Meir and Keeler Type Fixed Point Theorem for Set-Valued Generalized Contractions in Metrically Convex Spaces

Ladlay Khan†,1 and M. Imdad‡

†Department of Applied Sciences, Mewat Engineering College (Wakf)
Palla, Nuh, Mewat 122 107, Haryana, India
e-mail : kladlay@yahoo.com
‡Department of Mathematics, Aligarh Muslim University
Aligarh 202 002, India
e-mail : mhimdad@yahoo.co.in

Abstract : A fixed point theorem for generalized set-valued contraction in metrically convex spaces has been proved which generalizes a fixed point theorem due to Rhoades [B.E. Rhoades, A fixed point theorem for some non-self mappings, Math. Japonica. 23 (4) (1978) 457–459]. An illustrative example is also discussed.

Keywords : metrically convex metric spaces; non-self mappings; set-valued mappings; metric convexity; Meir-Keeler type condition.

2010 Mathematics Subject Classification : 54H25; 47H10.

1 Introduction

Meir and Keeler [1] established that classical Banach contraction principle remains true for weakly uniformly strict contractions:

Given \( \epsilon > 0 \) there exists a \( \delta > 0 \) such that

\[
\epsilon \leq d(x, y) < \epsilon + \delta \quad \text{implies} \quad d(Tx, Ty) < \epsilon.
\] (1.1)

In recent years this result due to Meir and Keeler [1] has been generalized,
extended and improved in various ways and by now there exists a considerable literature in this direction for self mappings. To mention a few we cite [2–8].

In this note, we establish a Meir and Keeler [1] type fixed point theorem for set-valued generalized contraction in metrically convex spaces. In proving our result we follow the definition and convention of Assad and Kirk [9] and Nadler [10]. Before formulating our result, for the sake of completeness we state the following result due to Rhoades [11].

**Theorem 1.1.** Let \((X, d)\) be a complete metrically convex metric space and \(K\) a nonempty closed convex subset of \(X\). Let \(T : K \to X\) be a map satisfying:

\[d(Tx, Ty) \leq M(x, y)\]

where

\[M(x, y) = h \max \left\{ \frac{1}{2}d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{q} \right\}\]  

(1.2)

for all \(x, y \in K\), with \(x \neq y\), where \(0 < h < 1\), \(q \geq 1 + 2h\), and

(i) \(Tx \in K\) for each \(x \in \partial K\).

Then \(T\) has a fixed point in \(K\).

We now state relevant definition and lemmas which are used in the sequel.

**Definition 1.2 ([9]).** A metric space \((X, d)\) is said to be metrically convex if for any \(x, y \in X\) with \(x \neq y\) there exists a point \(z \in X, x \neq z \neq y\) such that

\[d(x, z) + d(z, y) = d(x, y)\]

**Lemma 1.3 ([9]).** Let \(K\) be a nonempty closed subset of a metrically convex metric space \(X\). If \(x \in K\) and \(y \notin K\) then there exists a point \(z \in \partial K\) (the boundary of \(K\)) such that

\[d(x, z) + d(z, y) = d(x, y)\]

In what follows, \(CB(X)\) denotes the set of all closed and bounded subsets of \((X, d)\), while \(C(X)\) for collection of all compact subsets of \((X, d)\). Also \(H\) denotes the Hausdorff distance between two sets.

**Lemma 1.4 ([10]).** Let \(A, B \in CB(X)\). Then for all \(\epsilon > 0\) and \(a \in A\) there exists \(b \in B\) such that \(d(a, b) \leq H(A, B) + \epsilon\). If \(A, B \in C(X)\), then one can choose \(b \in B\) such that \(d(a, b) \leq H(A, B)\).

**2 Main Results**

We prove the following.
Theorem 2.1. Let \((X, d)\) be a complete metrically convex metric space and \(K\) a nonempty closed subset of \(X\). Let \(T : K \to C(X)\) be a set-valued map which satisfies (i) and for a given \(\epsilon > 0\) there exists \(\delta(\epsilon) > 0, \delta(\epsilon)\) being a nondecreasing function of \(\epsilon\) with \(q \geq 1 + 2h\) where \(0 < h < 1\) such that
\[
\epsilon \leq M(x, y) < \epsilon + \delta \text{ implies } H(Tx, Ty) < \epsilon.
\] (2.1)

Then \(T\) has a fixed point in \(K\).

Proof. Firstly, we proceed to construct two sequences \(\{x_n\}\) and \(\{x'_n\}\) in the following way. Let \(x_0 \in K\). Define \(x'_1 \in Tx_0\). If \(x'_1 \in K\) then set \(x'_1 = x_1\). If \(x'_1 \notin K\) choose \(x_1 \in \delta K\) so that
\[
d(x_0, x_1) + d(x_1, x'_1) = d(x_0, x'_1).
\]
Then \(x_1 \in K\). By using Lemma 1.4, select \(x'_2 \in Tx_1\) such that \(d(x'_1, x'_2) \leq H(Tx_0, Tx_1)\). If \(x'_2 \in K\) then \(x'_2 = x_2\). Otherwise choose \(x_2 \in \delta K\) such that
\[
d(x_1, x_2) + d(x_2, x'_2) = d(x_1, x'_2).
\]
Thus by induction, one obtains two sequences \(\{x_n\}\) and \(\{x'_n\}\) such that
\[
i (i) \quad x'_{n+1} \in Tx_n
\]
\[
(ii) \quad d(x'_{n+1}, x'_n) \leq H(Tx_n, Tx_{n-1}).
\]
\[
(iii) \quad d(x'_{n+1}, x'_n) \leq H(Tx_n, Tx_{n-1}).
\]
\[
(iv) \quad x'_{n+1} \in K \Rightarrow x'_{n+1} = x_{n+1},
\]
\[
(v) \quad x'_{n+1} \notin K \Rightarrow x_{n+1} \in \delta K \text{ and } d(x_n, x_{n+1}) + d(x_{n+1}, x'_{n+1}) = d(x_n, x'_{n+1}).
\]

Now define
\[
P = \{x_i \in \{x_n\} : x'_i = x_i, i = 1, 2, 3, \ldots\}
\]
\[
Q = \{x_i \in \{x_n\} : x'_i \neq x_i, i = 1, 2, 3, \ldots\}.
\]

Obviously, the two consecutive terms cannot lie in \(Q\).

Now we distinguish the following three cases.

Case 1. If \(x_n, x_{n+1} \in P\), then
\[
d(x_n, x_{n+1}) = H(Tx_{n-1}, Tx_n) \leq M(x_{n-1}, x_n)
\]
\[
\leq h \max \left\{ \frac{1}{2}d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), \frac{d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})}{q} \right\},
\]
\[
\leq h \max \left\{ \frac{1}{2}d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_{n+1}) + d(x_n, x_n)}{q} \right\},
\]
\[
\leq h \max \{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}.
\]
If \(d(x_{n-1}, x_n) \leq d(x_n, x_{n+1})\) then we get \(d(x_n, x_{n+1}) \leq d(x_n, x_{n+1})\), which is a contradiction. Otherwise, if \(d(x_n, x_{n+1}) \leq h\) \(d(x_{n-1}, x_n)\) then one obtains \(d(x_n, x_{n+1}) \leq M(x_{n-1}, x_n) \leq h\) \(d(x_{n-1}, x_n)\).

**Case 2.** If \(x_n \in P\) and \(x_{n+1} \in Q\) then

\[
d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+1}^\prime) = d(x_n, x_{n+1}^\prime),
\]

which in turn yields

\[
d(x_n, x_{n+1}) \leq d(x_n, x_{n+1}^\prime).
\]

Now, proceeding as in Case 1, we have

\[
d(x_n, x_{n+1}) \leq M(x_{n-1}, x_n) \leq h\) \(d(x_{n-1}, x_n)\).
\]

**Case 3.** If \(x_n \in Q\) and \(x_{n+1} \in P\) then \(x_{n-1} \notin P\). Since \(x_n\) is a convex linear combination of \(x_{n-1}\) and \(x_{n-1}^\prime\), it follows that

\[
d(x_n, x_{n+1}) \leq \max\{d(x_{n-1}, x_{n+1}), d(x_{n+1}, x_{n+1}^\prime)\}.
\]

Now, if \(d(x_{n-1}, x_{n+1}) \leq d(x_{n+1}, x_{n+1}^\prime)\), then proceeding as in Case 1, one obtains

\[
d(x_n, x_{n+1}) \leq M(x_{n-1}, x_n) \leq h\) \(d(x_{n-1}, x_n)\).
\]

Otherwise if \(d(x_{n+1}, x_{n+1}^\prime) \leq d(x_{n-1}, x_{n+1})\), then we have

\[
d(x_n, x_{n+1}) \leq d(x_{n-1}, x_{n+1}) = H(Tx_{n-2}, Tx_n) \leq M(x_{n-2}, x_n)
\]

\[
\leq h\) \max\left\{\frac{1}{2}d(x_{n-2}, x_n), d(x_{n-2}, Tx_{n-2}), d(x_n, Tx_{n-2}), \frac{d(x_{n-2}, Tx_n) + d(x_n, Tx_{n-2})}{q}\right\}
\]

\[
\leq h\) \max\left\{\frac{1}{2}d(x_{n-2}, x_n), d(x_{n-2}, x_{n-1}), d(x_n, x_{n+1}), \frac{d(x_{n-2}, x_{n+1}) + d(x_n, x_{n-1})}{q}\right\}.
\]

Since

\[
\frac{1}{2}d(x_{n-2}, x_n) = \max\{d(x_{n-2}, x_{n-1}), d(x_{n-1}, x_n)\}.
\]

Therefore, one obtains

\[
d(x_n, x_{n+1}) \leq h\) \max\left\{d(x_{n-2}, x_{n-1}), d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-2}, x_{n+1}) + d(x_n, x_{n-1})}{q}\right\}
\]
which in turn yields
\[ d(x_n, x_{n+1}) \leq \begin{cases} 
  h \, d(x_{n-1}, x_n), & \text{if } d(x_{n-1}, x_n) \geq d(x_{n-2}, x_{n-1}) \\
  h \, d(x_{n-2}, x_{n-1}), & \text{if } d(x_{n-1}, x_n) \leq d(x_{n-2}, x_{n-1}). 
\end{cases} \]

Thus in all the cases, we have
\[ d(x_n, x_{n+1}) \leq h \max\{d(x_{n-1}, x_n), d(x_{n-2}, x_{n-1})\}. \]

It can be easily shown by induction that for \( n \geq 1 \), we have
\[ d(x_n, x_{n+1}) \leq h \max\{d(x_0, x_1), d(x_1, x_2)\}. \]

Thus \( d(x_n, x_{n+1}) \) is a decreasing sequence and tending to \( t \in [0, \infty) \) as \( n \to \infty \).

Let on contrary
\[ d(x_n, x_{n+1}) > t \text{ for } n = 0, 1, 2\ldots. \] (2.2)

Suppose \( t > 0 \). Then there exists a \( \delta = \delta(\epsilon) \) and a positive integer \( k \) such that \( t \leq d(x_k, x_{k+1}) < \delta + t \). Hence by (2.1), one obtains
\[ d(x_{k+1}, x_{k+2}) = d(Tx_k, Tx_{k+1}) < t, \]
which contradicts (2.2) therefore \( d(x_n, x_{n+1}) \to 0 \) as \( n \to \infty \).

Now we wish to show that the sequence \( \{x_n\} \) is Cauchy. If it is not Cauchy then there exists \( 2\epsilon > 0 \) such that \( d(x_m, x_n) > 2\epsilon \). Choose \( \delta > 0 \) with \( \delta < \epsilon \) for which (2.1) is satisfied. Since \( d(x_n, x_{n+1}) \to 0 \) there exists a positive integer \( N = N(\delta) \) such that \( d(x_i, x_{i+1}) \leq \frac{\delta}{6} \) for all \( i \geq N \). With this choice of \( N \), let us choose \( m, n \) with \( m > n > N \) such that
\[ d(x_m, x_n) \geq 2\epsilon > \epsilon + \delta. \] (2.3)

By (2.3), \( m - n > 6 \), otherwise
\[ d(x_m, x_n) \leq d(x_n, x_{n+1}) + \cdots + d(x_{n+4}, x_{n+5}) \leq \frac{5\delta}{6} < \delta, \]
a contradiction. Now suppose that \( d(x_n, x_{m-1}) \leq \epsilon + \frac{\delta}{3} \). Then
\[ d(x_n, x_m) \leq d(x_n, x_{m-1}) + d(x_{m-1}, x_m) \leq \epsilon + \frac{\delta}{3} + \frac{\delta}{6} < \epsilon + \delta, \]
a contradiction. Similarly, suppose \( d(x_n, x_{m-2}) \leq \epsilon + \frac{\delta}{4} \). Then
\[
\begin{align*}
  d(x_n, x_m) & \leq d(x_n, x_{m-2}) + d(x_{m-2}, x_{m-1}) + d(x_{m-1}, x_m) \\
  & \leq \epsilon + \frac{\delta}{3} + \frac{\delta}{6} + \frac{\delta}{6} < \epsilon + \delta.
\end{align*}
\]
Let for the smallest integer $j \in (m, n)$ with $d(x_m, x_j) > \epsilon + \frac{\delta}{3}$, whereas

$$d(x_n, x_j) \leq d(x_n, x_{j-1}) + d(x_{j-1}, x_j) \leq \epsilon + \frac{\delta}{3} + \frac{\delta}{6} < \epsilon + \frac{2\delta}{3}.$$ 

Thus there exists a $j \in (n, m)$ such that

$$\epsilon + \frac{\delta}{3} < d(x_n, x_j) < \epsilon + \frac{2\delta}{3}.$$ 

Then

$$d(x_n, x_j) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{j+1}) + d(x_{j+1}, x_j)$$

$$\leq \frac{\delta}{6} + \epsilon + \frac{\delta}{6} = \epsilon + \frac{\delta}{3},$$

which is indeed a contradiction, therefore one may conclude that the sequence $\{x_n\}$ is Cauchy and it converges to a point $z$ in $X$.

Now, we assume that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which is contained in $P$. Using (2.1), one can write

$$H(Tx_{n_k-1}, Tz) \leq h \max \left\{ \frac{1}{2}d(x_{n_k-1}, z), d(x_{n_k-1}, Tx_{n_k-1}), d(z, Tz), \right\}$$

$$\frac{d(z, Tx_{n_k-1}) + d(x_{n_k-1}, Tz)}{q}$$

which on letting $k \to \infty$ we get $H(Tz, z) \leq hd(Tz, z)$, yielding thereby $z \in Tz$. This completes the proof. 

**Remark 2.2.** By setting $\delta(\epsilon) = \frac{2(1-h)\epsilon}{h}$, $0 < h < 1$ in the Theorem 2.1 then $\delta(\epsilon)$ is nondecreasing function of $\epsilon > 0$, one obtains

$$\epsilon' < \epsilon = \epsilon' + \frac{1}{2}\delta(\epsilon') < \epsilon' + \delta(\epsilon')$$

by choosing $\epsilon' = h\epsilon$. The condition (2.1) of Theorem 2.1 reduces to (1.2) due to Rhoades [11].

Finally, we furnish an example to discuss the validity of the hypotheses of Theorem 2.1 proved in this note which also establish the genuineness of our result.

**Example 2.3.** Let $X = \mathbb{R}$ with Euclidean metric and $K = [0, 16] \cup \{-4\}$. Define $T : K \to X$ as

$$Tx = \begin{cases} 
-4, & \text{if } 0 \leq x \leq 16 \\
1, & \text{if } x = -4.
\end{cases}$$
Since $\delta K (\text{boundary of } K) = \{-4, 0, 16\}$. Also $-4 \in \delta K \Rightarrow T(-4) = 1 \in K$, $0 \in \delta K \Rightarrow T0 = 0 \in K$, $16 \in \delta K \Rightarrow T16 = -4 \in K$. Moreover, if $0 \leq x, y \leq 16$, then
\[
d(Tx, Ty) = \frac{1}{4}|x - y| = \frac{1}{2} \left( \frac{1}{2}d(x, y) \right)
\leq h \max \left\{ \frac{1}{2}d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{q} \right\}.
\]

Next, if $x \in [0, 16]$ and $y = -4$ then
\[
d(Tx, Ty) = \frac{1}{4}|x + y| = \frac{1}{2} \left( \frac{1}{2}d(x, y) \right)
\leq h \max \left\{ \frac{1}{2}d(x, y), d(x, Tx), (y, Ty), \frac{d(x, Ty) + d(y, Tx)}{q} \right\},
\]
which shows that the contraction condition (2.1) is satisfied for every $x, y \in K$. Thus all the conditions of the Theorem 2.1 are satisfied and 0 is the fixed point of $T$.

Acknowledgement: Authors express their sincere thanks to the learned referee for their critical reading and his kind suggestions towards the improvement of entire manuscript.

References


(Received 23 February 2011)
( Accepted 13 January 2012)