can see that the \( K_{7,14} \) contains red \( K_{2,4} \). Similarly, if \( r_3 \) and \( r_4 \) are joined by red lines to \( c_{14} \), we can see that red \( K_{2,4} \) is also formed.

(b) \( r_3 \) is joined by red lines to \( c_{13} \) and \( r_4 \) is joined by red lines to \( c_{14} \).

Since \( r_3 \) is joined to \( c_{13} \) and is not joined to \( c_6, c_7, \) and \( c_8 \), there are six other red lines joining \( r_3 \). From these six red lines joining \( r_3 \), either at least three of the red lines are joined to points among \( c_1, c_2, \ldots, c_5 \) or at least three of the red lines are joined to points among \( c_9, c_{10}, \ldots, c_{12} \). Consider the cases when there are at least four points among \( c_1, c_2, \ldots, c_5 \) or at least four points among \( c_9, c_{10}, \ldots, c_{12} \) that are joined to \( r_3 \). In either case, we can see that red \( K_{2,4} \) is contained in the \( K_{7,14} \).

Thus we consider the cases when there are three points among \( c_1, c_2, \ldots, c_5 \) and three points among \( c_9, c_{10}, \ldots, c_{12} \) that are joined to \( r_3 \) by red lines. Suppose these six points are \( c_1, c_2, c_3, c_{10}, c_{11}, c_{12} \), we can see that there are cases when the \( K_{7,14} \) does not contain red \( K_{2,4} \), so we consider \( r_4 \).

Since \( r_4 \) is joined to \( c_{14} \), there are six other red lines joining \( r_4 \). From these six red lines, we can see that there are at least three lines are joined to points among \( c_1, c_2, \ldots, c_5, c_9, c_{10}, \ldots, c_{12} \) that are joined to both \( r_3 \) and \( r_4 \). There are cases when the \( K_{7,14} \) does not contain red \( K_{2,4} \). See Table 2.37 for example, so we consider \( r_i \)'s \((i = 5, 6, 7)\). We consider three subcases.

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</tbody>
</table>

Table 2.37.

From what we have mentioned at the end of (ii) of (b) in (2.2.1), here we need to consider only the cases when \( r_5, r_6 \), and \( r_7 \) are joined to one and none of \( c_6, c_7, \) and \( c_8 \).

(i) All \( r_i \)'s \((i = 5, 6, 7)\) are not joined to \( c_6, c_7, c_8 \).

From Table 2.37, we can see that \( r_3 \) is joined to \( c_1, c_2, c_3, c_{10}, c_{11}, c_{12} \) and \( r_4 \) is joined to \( c_9, c_5, c_6, c_8, c_{11}, c_{12} \). We consider \( r_5 \). Since \( r_5 \) is joined to \( c_{13}, c_{14} \), then there are five other red lines joining \( r_5 \). From these five red lines, either at least three lines are joined to points among \( c_1, c_2, c_3, c_{10}, c_{11}, c_{12} \) or at least three lines are joined to points among \( c_3, c_4, c_5, c_9, c_{11}, c_{12} \). Suppose that these five points are \( c_1, c_2, c_4, c_6, c_{10} \). Since \( r_3 \) and \( r_5 \) are joined to \( c_{13} \), we can see that the \( K_{7,14} \)
contains red $K_{2,4}$ see Table 2.38 for example.

\begin{center}
\begin{tabular}{cccccccccc}
  & c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 & c_8 & c_9 & c_{10} & c_{11} & c_{12} & c_{13} & c_{14} \\
  $r_1$ & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
  $r_2$ & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
  $r_3$ & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
  $r_4$ & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
  $r_5$ & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{tabular}
\end{center}

\begin{center}
Table 2.38.
\end{center}

(ii) Each of all $r_i$'s ($i = 5, 6, 7$) is joined to one of $c_6, c_7, c_8$.

Since $r_5$ is joined to $c_{13}, c_{14}$ and is joined to one of $c_6, c_7, c_8$, say $c_8$, then there are four other red lines joining $r_5$. From these four red lines, either at least two lines are joined to points among $c_1, c_2, c_3, c_{10}, c_{11}, c_{12}$ or at least two lines are joined to points among $c_3, c_4, c_5, c_9, c_{11}, c_{12}$. For this case there are cases when the $K_{7,14}$ does not contain red $K_{2,4}$, see Table 2.39 for example, so we consider $r_6$.

\begin{center}
\begin{tabular}{cccccccccc}
  & c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 & c_8 & c_9 & c_{10} & c_{11} & c_{12} & c_{13} & c_{14} \\
  $r_1$ & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
  $r_2$ & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
  $r_3$ & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
  $r_4$ & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
  $r_5$ & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
  $r_6$ & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{tabular}
\end{center}

\begin{center}
Table 2.39.
\end{center}

Since $r_6$ is joined to $c_{13}, c_{14}$ and is joined by red line to one point among $c_6, c_7, c_8$, say $c_7$, there are four other red lines joining $r_6$. From these four red lines, either there are at least three red lines that are joined to points among $c_1, c_2, c_3, c_{10}, c_{11}, c_{12}$ or there are at least three red lines that are joined to points among $c_3, c_4, c_5, c_9, c_{11}, c_{12}$ or there are at least two red lines that are joined to points among $c_2, c_4, c_9, c_{10}$. We can see that the $K_{7,14}$ contains red $K_{2,4}$, see Table 2.40 for example.

\begin{center}
\begin{tabular}{cccccccccc}
  & c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 & c_8 & c_9 & c_{10} & c_{11} & c_{12} & c_{13} & c_{14} \\
  $r_1$ & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
  $r_2$ & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
  $r_3$ & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
  $r_4$ & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
  $r_5$ & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
  $r_6$ & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{tabular}
\end{center}

\begin{center}
Table 2.40.
\end{center}

(iii) Some $r_i$'s ($i = 5, 6, 7$) are joined to one of $c_6, c_7, c_8$ and some $r_i$'s ($i = 5, 6, 7$) are not joined to $c_6, c_7, c_8$. 


For this case we can see that there is at least one of $r_5$, $r_6$, and $r_7$ that is not joined to $c_6$, $c_7$, $c_8$. From subcase (i) and Table 2.38, we can see that the $K_{7,14}$ contains red $K_{2,4}$.

(2.2.3) None $r_i$'s ($i = 3, 4, 5, 6, 7$) are joined to one of $c_{13}$ and $c_{14}$.

In these cases all $r_i$'s ($i = 3, 4, 5, 6, 7$) are joined to both $c_{13}$ and $c_{14}$.

From the case (2.2.1), since if four of $r_i$'s ($i = 3, 4, 5, 6, 7$) are joined to $c_{13}$ and $c_{14}$ then the $K_{7,14}$ contains red $K_{2,4}$. Therefore in the case (2.2.3) the $K_{7,14}$ contains red $K_{2,4}$.

Hence,

$$rt(K(2, 4), K(2, 4)) \leq 7. \quad (2.3)$$

Therefore, from the inequalities (2.1) and (2.3), we have $rt(K(2, 4), K(2, 4)) = 7$ as required. $\square$

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References


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