Thai Journal of Mathematics
Volume 10 (2012) Number 3 : 661–683
http://thaijmath.in.cmu.ac.th
Online ISSN 1686-0209

Existence and Iterative Approximation of Solutions of a System of Random Variational Inclusions with Random Fuzzy Mappings

K.R. Kazmi[†,1], F.A. Khan[‡] and Naeem Ahmad[†,2]

[†]Department of Mathematics
Aligarh Muslim University, Aligarh-202002, India
e-mail : krkazmi@gmail.com (K.R. Kazmi)
nahmadamu@gmail.com (N. Ahmad)

[‡]Department of Mathematics, Faculty of Science
University of Tabuk, Tabuk-71491, Kingdom of Saudia Arabia
e-mail : faizan_math@yahoo.com

Abstract : In this paper, we introduce random P-monotone mapping and its associated proximal-point mapping in Hilbert space and discuss their some properties. Further, we consider a system of random variational inclusions with random fuzzy mappings in real Hilbert spaces. Using proximal-point mapping technique, we construct an iterative algorithm for the system of random variational inclusions. Furthermore, we prove the existence of solution of the system of random variational inclusions and discuss the convergence analysis of the iterative algorithm. The results presented in this paper generalize, improve and unify the known results of recent works [1–11].

Keywords : system of random variational inclusions; random fuzzy mappings; random P-monotone mappings; iterative algorithm; convergence analysis.
2010 Mathematics Subject Classification : 60D05.

---

[1]Corresponding author.
[2]Present address: Department of Mathematics, Al-Jouf University, P.O. Box 2014, Skaka, Kingdom of Saudia Arabia

1 Introduction

Variational inclusion problems, as the generalization of variational inequality problems, are among the most interesting and intensively studied classes of mathematical problems and have wide applications in the field of optimization and control, economics, transportation equilibrium and engineering sciences.

Variational inequalities (inclusions) are used as a mathematical tool in modeling many optimization and decision making problems. However, facing uncertainty is a constant challenge for optimization and decision making. The fuzzy set theory, introduced by Zadeh [12], is useful in treating uncertainty in the study of fuzzy optimization and decision making.

In 1989, Chang and Zhu [13] introduced the concept of variational inequalities for fuzzy mappings. Since then several classes of variational inequalities (inclusions) have been studied by many authors, see for example [4, 6, 7, 13–15]. It is well known that the study of random equations involving random mappings in view of their need in dealing with probabilistic models in applied sciences is very important. In 1999, Huang [6] introduced and studied a class of random variational inclusions with random fuzzy mappings in Hilbert spaces. For related work, see [1, 3, 16, 17]. Very recently, Wu and Zou [18] and Zhang [19] have studied some classes of variational inequalities (inclusions) with random fuzzy mappings, see also [2, 9, 20].

In 1985, Pang [21] showed that a variety of equilibrium models, for example, the traffic equilibrium problem, the spatial equilibrium problem, the Nash equilibrium problem and the general equilibrium programming problem can be uniformly modelled as a variational inequality defined on the product sets. He decomposed the original variational inequality into a system of variational inequalities and discussed the convergence of the method of decomposition for a system of variational inequalities. Later, it was noticed that variational inequality over product sets and the system of variational inequalities both are equivalent, see for applications [21–23]. Since then many authors, see for example [23–25] studied the existence theory of various classes of system of variational inequalities by exploiting fixed point theorems and minimax theorems. On the other hand, only a few iterative algorithms have been constructed for approximating the solution of the system of variational inequalities (inclusions) using the system of proximal-point methods, see for example [5, 10, 26–29]. One of the most important tasks is to construct an efficient method to solve variational inclusions. One of the such methods is the method based on proximal-point mapping. In recent past, the methods based on different classes of proximal-point mappings have been developed to study the existence of solutions and to discuss the convergence analysis of constructed iterative algorithms for various classes of variational inclusions, see for example [4, 7, 27, 30–48].

Very recently, Sun et al. [49] introduced the notion of M-proximal-point mapping and developed a method to solve the variational inequalities. Zou and Huang [50] and Kazmi et al. [43] extended the concept of M-proximal-point mappings and used these concepts in developing the iterative methods to solve the systems of variational inclusions.

Motivated and inspired by the recent research works in this area, we introduce random P-monotone mapping, and its associated proximal-point mapping in Hilbert space and discuss their some properties. Further, we consider a system of random variational inclusions with random fuzzy mappings (in short, SRVI) in real Hilbert spaces. Using proximal-point mapping technique, we construct an iterative algorithm for SRVI. Furthermore, we prove the existence of solution of SRVI and discuss the convergence analysis of the iterative algorithm. To the best of our knowledge, the work presented in this paper is the first attempt to study the system of random variational inclusions involving random fuzzy mappings. The results presented in this paper generalize, improve and unify the known results of recent works [1–11].

2 Preliminaries

Throughout the paper unless otherwise stated, let $I = \{1, 2\}$ be an index set and for each $i \in I$, let H_i be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle_i$ and $\|\cdot\|_i$, respectively and let H be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, respectively. We denote by (Ω, Σ) a measurable space, where Ω is a set and σ-algebra of subsets of Ω and by $\mathcal{B}(H), 2^H, CB(H)$ and $\mathcal{F}(H)$, the class of Borel σ-field in H, the family of all nonempty subsets of H, the family of all nonempty closed bounded subsets of H and the collection of a fuzzy sets over H, respectively. Let $N \in \mathcal{F}(H), q \in [0, 1]$, then the set $(N)_q = \{x \in H : N(x) \geq q\}$ is called a q-cut set of N.

The following definitions and concepts are needed in the sequel.

Definition 2.1. A mapping $x : \Omega \to H$ is said to be *measurable* if for any $B \in \mathcal{B}(H), \{t \in \Omega : x(t) \in B\} \in \Sigma$.

Definition 2.2. A mapping $f : \Omega \times H \to H$ is called a *random mapping* if for any $x \in H, f(t, x) = x(t)$ is measurable. A random mapping f is said to be continuous if for any $t \in \Omega$, the mapping $f(t, \cdot) : H \to H$ is continuous.

Similarly, we can define a random mapping $P : \Omega \times H \times H \to H$. It is well known that a measurable mapping is necessarily a random mapping.

Definition 2.3. A multi-valued mapping $T : \Omega \to 2^H$ is said to be *measurable* if for any $B \in \mathcal{B}(H), T^{-1}(B) = \{t \in \Omega : T(t) \cap B \neq \emptyset\} \in \Sigma$.

Definition 2.4. A mapping $u : \Omega \to H$ is called a *measurable selection of a multi-valued measurable mapping* $T : \Omega \to 2^H$ if u is measurable and for any $t \in \Omega$, $u(t) \in T(t)$.

Definition 2.5. A mapping $T : \Omega \times H \to 2^H$ is called a *random multi-valued mapping* if for any $x \in H, T(\cdot, x)$ is measurable. A random multi-valued mapping $T : \Omega \times H \to CB(H)$ is said to be $\mathcal{H}$-*continuous* if for any $t \in \Omega, T(t, \cdot)$ is continuous in $\mathcal{H}(\cdot,\cdot)$, where $\mathcal{H}(\cdot,\cdot)$ is the Hausdorff metric on $CB(H)$ defined as follows: for any given $A, B \in CB(H)$,

$$\mathcal{H}(A,B) = \max\left\{\sup_{x\in A}\inf_{y\in B} d(x,y),\ \sup_{y\in B}\inf_{x\in A} d(x,y)\right\}.$$

Definition 2.6. A fuzzy mapping $F : \Omega \to \mathcal{F}(H)$ is called *measurable* if for any $\alpha \in (0,1], (F(\cdot))_\alpha : \Omega \to 2^H$ is a measurable multi-valued mapping.

Definition 2.7. A fuzzy mapping $F : \Omega \times H \to \mathcal{F}(H)$ is called a *random fuzzy mapping*, if for any $x \in H, F(\cdot, x) : \Omega \to \mathcal{F}(H)$ is a measurable fuzzy mapping.

We note that, the random fuzzy mappings include multi-valued mappings, random multi-valued mappings and fuzzy mappings as special cases.

For each $i = 1, 2$, let $N_i : \Omega \times H_1 \to \mathcal{F}(H_1),\ S_i : \Omega \times H_2 \to \mathcal{F}(H_2),\ T_i : \Omega \times H_i \to \mathcal{F}(H_i)$ be random fuzzy mappings satisfying the following condition (C):

(C) There exist mappings $a_i : H_1 \to [0,1], b_i : H_2 \to [0,1], c_i : H_i \to [0,1]$ such that $N_i(t, x_1(t))_{a_i(x_1)} \in CB(H_1), S_i(t, x_2(t))_{b_i(x_2)} \in CB(H_2), T_i(t, x_i(t))_{c_i(x_i)} \in CB(H_i), \forall (t, x_1(t) \in \Omega \times H_1,\ (t, x_2(t)) \in \Omega \times H_2$.

By using the random fuzzy mappings N_i, S_i and T_i, we can define random multi-valued mappings $\tilde{N}_i,\ \tilde{S}_i$ and $\tilde{T}_i$, as follows:

$$\tilde{N}_i : \Omega \times H_1 \to CB(H_1),\ x_1 \to (N_i(t, x_1(t)))_{a_i(x_1)},\ \forall (t, x_1) \in \Omega \times H_1,$$

$$\tilde{S}_i : \Omega \times H_2 \to CB(H_2),\ x_2 \to (S_i(t, x_2(t)))_{b_i(x_2)},\ \forall (t, x_2) \in \Omega \times H_2,$$

and

$$\tilde{T}_i : \Omega \times H_i \to CB(H_i),\ x_i \to (T_i(t, x_i(t)))_{c_i(x_i)},\ \forall (t, x_i) \in \Omega \times H_i.$$

In the sequel, $\tilde{N}_i,\ \tilde{S}_i$ and $\tilde{T}_i$ are called the random multi-valued mappings induced by the random fuzzy mappings N_i, S_i and T_i, respectively.

Given mappings $a_i : H_1 \to [0,1],\ b_i : H_2 \to [0,1],\ c_i : H_i \to [0,1]$, random fuzzy mappings $N_i : \Omega\times H_1 \to \mathcal{F}(H_1),\ S_i : \Omega\times H_2 \to \mathcal{F}(H_2),\ T_i : \Omega\times H_i \to \mathcal{F}(H_i)$ and random mappings $f_i, g_i, h_i : \Omega \times H_i \to H_i,\ F_i : \Omega \times H_1 \times H_2 \to H_i\ M_i : \Omega \times H_i \to 2^{H_i}$ with range $g_i(t,\cdot) \cap \text{dom } (M_i(t,\cdot)) \neq \emptyset$, for $t \in \Omega$. We consider the following system of random variational inclusions (SRVI):

Find measurable mappings $x_1, u_1, u_2, w_1 : \Omega \to H_1;\ x_2, v_1, v_2, w_2 : \Omega \to H_2$ such that for all $t \in \Omega,\ x_1(t) \in H_1,\ x_2(t) \in H_2,\ N_i(t, x_1(t))(u_i(t)) \geq a_i(x_1(t)),\ S_i(t, x_2(t))(v_i(t)) \geq b_i(x_2(t)),\ T_i(t, x_i(t))(w_i(t)) \geq c_i(x_i(t))$ and range $g_i(t,\cdot) \cap \text{dom } (M_i(t,\cdot)) \neq \emptyset$, for $t \in \Omega$ such that

$$\Theta_1 \in F_1(t, u_1(t), v_1(t)) - \Big\{f_1(t, w_1(t)) - h_1(t, x_1(t))\Big\} + M_1(t, g_1(t, x_1(t))), \quad (2.1)$$

$$\Theta_2 \in F_2(t, u_2(t), v_2(t)) - \Big\{f_2(t, w_2(t)) - h_2(t, x_2(t))\Big\} + M_2(t, g_2(t, x_2(t))), \quad (2.2)$$

where Θ_1, Θ_2 are zero vectors of H_1, H_2, respectively. The set of measurable mappings $(x_1, x_2, u_1, u_1, v_1, v_2, w_1, w_2)$ is called a random solution of SRVI (2.1)-(2.2).

Special Cases:

(1) For each $i = 1, 2$, if $H \equiv H_i$; $F \equiv F_i$; $f \equiv f_i$; $h \equiv h_i$; $M \equiv M_i$; $N \equiv N_i$; $S \equiv S_i$ and $T \equiv T_i$ and $g \equiv g_i$, then SRVI (2.1)-(2.2) reduces to the problem of finding measurable mappings $x, u, w, v : \Omega \to H$ such that for $t \in \Omega, x(t) \in H$, $N(t, x_1(t))(u(t)) \geq a(x(t))$, $S(t, x(t))(v(t)) \geq b(x(t))$, $T(t, x(t))(w(t)) \geq c(x(t))$ and range $g(t, .) \cap \text{dom } (M(t, .)) \neq \emptyset$, for $t \in \Omega$ such that

$$\Theta \in F(t, u(t), v(t)) - \Big\{ f(t, w(t)) - h(t, x(t)) \Big\} + M(t, g(t, x(t))), \qquad (2.3)$$

where Θ is zero vector of H. A problem similar to (2.3) is studied by Ahmad and Farajzadeh [2].

(2) If $a(x) = b(x) = c(x) = 1$, $F(t, u(t), v(t)) = F(t, u(t))$, $u(t) = w(t) = x(t)$ and $f(t, w(t)) - h(t, x(t)) = v(t)$, then Problem (2.3) reduces to the problem of finding measurable mappings $x, v : \Omega \to H$ such that

$$\Theta \in F(t, x(t)) + v(t) + M(t, g(t, x(t))), \qquad (2.4)$$

where $g, F : \Omega \times H \to H$ are single-valued random mappings and $M : \Omega \times H \to 2^H$ is multi-valued random mapping. A problem similar to (2.4) is considered by Cho and Lan [16].

3 P-proximal-point Mappings

The following definitions and results are needed in the sequel.

Definition 3.1. A random mapping $g : \Omega \times H \to H$ is said to be *Lipschitz continuous* if there exists a measurable function $\lambda_g : \Omega \to (0, \infty)$ such that

$$\|g(t, x_1(t)) - g(t, x_2(t))\| \leq \lambda_g(t) \|x_1(t) - x_2(t)\|, \quad \forall t \in \Omega, x_1, (t), x_2(t) \in H.$$

Lemma 3.2 ([24]). *Let $T : \Omega \times H \to CB(H)$ be a $\mathcal{H}$-continuous random multi-valued mapping. Then for any measurable mapping $w : \Omega \to H$, the multi-valued mapping $T(., w(.)) : \Omega \to CB(H)$ is measurable.*

Lemma 3.3 ([24]). *Let $S, T : \Omega \to CB(H)$ be two measurable multi-valued mappings, $\epsilon > 0$ be a constant and let $v : \Omega \to H$ be a measurable selection of S. Then there exists a measurable selection $w : \Omega \to H$ of T such that for all $t \in \Omega$,*

$$\|v(t) - w(t)\| \leq (1 + \epsilon) \mathcal{H}(S(t), T(t)).$$

Definition 3.4. Let $A, B : \Omega \times H \to H$ be single-valued random mappings. A random mapping $P : \Omega \times H \times H \to H$ is said to be

(i) *α-strongly monotone with respect to A*, if there exists a measurable function $\alpha : \Omega \to (0, \infty)$ such that

$$\langle P(t, A(t, x(t)), z(t)) - P(t, A(t, y(t)), z(t)), x(t) - y(t)\rangle \geq \alpha(t)\|x(t) - y(t)\|^2,$$

$\forall t \in \Omega, x(t), y(t), z(t) \in H$;

(ii) *β-relaxed monotone with respect to B*, if there exists a measurable function $\beta : \Omega \to (0, \infty)$ such that

$$\langle P(t, z(t), B(t, x(t))) - P(t, z(t), B(t, y(t))), x(t) - y(t)\rangle \geq -\beta(t)\|x(t) - y(t)\|^2,$$

$\forall t \in \Omega, x(t), y(t), z(t) \in H$;

(iii) *$\alpha\beta$-symmetric monotone with respect to A and B*, if P is α-strongly monotone with respect to A and β-relaxed monotone with respect to B with $\alpha(t) > \beta(t)$ and $\alpha(t) = \beta(t)$ if and only if $x(t) = y(t)$ $\forall t \in \Omega, x(t), y(t) \in H$.

(iv) (ξ_1, ξ_2)*-mixed Lipschitz continuous* if there exist measurable functions $\xi_1, \xi_2 : \Omega \to (0, \infty)$ such that

$$\|P(t, x(t), z_1(t)) - P(t, y(t), z_2(t))\| \leq \xi_1(t)\|x(t) - y(t)\| + \xi_2(t)\|z_1(t) - z_2(t)\|,$$

$\forall t \in \Omega, x(t), y(t), z_1(t), z_2(t) \in H$.

Definition 3.5. A random multi-valued mapping $M : \Omega \times H \to 2^H$ is said to be

(i) *monotone*, if

$$\langle u(t) - v(t), x_1(t) - x_2(t)\rangle \geq 0,$$

$\forall t \in \Omega, x_1(t), x_2(t) \in H, u(t) \in M(t, x_1(t)), v(t) \in M(t, x_2(t))$;

(ii) *r-strongly monotone*, if there exists a measurable function $r : \Omega \to (0, \infty)$ such that

$$\langle u(t) - v(t), x_1(t) - x_2(t)\rangle \geq r(t)\|x_1(t) - x_2(t)\|^2,$$

$\forall t \in \Omega, x_1(t), x_2(t) \in H, u(t) \in M(t, x_1(t)), v(t) \in M(t, x_2(t))$;

(iii) *m-relaxed monotone*, if there exists a measurable function $m : \Omega \to (0, \infty)$ such that

$$\langle u(t) - v(t), x_1(t) - x_2(t)\rangle \geq -m(t)\|x_1(t) - x_2(t)\|^2, \forall x_1(t), x_2(t) \in H,$$

$\forall t \in \Omega, x_1(t), x_2(t) \in H, u(t) \in M(t, x_1(t)), v(t) \in M(t, x_2(t))$.

Lemma 3.6. *Let H be a Hilbert space. Then for any $x, y \in H$,*

$$\|x + y\|^2 \leq \|x\|^2 + \langle y, x + y\rangle.$$

Definition 3.7. Let $A, B : \Omega \times H \to H$, $P : \Omega \times H \times H \to H$ be single-valued random mappings. A random multi-valued mapping $M : \Omega \times H \to 2^H$ is said to be *P-monotone* if

(a) M is m-relaxed monotone;

(b) $(P_t(A_t, B_t) + \rho(t)M_t)(H) = H, \ \forall t \in \Omega,$

where $A_t(x) = A(t, x(t)), B_t(x) = B(t, x(t)), P_t(A_t, B_t)(x) = P(t, A(t, x(t)), B(t, x(t)))$ for $\rho(t) > 0$, is a real valued random variable.

The following theorem gives some properties of random P-monotone mappings.

Theorem 3.8. *Let $A, B : \Omega\times H \to H$ be random mappings, let $P : \Omega\times H\times H \to H$ be a random $\alpha\beta$-symmetric monotone mapping, and let $M : \Omega \times H \to 2^H$ be a random P-monotone multi-valued mapping. Then*

(a) $\langle u(t)-v(t), x(t)-y(t)\rangle \geq 0, \forall (v(t), y(t)) \in \mathrm{Graph}(M), t \in \Omega$ implies $(u(t), v(t)) \in \mathrm{Graph}(M)$, where $\mathrm{Graph}(M) := \{(u(t), x(t)) \in H \times H : u(t) \in M_t(x)\}$;

(b) the mapping $(P_t(A_t, B_t) + \rho(t)M_t)^{-1}$ is single-valued for all $\rho(t) > 0$, real valued random variables, such that $\rho(t) \in (0, \frac{\alpha(t)-\beta(t)}{m(t)}), t \in \Omega$.

Proof. (a): Suppose on contrary that, there exists $(u_0(t), x_0(t)) \notin \mathrm{Graph}(M)$, such that

$$\langle u_0(t) - v(t), x_0(t) - y(t)\rangle \geq 0, \forall (v(t), y(t)) \in \mathrm{Graph}(M), t \in \Omega. \tag{3.1}$$

Since M is P-monotone, we have $(P_t(A_t, B_t) + \rho(t)M_t)(H) = H$, and hence there exists $(u_1(t), x_1(t)) \in \mathrm{Graph}(M)$ such that

$$P_t(A_t, B_t)x_1 + \rho(t)u_1(t) = P_t(A_t, B_t)x_0 + \rho(t)u_0(t). \tag{3.2}$$

Now set $(v(t), y(t)) = (u_1(t), x_1(t))$ in (3.1) and then, from the resultant inequality (3.2) and from the fact that $\rho(t) > 0$, we obtain

$$\begin{aligned} 0 &\leq \rho(t)\langle u_0(t) - u_1(t), x_0(t) - x_1(t)\rangle \\ &= \langle P(t, A(t, x_1(t)), B(t, x_1(t))) - P(t, A(t, x_0(t)), B(t, x_1(t))), x_0(t) - x_1(t)\rangle \\ &\quad + \langle P(t, A(t, x_0(t)), B(t, x_1(t))) - P(t, A(t, x_0(t)), B(t, x_0(t))), x_0(t) - x_1(t)\rangle \end{aligned}$$

which implies that

$$\langle P(t, A(t, x_0(t)), B(t, x_1(t)))-P(t, A(t, x_1(t)), B(t, x_1(t))), x_0(t)-x_1(t)\rangle$$

$$+\langle P(t, A(t, x_0(t)), B(t, x_0(t))) - P(t, A(t, x_0(t)), B(t, x_1(t))), x_0(t) - x_1(t)\rangle \leq 0$$

or

$$(\alpha(t) - \beta(t))\|x_0(t) - x_1(t)\|^2 \leq 0,$$

where P is $\alpha\beta$-symmetric monotone with respect to A and B, so we have $x_0(t) = x_1(t), \forall t \in \Omega$ and hence from (3.2), we have $u_1(t) = u_0(t), t \in \Omega$, a contradiction. This completes the proof (a).

(b): For any given $z(t) \in H$, let $x(t), y(t) \in (P_t(A_t, B_t)+\rho(t)M_t)^{-1}z$. It follows that

$$\frac{1}{\rho(t)}(z(t) - P_t(A_t, B_t)x) \in M_t(x) \text{ and } \frac{1}{\rho(t)}(z(t) - P_t(A_t, B_t)y) \in M_t(y).$$

Since M is m-relaxed monotone and P is $\alpha\beta$-symmetric monotone, it implies that

$$\begin{aligned}
&-m(t)\|x(t) - y(t)\|^2\\
&\quad= \frac{1}{\rho(t)}\langle P(t, A(t, y(t)), B(t, y(t))) - P(t, A(t, x(t)), B(t, x(t))), x(t) - y(t)\rangle\\
&\quad= -\frac{1}{\rho(t)}\Big[\langle P(t, A(t, x(t)), B(t, x(t))) - P(t, A(t, y(t)), B(t, x(t))), x(t) - y(t)\rangle\\
&\qquad\quad + \langle P(t, A(t, y(t)), B(t, x(t))) - P(t, A(t, y(t)), B(t, y(t))), x(t) - y(t)\rangle\Big]\\
&\quad\leq -\frac{1}{\rho(t)}\Big[(\alpha(t) - \beta(t))\|x(t) - y(t)\|^2\Big],
\end{aligned}$$

i.e.

$$\Big[(\alpha(t) - \beta(t)) - \rho(t)m(t)\Big]\|x(t) - y(t)\|^2 \leq 0.$$

This implies that $x(t) = y(t), \forall t \in \Omega$. Thus $(P_t(A_t, B_t) + \rho(t)M_t)^{-1}$ is a single-valued mapping, $\forall t \in \Omega$. □

By Theorem 3.8, we can define the following random proximal-point mapping $J_{M_t}^{\rho(t),P_t(A_t,B_t)}$.

Definition 3.9. Let $A, B : \Omega \times H \to H$ be single-valued random mappings and $P : \Omega\times H\times H \to H$ be a random $\alpha\beta$-symmetric monotone mapping with respect to A and B. Let $M : \Omega \times H \to 2^H$ be a random multi-valued P-monotone mapping. Then proximal-point mapping $J_{M(\cdot,\cdot)}^{\rho(\cdot),P(\cdot,A(\cdot,\cdot),B(\cdot,\cdot))} : \Omega \times H \to H$ associated with P and M is defined by

$$\begin{aligned}
J_{M(\cdot,\cdot)}^{\rho(\cdot),P(\cdot,A(\cdot,\cdot),B(\cdot,\cdot))}(t, x(t)) &= J_{M(t,x)}^{\rho(t),P(t,A(t,\cdot),B(t,\cdot))}(x(t)) = J_{M_t}^{\rho(t),P_t(A_t,B_t)}(x)\\
&= (P_t(A_t, B_t) + \rho(t)M_t)^{-1}(x), \qquad (3.3)
\end{aligned}$$

where $\rho(t) > 0$ is real valued random variable and $A_t(x) = A(t, x(t)), B_t(x) = B(t, x(t))$,

$$M_t(x) = M(t, x(t)), P_t(A_t, B_t)(x) = P(t, A(t, x(t)), B(t, x(t))), \forall t \in \Omega, x(t) \in H.$$

Next, we prove that P-proximal-point mapping is Lipschitz continuous.

Theorem 3.10. *Let $A, B : \Omega \times H \to H$ be single-valued random mappings and $P : \Omega\times H\times H \to H$ be a random $\alpha\beta$-symmetric monotone mapping with respect to A and B. Let $M : \Omega \times H \to 2^H$ be a random multi-valued P-monotone mapping.*

Then proximal-point mapping $J_{M_t}^{\rho(t),P_t(A_t,B_t)}: H \to H$ *is* $\frac{1}{[(\alpha(t)-\beta(t))-\rho(t)m(t)]}$-*Lipschitz continuous*

$$\left\| J_{M_t}^{\rho(t),P_t(A_t,B_t)}(x^*) - J_{M_t}^{\rho(t),P_t(A_t,B_t)}(y^*) \right\| \leq \frac{1}{[(\alpha(t)-\beta(t))-\rho(t)m(t)]}\|x^*(t)-y^*(t)\|, \tag{3.4}$$

where $\rho(t) \in (0, \frac{\alpha(t)-\beta(t)}{m(t)}),\ \forall t \in \Omega, x^*(t), y^*(t) \in H$.

Proof. Let $x^*(t)$ and $y^*(t)$ be given points in H. It follows from (3.3) that

$$\frac{1}{\rho(t)}\Big[x^*(t) - P_t(A_t,B_t)\Big(J_{M_t}^{\rho(t),P_t(A_t,B_t)}(x^*)\Big)\Big] \in M_t\Big(J_{M_t}^{\rho(t),P_t(A_t,B_t)}(x^*)\Big), \tag{3.5}$$

$$\frac{1}{\rho(t)}\Big[y^*(t) - P_t(A_t,B_t)\Big(J_{M_t}^{\rho(t),P_t(A_t,B_t)}(y^*)\Big)\Big] \in M_t\Big(J_{M_t}^{\rho(t),P_t(A_t,B_t)}(y^*)\Big). \tag{3.6}$$

Since M is m-relaxed monotone, we get

$$\begin{aligned}
&- m(t)\Big\|J_{M_t}^{\rho(t),P_t(A_t,B_t)}(x^*) - J_{M_t}^{\rho(t),P_t(A_t,B_t)}(y^*)\Big\|^2 \\
&\quad\leq \frac{1}{\rho(t)}\Big\langle x^*(t) - P_t(A_t,B_t)\Big(J_{M_t}^{\rho(t),P_t(A_t,B_t)}(x^*)\Big) \\
&\qquad - \Big(y^*(t) - P_t(A_t,B_t)\Big(J_{M_t}^{\rho(t),P_t(A_t,B_t)}(y^*)\Big)\Big), \\
&\qquad J_{M_t}^{\rho(t),P_t(A_t,B_t)}(x^*) - J_{M_t}^{\rho(t),P_t(A_t,B_t)}(y^*)\Big\rangle \\
&\quad= \frac{1}{\rho(t)}\Big\langle x^*(t) - y^*(t) - \Big(P_t(A_t,B_t)\Big(J_{M_t}^{\rho(t),P_t(A_t,B_t)}(x^*)\Big) \\
&\qquad - P_t(A_t,B_t)\Big(J_{M_t}^{\rho(t),P_t(A_t,B_t)}(y^*)\Big)\Big), \\
&\qquad J_{M_t}^{\rho(t),P_t(A_t,B_t)}(x^*) - J_{M_t}^{\rho(t),P_t(A_t,B_t)}(y^*)\Big\rangle.
\end{aligned}$$

Therefore,

$$\left\| J_{M_t}^{\rho(t),P_t(A_t,B_t)}(x^*) - J_{M_t}^{\rho(t),P_t(A_t,B_t)}(y^*) \right\| \leq \frac{1}{[(\alpha(t)-\beta(t))-\rho(t)m(t)]}\|x^*(t)-y^*(t)\|. \tag{3.7}$$

This completes the proof. □

We remark that the concepts and results presented in this section generalize the concepts and related results given in [49] and the relevant references cited therein.

4 Random Iterative Algorithm

Definition 4.1. A random multi-valued mapping $T : \Omega \times H \to CB(H)$ is said to be $\mathcal{H}$-*Lipschitz continuous* if there exists a measurable function $\lambda_T : \Omega \to (0, \infty)$ such that

$$\mathcal{H}(T(t,x_1(t)) - T(t,x_2(t))) \leq \lambda_T(t)\|x_1(t) - x_2(t)\|, \quad \forall t \in \Omega, x_1(t), x_2(t) \in H.$$

First, we give the following technical lemma.

Lemma 4.2. *For each $i = 1, 2$, let $M_i : \Omega \times H_i \to 2^{H_i}$ be random multi-valued P_i-monotone mapping and let $A_i : \Omega\times H_1 \to H_1,\ B_i : \Omega\times H_2 \to H_2$ random mappings. The set of measurable mappings $(x_1, x_2, u_1, u_2, v_1, v_2, w_1, w_2)$ is a random solution of SRVI (2.1)-(2.2) if and only if for all $(t, x_1(t), x_2(t)) \in \Omega \times H_1 \times H_2,\ u_i(t) \in \tilde{N}_i(t, x_1(t)),\ v_i(t) \in \tilde{S}_i(t, x_2(t)),\ w_i(t) \in \tilde{T}_i(t, x_i(t))$ satisfy*

$$g_1(t, x_1(t)) = J^{\rho_1(t),P_{1t}(A_{1t},B_{1t})}_{M_{1t}}\Big[P_{1t}(A_{1t}, B_{1t})(g_{1t}(x_1)) - \rho_1(t)\Big\{F_1(t, u_1(t), v_1(t)) - \Big(f_1(t, w_1(t)) - h_1(t, x_1(t))\Big)\Big\}\Big], \tag{4.1}$$

$$g_2(t, x_2(t)) = J^{\rho_2(t),P_{2t}(A_{2t},B_{2t})}_{M_{2t}}\Big[P_{2t}(A_{2t}, B_{2t})(g_{2t}(x_2)) - \rho_2(t)\Big\{F_2(t, u_2(t), v_2(t)) - \Big(f_2(t, w_2(t)) - h_2(t, x_2(t))\Big)\Big\}\Big], \tag{4.2}$$

where $\rho_i : \Omega \to (0, \infty)$ is a measurable function.

Proof. The proof directly follows from the definition of $J^{\rho_i(t),P_{it}(A_{it},B_{it})}_{M_{it}}$ for $i = 1, 2$. □

Based on Lemma 4.2, we construct the following iterative algorithm for finding the approximate solution of SRVI (2.1)-(2.2).

Iterative Algorithm 4.1. For each $i = 1, 2$, let $N_i : \Omega \times H_1 \to \mathcal{F}(H_1),\ S_i : \Omega \times H_2 \to \mathcal{F}(H_2),\ T_i : \Omega \times H_i \to \mathcal{F}(H_i)$ be random fuzzy mappings satisfying the condition (C). Let $\tilde{N}_i : \Omega \times H_1 \to CB(H_1),\ \tilde{S}_i : \Omega \times H_2 \to CB(H_2),\ \tilde{T}_i : \Omega \times H_i \to CB(H_i)$ be $\mathcal{H}$-continuous random multi-valued mappings induced by N_i, S_i and T_i, respectively. Let $f_i, g_i, h_i : \Omega\times H_i \to H_i$ be single-valued random mappings and $M_i : \Omega \times H_i \to 2^{H_i}$ be multi-valued random mapping such that for each fixed $t \in \Omega,\ M_i(t, \cdot) : H_i \to 2^{H_i}$ is P_i-monotone mapping with $g_i(t, H)\cap\mathrm{dom}(M_i(t, H)) \neq \emptyset$. Let $F_i : \Omega \times H_i \times H_i \to H_i$ be a random mapping.

For each $(x_1, x_2) \in H_1 \times H_2$, let $Q_1(t, x_1(t), x_2(t)) \subseteq g_1(\Omega, H_1)$ and $Q_2(t, x_1(t), x_2(t)) \subseteq g_2(\Omega, H_2)$, where $Q_1 : \Omega \times H_1 \times H_2 \to 2^{H_1},\ Q_2 : \Omega \times H_1 \times H_2 \to 2^{H_2}$ be multi-valued mappings defined by

$$\begin{aligned} &Q_1(t, x_1(t), x_2(t)) \\ &= \bigcup_{u_1(t)\in\tilde{N}_1(t,x_1(t))} \bigcup_{v_1(t)\in\tilde{S}_1(t,x_2(t))} \bigcup_{w_1(t)\in\tilde{T}_1(t,x_1(t))} J^{\rho_1(t),P_{1t}(A_{1t},B_{1t})}_{M_{1t}}\Big[P_{1t}(A_{1t}, B_{1t})(g_{1t}(x_1)) \\ &\quad - \rho_1(t)\Big\{F_1(t, u_1(t), v_1(t)) - \Big(f_1(t, w_1(t)) - h_1(t, x_1(t))\Big)\Big\}\Big], \end{aligned} \tag{4.3}$$

$$\begin{aligned} &Q_2(t, x_1(t), x_2(t)) \\ &= \bigcup_{u_2(t)\in\tilde{N}_2(t,x_1(t))} \bigcup_{v_2(t)\in\tilde{S}_2(t,x_2(t))} \bigcup_{w_2(t)\in\tilde{T}_2(t,x_2(t))} J^{\rho_2(t),P_{2t}(A_{2t},B_{2t})}_{M_{2t}}\Big[P_{2t}(A_{2t}, B_{2t})(g_{2t}(x_2)) \\ &\quad - \rho_2(t)\Big\{F_2(t, u_2(t), v_2(t)) - \Big(f_2(t, w_2(t)) - h_2(t, x_2(t))\Big)\Big\}\Big], \end{aligned} \tag{4.4}$$

where $\rho_i(t)$ is same as in Lemma 4.2.

Let, for any given measurable mappings $x_i^0 : \Omega \to H_i,\ (i = 1, 2)$, the multi-valued random mappings $\tilde{N}_i(\cdot, x_1^0(\cdot)) : \Omega \to CB(H_1), \tilde{S}_i(\cdot, x_2^0(\cdot)) : \Omega \to CB(H_2)$, $\tilde{T}_i(\cdot, x_i^0(\cdot)) : \Omega \to CB(H_i)$ are measurable by Lemma 3.2. Hence, there exist measurable selections $u_i^0 : \Omega \to H_1$ of $\tilde{N}_i(\cdot, x_1^0(\cdot))$, $v_i^0 : \Omega \to H_2$ of $\tilde{S}_i(\cdot, x_2^0(\cdot))$ and $w_i^0 : \Omega \to H_i$ of $\tilde{T}_i(\cdot, x_i^0(\cdot))$, by Himmelberg [51]. Let

$$a_0 = J_{M_{1t}}^{\rho_1(t), P_{1t}(A_{1t}, B_{1t})}\Big[P_{1t}(A_{1t}, B_{1t})(g_{1t}(x_1^0)) - \rho_1(t)\Big\{F_1(t, u_1^0(t), v_1^0(t)) - \Big(f_1(t, w_1^0(t)) - h_1(t, x_1^0(t))\Big)\Big\}\Big] \in Q_1(t, x_1^0(t), x_2^0(t)) \subseteq g_1(\Omega, H_1), \quad (4.5)$$

$$b_0 = J_{M_{2t}}^{\rho_2(t), P_{2t}(A_{2t}, B_{2t})}\Big[P_{2t}(A_{2t}, B_{2t})(g_{2t}(x_1^0)) - \rho_2(t)\Big\{F_2(t, u_2^0(t), v_2^0(t)) - \Big(f_2(t, w_2^0(t)) - h_2(t, x_2^0(t))\Big)\Big\}\Big] \in Q_2(t, x_1^0(t), x_2^0(t)) \subseteq g_2(\Omega, H_2). \quad (4.6)$$

Hence there exists $(t, x_1^1(t)) \in \Omega \times H_1$ such that $a_0 = g_1(t, x_1^1(t))$ and $(t, x_2^1(t)) \in \Omega \times H_2$ such that $b_0 = g_2(t, x_2^1(t))$, and we observe that, for each $i = 1, 2, x_i^1 : \Omega \to H_i$ is measurable. Further, by Lemma 3.3, there exist measurable selections $u_i^1 : \Omega \to H_1$ of $\tilde{N}_i(\cdot, x_1^1(\cdot))$, $v_i^1 : \Omega \to H_2$ of $\tilde{S}_i(\cdot, x_2^1(\cdot))$ and $w_i^1 : \Omega \to H_i$ of $\tilde{T}_i(\cdot, x_i^1(\cdot))$ such that $\forall t \in \Omega$,

$$\begin{aligned}
\|u_1^0(t) - u_1^1(t)\|_1 &\leq (1+1)\ \mathcal{H}_1\Big(\tilde{N}_1(t, x_1^0(t)), \tilde{N}_1(t, x_1^1(t))\Big),\\
\|u_2^0(t) - u_2^1(t)\|_1 &\leq (1+1)\ \mathcal{H}_1\Big(\tilde{N}_2(t, x_1^0(t)), \tilde{N}_2(t, x_1^1(t))\Big),\\
\|v_1^0(t) - v_1^1(t)\|_2 &\leq (1+1)\ \mathcal{H}_2\Big(\tilde{S}_1(t, x_2^0(t)), \tilde{S}_1(t, x_2^1(t))\Big),\\
\|v_2^0(t) - v_2^1(t)\|_2 &\leq (1+1)\ \mathcal{H}_2\Big(\tilde{S}_2(t, x_2^0(t)), \tilde{S}_2(t, x_2^1(t))\Big),\\
\|w_1^0(t) - w_1^1(t)\|_1 &\leq (1+1)\ \mathcal{H}_1\Big(\tilde{T}_1(t, x_1^0(t)), \tilde{T}_1(t, x_1^1(t))\Big),\\
\|w_2^0(t) - w_2^1(t)\|_2 &\leq (1+1)\ \mathcal{H}_2\Big(\tilde{T}_2(t, x_2^0(t)), \tilde{T}_2(t, x_2^1(t))\Big). \quad (4.7)
\end{aligned}$$

Let

$$a_1 = J_{M_{1t}}^{\rho_1(t), P_{1t}(A_{1t}, B_{1t})}\Big[P_{1t}(A_{1t}, B_{1t})(g_{1t}(x_1^1)) - \rho_1(t)\Big\{F_1(t, u_1^1(t), v_1^1(t)) - \Big(f_1(t, w_1^1(t)) - h_1(t, x_1^1(t))\Big)\Big\}\Big] \in Q_1(t, x_1^1(t), x_2^1(t)) \subseteq g_1(\Omega, H_1), \quad (4.8)$$

$$b_1 = J_{M_{2t}}^{\rho_2(t), P_{2t}(A_{2t}, B_{2t})}\Big[P_{2t}(A_{2t}, B_{2t})(g_{2t}(x_1^1)) - \rho_2(t)\Big\{F_2(t, u_2^1(t), v_2^1(t)) - \Big(f_2(t, w_2^1(t)) - h_2(t, x_2^1(t))\Big)\Big\}\Big] \in Q_2(t, x_1^1(t), x_2^1(t)) \subseteq g_2(\Omega, H_2). \quad (4.9)$$

Hence there exist $(t, x_1^2(t)) \in \Omega \times H_1$ such that $a_1 = g_1(t, x_1^2(t))$ and $(t, x_2^2(t)) \in \Omega \times H_2$ such that $b_1 = g_2(t, x_2^2(t))$. It is easy to observe that $x_i^2 : \Omega \to H_i$ is measurable. Continuing the above process, we can define the following random iterative sequences $\{x_1^n(t)\}, \{x_2^n(t)\}, \{u_1^n(t)\}, \{u_2^n(t)\}, \{v_1^n(t)\}, \{v_2^n(t)\}$, $\{w_1^n(t)\}$ and $\{w_2^n(t)\}$ for solving SRVI (2.1)-(2.2) as follows:

$$g_1\left(t, x_1^{n+1}(t)\right) = J_{M_{1t}}^{\rho_1(t), P_{1t}(A_{1t}, B_{1t})}\Big[P_{1t}(A_{1t}, B_{1t})(g_{1t}(x_1^n)) - \rho_1(t)\Big\{F_1(t, u_1^n(t), v_1^n(t)) - \Big(f_1(t, w_1^n(t)) - h_1(t, x_1^n(t))\Big)\Big\}\Big], \tag{4.10}$$

$$g_2\left(t, x_2^{n+1}(t)\right) = J_{M_{2t}}^{\rho_2(t), P_{2t}(A_{2t}, B_{2t})}\Big[P_{2t}(A_{2t}, B_{2t})(g_{2t}(x_1^n)) - \rho_2(t)\Big\{F_2(t, u_2^n(t), v_2^n(t)) - \Big(f_2(t, w_2^n(t)) - h_2(t, x_2^n(t))\Big)\Big\}\Big], \tag{4.11}$$

$$\begin{aligned}
\|u_1^n(t) - u_1^{n+1}(t)\|_1 &\leq (1 + (1+n)^{-1})\ \mathcal{H}_1\Big(\tilde{N}_1(t, x_1^n(t)), \tilde{N}_1(t, x_1^{n+1}(t))\Big),\\
\|u_2^n(t) - u_2^{n+1}(t)\|_1 &\leq (1 + (1+n)^{-1})\ \mathcal{H}_1\Big(\tilde{N}_2(t, x_1^n(t)), \tilde{N}_2(t, x_1^{n+1}(t))\Big),\\
\|v_1^n(t) - v_1^{n+1}(t)\|_2 &\leq (1 + (1+n)^{-1})\ \mathcal{H}_2\Big(\tilde{S}_1(t, x_2^n(t)), \tilde{S}_1(t, x_2^{n+1}(t))\Big),\\
\|v_2^n(t) - v_2^{n+1}(t)\|_2 &\leq (1 + (1+n)^{-1})\ \mathcal{H}_2\Big(\tilde{S}_2(t, x_2^n(t)), \tilde{S}_2(t, x_2^{n+1}(t))\Big),\\
\|w_1^n(t) - w_1^{n+1}(t)\|_1 &\leq (1 + (1+n)^{-1})\ \mathcal{H}_1\Big(\tilde{T}_1(t, x_1^n(t)), \tilde{T}_1(t, x_1^{n+1}(t))\Big),\\
\|w_2^n(t) - w_2^{n+1}(t)\|_2 &\leq (1 + (1+n)^{-1})\ \mathcal{H}_2\Big(\tilde{T}_2(t, x_2^n(t)), \tilde{T}_2(t, x_2^{n+1}(t))\Big).
\end{aligned} \tag{4.12}$$

where $n = 0, 1, 2, \ldots$, and $\rho_i(t)$ is given as in Lemma 4.1.

5 Convergence Analysis

Theorem 5.1. *For each $i = 1, 2$, let $N_i : \Omega \times H_1 \to \mathcal{F}(H_1)$, $S_i : \Omega \times H_2 \to \mathcal{F}(H_2)$, $T_i : \Omega \times H_i \to \mathcal{F}(H_i)$ be random fuzzy mappings satisfying the condition (C). Let $\tilde{N}_i : \Omega \times H_1 \to CB(H_1)$ be $\lambda_{N_i}(t)$-$\mathcal{H}_1$-Lipschitz continuous random multi-valued mapping induced by N_i; let $\tilde{S}_i : \Omega \times H_2 \to CB(H_2)$ be $\lambda_{S_i}(t)$-$\mathcal{H}_2$-Lipschitz continuous random multi-valued mapping induced by S_i and $\tilde{T}_i : \Omega \times H_i \to CB(H_i)$ be $\lambda_{T_i}(t) - \mathcal{H}_i$-Lipschitz continuous random multi-valued mappings induced by T_i. Let $f_i, g_i, h_i : \Omega \times H_i \to H_i$ be single-valued random mappings where f_i is λ_{f_i} Lipschitz continuous, g_i is (d_i, e_i)-relaxed cocoercive mappings and h_i is λ_{h_i}-Lipschitz continuous. Let $F_i : \Omega \times H_i \times H_i \to H_i$ is random (μ_i, η_i)-mixed Lipschitz continuous. Let $M_i : \Omega \times H_i \to 2^{H_i}$ be multi-valued random mappings such that for each fixed $t \in \Omega$, $M_i(t, \cdot) : H_i \to 2^{H_i}$ is P_i-monotone mappings with $g_i(t, H) \cap \mathrm{dom}(M_i(t, H)) \neq \emptyset$. Suppose that, there are measurable functions*

$\xi_i : \Omega \to (0,1)$ *with the assumption*

$$\|P_{1t}(A_{1t},B_{1t})(g_{1t}(x_1^n)) - P_{1t}(A_{1t},B_{1t})(g_{1t}(x_1^{n-1}))\|_1 \le \xi_1(t)\|x_1^n(t) - x_1^{n-1}(t)\|_1, \tag{5.1}$$

$$\|P_{2t}(A_{2t},B_{2t})(g_{2t}(x_2^n)) - P_{2t}(A_{2t},B_{2t})(g_{2t}(x_2^{n-1}))\|_2 \le \xi_2(t)\|x_2^n(t) - x_2^{n-1}(t)\|_2. \tag{5.2}$$

If the following conditions hold:

$$\begin{aligned}\theta(t) := \max\Big\{&L_1(t)\Big(\xi_1(t) + \rho_1(t)\Big[\lambda_{h_1}(t) + \Big(\mu_1(t)\lambda_{N_1}(t) + \lambda_{f_1}(t)\lambda_{T_1}(t)\Big)\Big]\Big)\\ &+ L_2(t)\mu_2(t)\lambda_{N_2}(t), L_2(t)\Big(\xi_2(t) + \rho_2(t)\Big[\lambda_{h_2}(t) + \Big(\eta_2(t)\lambda_{S_2}(t)\\ &+ \lambda_{f_2}(t)\lambda_{T_2}(t)\Big)\Big]\Big) + L_1(t)\eta_1(t)\lambda_{S_1}(t)\Big\} < 1,\end{aligned}$$

and

$$\begin{aligned}L_1(t) &:= \frac{1}{[\alpha_1(t) - \beta_1(t) - \rho_1(t)m_1(t)]}\sqrt{\frac{1+2d_1(t)}{2e_1(t)+3}},\\ L_2(t) &:= \frac{1}{[\alpha_2(t) - \beta_2(t) - \rho_2(t)m_2(t)]}\sqrt{\frac{1+2d_2(t)}{2e_2(t)+3}}.\end{aligned} \tag{5.3}$$

Then there exist measurable mappings $(x_1, x_2, u_1, u_1, v_1, v_2, w_1, w_2)$ such that SRVI (2.1)-(2.2) hold. Moreover, $u_1^n(t) \to u_1(t)$, $u_2^n(t) \to u_2(t)$, $v_1^n(t) \to v_1(t)$, $v_2^n(t) \to v_2(t)$, $w_1^n(t) \to w_1(t)$ and $w_2^n(t) \to w_2(t)$ as $n \to \infty$, where $\{u_1^n(t)\}, \{u_2^n(t)\}, \{v_1^n(t)\}, \{v_2^n(t)\}, \{w_1^n(t)\}, \{w_2^n(t)\}$ are the random sequences obtained by Iterative Algorithm 4.1.

Proof. Since g_i is (d_i, e_i)-relaxed cocoercive, for $i = 1, 2$, by using Lemma 3.2, we have the following estimate:

$$\begin{aligned}&\|x_1^{n+1}(t) - x_1^n(t)\|_1\\ &= \|g_1(t,x_1^{n+1}(t)) - g_1(t,x_1^n(t)) + x_1^{n+1}(t) - x_1^n(t) - (g_1(t,x_1^{n+1}(t)) - g_1(t,x_1^n(t)))\|_1\\ &\le \|g_1(t,x_1^{n+1}(t)) - g_1(t,x_1^n(t))\|_1^2\\ &\qquad - 2\langle g_1(t,x_1^{n+1}(t)) - g_1(t,x_1^n(t)) - x_1^{n+1}(t) + x_1^n(t), x_1^{n+1}(t) - x_1^n(t)\rangle_1\\ &\le (1+2d_1(t))\|g_1(t,x_1^{n+1}(t)) - g_1(t,x_1^n(t))\|_1^2 - (2+2e_1(t))\|x_1^{n+1}(t) - x_1^n(t)\|_1^2,\end{aligned}$$

and

$$\begin{aligned}&\|x_2^{n+1}(t) - x_2^n(t)\|_2\\ &= \|g_2(t,x_2^{n+1}(t)) - g_2(t,x_2^n(t)) + x_2^{n+1}(t) - x_2^n(t) - (g_2(t,x_2^{n+1}(t)) - g_2(t,x_1^n(t)))\|_2\\ &\le \|g_2(t,x_2^{n+1}(t)) - g_2(t,x_2^n(t))\|_2^2\\ &\qquad - 2\langle g_2(t,x_2^{n+1}(t)) - g_2(t,x_2^n(t)) - x_2^{n+1}(t) + x_2^n(t), x_2^{n+1}(t) - x_2^n(t)\rangle_2\\ &\le (1+2d_2(t))\|g_2(t,x_2^{n+1}(t)) - g_2(t,x_2^n(t))\|_2^2 - (2+2e_2(t))\|x_2^{n+1}(t) - x_2^n(t)\|_2^2,\end{aligned}$$

which implies that

$$\|x_1^{n+1}(t) - x_1^n(t)\|_1 \leq \sqrt{\frac{1+2d_1(t)}{2e_1(t)+3}}\ \|g_1(t, x_1^{n+1}(t)) - g_1(t, x_1^n(t))\|_1, \tag{5.4}$$

$$\|x_2^{n+1}(t) - x_2^n(t)\|_2 \leq \sqrt{\frac{1+2d_2(t)}{2e_2(t)+3}}\ \|g_2(t, x_2^{n+1}(t)) - g_2(t, x_2^n(t))\|_2. \tag{5.5}$$

Now by using Theorem 3.10 and Iterative Algorithm 4.1, we have

$$\begin{aligned}
&\|g_1(t, x_1^{n+1}(t)) - g_1(t, x_1^n(t))\|_1 \\
&= \|J_{M_{1t}}^{\rho_1(t), P_{1t}(A_{1t}, B_{1t})}[P_{1t}(A_{1t}, B_{1t})(g_{1t}(x_1^n)) - \rho_1(t)\{F_1(t, u_1^n(t), v_1^n(t)) \\
&\quad - (f_1(t, w_1^n(t)) - h_1(t, x_1^n(t)))\}] - J_{M_{1t}}^{\rho_1(t), P_{1t}(A_{1t}, B_{1t})}[P_{1t}(A_{1t}, B_{1t})(g_{1t}(x_1^{n-1})) \\
&\quad - \rho_1(t)\{F_1(t, u_1^{n-1}(t), v_1^{n-1}(t)) - (f_1(t, w_1^{n-1}(t)) - h_1(t, x_1^{n-1}(t)))\}]\|_1 \\
&\leq \frac{1}{[\alpha_1(t) - \beta_1(t) - \rho_1(t)m_1(t)]}\left[\|P_{1t}(A_{1t}, B_{1t})(g_{1t}(x_1^n)) - P_{1t}(A_{1t}, B_{1t})(g_{1t}(x_1^{n-1}))\|_1\right. \\
&\quad + \rho_1(t)\|F_1(t, u_1^n(t), v_1^n(t)) - F_1(t, u_1^{n-1}(t), v_1^{n-1}(t))\|_1 \\
&\quad \left. +\rho_1(t)\|f_1(t, w_1^n(t)) - f_1(t, w_1^{n-1}(t))\|_1 + \rho_1(t)\|h_1(t, x_1^n(t))) - h_1(t, x_1^{n-1}(t))\|_1\right]
\end{aligned} \tag{5.6}$$

and

$$\begin{aligned}
&\|g_2(t, x_2^{n+1}(t)) - g_2(t, x_2^n(t))\|_2 \\
&= \|J_{M_{2t}}^{\rho_2(t), P_{2t}(A_{2t}, B_{2t})}[P_{2t}(A_{2t}, B_{2t})(g_{2t}(x_2^n)) - \rho_2(t)\{F_2(t, u_2^n(t), v_2^n(t)) \\
&\quad - (f_2(t, w_2^n(t)) - h_2(t, x_2^n(t)))\}] - J_{M_{2t}}^{\rho_2(t), P_{2t}(A_{2t}, B_{2t})}[P_{2t}(A_{2t}, B_{2t})(g_{2t}(x_2^{n-1})) \\
&\quad - \rho_2(t)\{F_2(t, u_2^{n-1}(t), v_2^{n-1}(t)) - (f_2(t, w_2^{n-1}(t)) - h_2(t, x_2^{n-1}(t)))\}]\|_2 \\
&\leq \frac{1}{[\alpha_2(t) - \beta_2(t) - \rho_2(t)m_2(t)]}\left[\|P_{2t}(A_{2t}, B_{2t})(g_{2t}(x_2^n)) - P_{2t}(A_{2t}, B_{2t})(g_{2t}(x_2^{n-1}))\|_2\right. \\
&\quad + \rho_2(t)\|F_2(t, u_2^n(t), v_2^n(t)) - F_2(t, u_2^{n-1}(t), v_2^{n-1}(t))\|_2 \\
&\quad \left. +\rho_2(t)\|f_2(t, w_2^n(t)) - f_2(t, w_2^{n-1}(t))\|_2 + \rho_2(t)\|h_2(t, x_2^n(t))) - h_2(t, x_2^{n-1}(t))\|_2\right].
\end{aligned} \tag{5.7}$$

Since F_1 is (μ_1, η_1)-mixed Lipschitz continuous and F_2 is (μ_2, η_2)-mixed Lipschitz continuous and $\mathcal{H}_1$-Lipschitz continuity of multi-valued mappings $\tilde{N}_1, \tilde{S}_1$ and $\mathcal{H}_2$-Lipschitz continuity of multi-valued mappings $\tilde{N}_2$, $\tilde{S}_2$, we have

$$\begin{aligned}
\|F_1(t, u_1^n(t), v_1^n(t)) - F_1(t, u_1^{n-1}(t), v_1^{n-1}(t))\|_1 \\
&\leq \mu_1(t)\|u_1^n(t) - u_1^{n-1}(t)\|_1 + \eta_1(t)\|v_1^n(t) - v_1^{n-1}(t)\|_2 \\
&\leq \mu_1(t)\lambda_{N_1}(t)\left(1 + (1+n)^{-1}\right)\|x_1^n(t) - x_1^{n-1}(t)\|_1 \\
&\quad + \eta_1(t)\lambda_{S_1}(t)\left(1 + (1+n)^{-1}\right)\|x_2^n(t) - x_2^{n-1}(t)\|_2,
\end{aligned} \tag{5.8}$$

and

$$\begin{aligned}
&\|F_2(t,u_2^n(t),v_2^n(t))-F_2(t,u_2^{n-1}(t),v_2^{n-1}(t))\|_2\\
&\qquad\le \mu_2(t)\|u_2^n(t)-u_2^{n-1}(t)\|_2+\eta_2(t)\|v_2^n(t)-v_2^{n-1}(t)\|_2\\
&\qquad\le \mu_2(t)\lambda_{N_2}(t)\left(1+(1+n)^{-1}\right)\|x_1^n(t)-x_1^{n-1}(t)\|_1\\
&\qquad\quad+\eta_2(t)\lambda_{S_2}(t)\left(1+(1+n)^{-1}\right)\|x_2^n(t)-x_2^{n-1}(t)\|_2. \qquad (5.9)
\end{aligned}$$

Since, for each $i=1,2$, f_i is λ_{f_i}-Lipschitz continuous and T_i is $\mathcal{H}_i$-Lipschitz continuous, we have

$$\|f_1(t,w_1^n(t))-f_1(t,w_1^{n-1}(t))\|_1\le \lambda_{f_1}(t)\lambda_{T_1}(t)\left(1+(1+n)^{-1}\right)\|x_1^n(t)-x_1^{n-1}(t)\|_1, \qquad (5.10)$$

and

$$\|f_2(t,w_2^n(t))-f_2(t,w_2^{n-1}(t))\|_2\le \lambda_{f_2}(t)\lambda_{T_2}(t)\left(1+(1+n)^{-1}\right)\|x_2^n(t)-x_2^{n-1}(t)\|_2. \qquad (5.11)$$

Since h_i is λ_{h_i}-Lipschitz continuous, we have

$$\|h_1(t,x_1^n(t))-h_1(t,x_1^{n-1}(t))\|_1\le \lambda_{h_1}(t)\|x_1^n(t)-x_1^{n-1}(t)\|_1, \qquad (5.12)$$

and

$$\|h_2(t,x_2^n(t))-h_2(t,x_2^{n-1}(t))\|_2\le \lambda_{h_2}(t)\|x_2^n(t)-x_2^{n-1}(t)\|_2. \qquad (5.13)$$

From (5.1), (5.4), (5.6), (5.8), (5.10) and (5.12), it follows that

$$\begin{aligned}
&\|x_1^{n+1}(t)-x_1^n(t)\|_1\\
&\quad\le L_1(t)\left[(\xi_1(t)+\rho_1(t)[\lambda_{h_1}(t)+L(n)(\mu_1(t)\lambda_{N_1}(t)+\lambda_{f_1}(t)\lambda_{T_1}(t))])\right.\\
&\qquad\left.\times\|x_1^n(t)-x_1^{n-1}(t)\|_1+\eta_1(t)\lambda_{S_1}(t)L(n)\|x_2^n(t)-x_2^{n-1}(t)\|_2\right], \qquad (5.14)
\end{aligned}$$

where

$$L_1(t)=\frac{1}{[\alpha_1(t)-\beta_1(t)-\rho_1(t)m_1(t)]}\sqrt{\frac{1+2d_1(t)}{2e_1(t)+3}};\ L(n)=\left(1+(1+n)^{-1}\right). \qquad (5.15)$$

Also, from (5.2), (5.5), (5.7), (5.9), (5.11) and (5.13), it follows that

$$\begin{aligned}
&\|x_2^{n+1}(t)-x_2^n(t)\|_2\\
&\quad\le L_2(t)\left[(\xi_2(t)+\rho_2(t)[\lambda_{h_2}(t)+L(n)(\eta_2(t)\lambda_{S_1}(t)+\lambda_{f_2}(t)\lambda_{T_2}(t))])\right.\\
&\qquad\left.\times\|x_2^n(t)-x_2^{n-1}(t)\|_2+\mu_2(t)\lambda_{N_2}(t)L(n)\|x_1^n(t)-x_1^{n-1}(t)\|_1\right], \qquad (5.16)
\end{aligned}$$

where

$$L_2(t)=\frac{1}{[\alpha_2(t)-\beta_2(t)-\rho_2(t)m_2(t)]}\sqrt{\frac{1+2d_2(t)}{2e_2(t)+3}};\ L(n)=\left(1+(1+n)^{-1}\right). \qquad (5.17)$$

From (5.14) and (5.17), we have

$$
\begin{aligned}
&\|x_1^{n+1}(t) - x_1^n(t)\|_1 + \|x_2^{n+1}(t) - x_2^n(t)\|_2 \\
&\le \big[L_1(t)(\xi_1(t) + \rho_1(t)[\lambda_{h_1}(t) + \left(1 + (1+n)^{-1}\right)(\mu_1(t)\lambda_{N_1}(t) + \lambda_{f_1}(t)\lambda_{T_1}(t))]) \\
&\qquad + L_2(t)\mu_2(t)\lambda_{N_2}(t)\left(1 + (1+n)^{-1}\right)]\|x_1^n(t) - x_1^{n-1}(t)\|_1 \\
&\qquad + [L_2(t)(\xi_2(t) + \rho_2(t)[\lambda_{h_2}(t) + \left(1 + (1+n)^{-1}\right)(\eta_2(t)\lambda_{S_2}(t) + \lambda_{f_2}(t)\lambda_{T_2}(t))]) \\
&\qquad + L_1(t)\eta_1(t)\lambda_{S_1}(t)\left(1 + (1+n)^{-1}\right)]\|x_2^n(t) - x_2^{n-1}(t)\|_2 \\
&\le \theta^n(t)\Big(\|x_1^n(t) - x_1^{n-1}(t)\|_1 + \|x_2^n(t) - x_2^{n-1}(t)\|_2\Big),
\end{aligned}
\tag{5.18}
$$

where

$$
\begin{aligned}
\theta^n(t) := \max\Big\{&L_1(t)\Big(\xi_1(t) + \rho_1(t)\Big[\lambda_{h_1}(t) + L(n)\Big(\mu_1(t)\lambda_{N_1}(t) + \lambda_{f_1}(t)\lambda_{T_1}(t)\Big)\Big]\Big) \\
&+ L_2(t)\mu_2(t)\lambda_{N_2}(t)L(n)\ ,\ L_2(t)\Big(\xi_2(t) + \rho_2(t)\Big[\lambda_{h_2}(t) + L(n)\Big(\eta_2(t)\lambda_{S_2}(t) \\
&+\lambda_{f_2}(t)\lambda_{T_2}(t)\Big)\Big]\Big) + L_1(t)\eta_1(t)\lambda_{S_1}(t)L(n)\Big\},
\end{aligned}
\tag{5.19}
$$

and

$$
L_1(t) := \frac{1}{[\alpha_1(t) - \beta_1(t) - \rho_1(t)m_1(t)]}\sqrt{\frac{1+2d_1(t)}{2e_1(t)+3}};
$$

$$
L_2(t) := \frac{1}{[\alpha_2(t) - \beta_2(t) - \rho_2(t)m_2(t)]}\sqrt{\frac{1+2d_2(t)}{2e_2(t)+3}};\quad L(n) := \left(1 + (1+n)^{-1}\right).
$$

Letting $n \to \infty$, we have

$$
\begin{aligned}
\theta(t) := \max\Big\{&L_1(t)\Big(\xi_1(t) + \rho_1(t)\Big[\lambda_{h_1}(t) + \mu_1(t)\lambda_{N_1}(t) + \lambda_{f_1}(t)\lambda_{T_1}(t)\Big]\Big) \\
&+ L_2(t)\mu_2(t)\lambda_{N_2}(t), L_2(t)\Big(\xi_2(t) + \rho_2(t)\Big[\lambda_{h_2}(t) + \eta_2(t)\lambda_{S_2}(t) \\
&+ \lambda_{f_2}(t)\lambda_{T_2}(t)\Big]\Big) + L_1(t)\eta_1(t)\lambda_{S_1}(t)\Big\}.
\end{aligned}
\tag{5.20}
$$

Define $\|.\|_*$ on $H_1 \times H_2$ by

$$
\|(x_1(t), x_2(t))\|_* = \|x_1(t)\|_1 + \|x_2(t)\|_2,\quad \forall (x_1(t), x_2(t)) \in H_1 \times H_2. \tag{5.21}
$$

It is observed that $(H_1 \times H_2, \|.\|_*)$ is a Banach space. Define $z^{n+1}(t) = (x_1^{n+1}(t), x_2^{n+1}(t))$. Then we have

$$
\|z^{n+1}(t) - z^n(t)\|_* = \|x_1^{n+1}(t) - x_1^n(t)\|_1 + \|x_2^{n+1}(t) - x_2^n(t)\|_2. \tag{5.22}
$$

From condition (5.3), we know that $0 < \theta(t) < 1$, and hence there exists an $n_0 > 0$ and $\theta_0(t) \in (0,1)$ such that $\theta^n(t) \le \theta_0(t)$ for all $n \ge n_0$. Therefore by (5.18) and (5.22), we have

$$
\|z^{n+1}(t) - z^n(t)\|_* \le \theta_0(t)\|z^n(t) - z^{n-1}(t)\|_*,\quad \forall n \ge n_0. \tag{5.23}
$$

It follows from (5.3) that

$$\|z^{n+1}(t) - z^n(t)\|_* \le (\theta_0(t))^{n-n_0} \|z^{n_0+1}(t) - z^{n_0}(t)\|_*.$$

Hence, for any $m \ge n > n_0$, it follows that

$$\begin{aligned} \|x_1^m(t) - x_1^n(t)\|_1 \le \|z^m(t) - z^n(t)\|_* &\le \sum_{i=n}^{m-1} \|z^{i+1}(t) - z^i(t)\|_* \\ &\le \sum_{i=n}^{m-1} (\theta_0(t))^{i-n_0} \|z^{n_0+1}(t) - z^{n_0}(t)\|_*. \end{aligned} \tag{5.24}$$

Since $0 < \theta_0(t) < 1$, it follows from (5.24) that $\|x_1^m(t) - x_1^n(t)\| \to 0$ as $n \to \infty$, and hence $\{x_1^n(t)\}$ is a Cauchy sequence in H_1. By the same argument, it follows that $\{x_2^n(t)\}$ is also Cauchy sequence in H_2. Thus, there exists $(x_1(t), x_2(t)) \in H_1 \times H_2$ such that $x_1^n(t) \to x_1(t)$ and $x_2^n(t) \to x_2(t)$ as $n \to \infty$.

Now, we prove that $u_1^n(t) \to u_1(t)$, $u_2^n(t) \to u_2(t)$, $v_1^n(t) \to v_1(t)$, $v_2^n(t) \to v_2(t)$, $w_1^n(t) \to w_1(t)$ and $w_2^n(t) \to w_2(t)$. In fact it follows from the Lipschitz continuity of $\tilde{N}_1$, $\tilde{S}_1$, $\tilde{N}_2$, $\tilde{S}_2$, $\tilde{T}_1$, $\tilde{T}_2$ and Iterative Algorithm 4.1 that,

$$\begin{aligned} \|u_1^n(t) - u_1^{n-1}(t)\|_1 &\le \lambda_{N_1}(t) \left(1 + (1+n)^{-1}\right) \|x_1^n(t) - x_1^{n-1}(t)\|_1, \\ \|u_2^n(t) - u_2^{n-1}(t)\|_1 &\le \lambda_{N_2}(t) \left(1 + (1+n)^{-1}\right) \|x_1^n(t) - x_1^{n-1}(t)\|_1, \\ \|v_1^n(t) - v_1^{n-1}(t)\|_2 &\le \lambda_{S_1}(t) \left(1 + (1+n)^{-1}\right) \|x_2^n(t) - x_2^{n-1}(t)\|_2, \\ \|v_2^n(t) - v_2^{n-1}(t)\|_2 &\le \lambda_{S_2}(t) \left(1 + (1+n)^{-1}\right) \|x_2^n(t) - x_2^{n-1}(t)\|_2, \\ \|w_1^n(t) - w_1^{n-1}(t)\|_1 &\le \lambda_{T_1}(t) \left(1 + (1+n)^{-1}\right) \|x_1^n(t) - x_1^{n-1}(t)\|_1, \\ \|w_2^n(t) - w_2^{n-1}(t)\|_1 &\le \lambda_{T_2}(t) \left(1 + (1+n)^{-1}\right) \|x_2^n(t) - x_2^{n-1}(t)\|_2. \end{aligned} \tag{5.25}$$

From (5.25), we know that $\{u_1^n(t)\}$, $\{u_2^n(t)\}$, $\{v_1^n(t)\}$, $\{v_2^n(t)\}$, $\{w_1^n(t)\}$, $\{w_2^n(t)\}$ are also Cauchy sequences. Therefore, there exist $u_i(t) \in \tilde{N}_i(t, x_1(t))$, $v_i(t) \in \tilde{S}_i(t, x_2(t))$, $w_i(t) \in \tilde{T}_i(t, x_i(t))$ $(i = 1, 2)$ such that $u_1^n(t) \to u_1(t)$, $u_2^n(t) \to u_2(t)$, $v_1^n(t) \to v_1(t)$, $v_2^n(t) \to v_2(t)$, $w_1^n(t) \to w_1(t)$ and $w_2^n(t) \to w_2(t)$ as $n \to \infty$. Further

$$\begin{aligned} d(u_1(t), N_1(t, x_1(t))) &\le \|u_1(t) - u_1^n(t)\|_1 + d(u_1^n(t), N_1(t, x_1(t))) \\ &\le \|u_1(t) - u_1^n(t)\|_1 + \mathcal{H}(N_1(\Omega, x_1^n(t)), N_1(t, x_1)) \\ &\le \|u_1(t) - u_1^n(t)\|_1 + t_1 \|x_1^n(t) - x_1(t)\|_1 \to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{5.26}$$

Since $\tilde{N}_1(t, x_1(t)) \subset H_1$ is closed, we have $u_1(t) \in \tilde{N}_1(t, x_1(t)) \subset H_1$. Similarly, we have $u_2(t) \in \tilde{N}_2(t, x_2(t))$, $v_1(t) \in \tilde{S}_1(t, x_1(t))$, $v_2(t) \in \tilde{S}_2(t, x_2(t))$, $w_1(t) \in \tilde{T}_1(t, x_1(t))$, $w_2(t) \in \tilde{T}_2(t, x_2(t)), \forall t \in \Omega, (x_1(t), x_2(t)) \in H_1 \times H_2$.

Finally, we define

$$\begin{aligned} w_1(t) = J_{M_{1t}}^{\rho_1(t), P_{1t}(A_{1t}, B_{1t})} \Big[P_{1t}(A_{1t}, B_{1t})(g_{1t}(x_1)) - \rho_1(t) \Big\{ & F_1(t, u_1(t), v_1(t)) \\ & - \Big(f_1(t, w_1(t)) - h_1(t, x_1(t)) \Big) \Big\} \Big], \end{aligned} \tag{5.27}$$

$$w_2(t) = J_{M_{2t}}^{\rho_2(t),P_{2t}(A_{2t},B_{2t})}\Big[P_{2t}(A_{2t},B_{2t})(g_{2t}(x_2)) - \rho_2(t)\Big\{F_2(t,u_2(t),v_2(t)) - \Big(f_2(t,w_2(t)) - h_2(t,x_2(t))\Big)\Big\}\Big]. \tag{5.28}$$

Now, we estimate

$$\begin{aligned}
&\|g_1(x_1^{n+1})(t) - w_1(t)\|_1 \\
&\quad \le \frac{1}{[\alpha_1(t) - \beta_1(t) - \rho_1(t)m_1(t)]}\Big[\xi_1(t) + \rho_1(t)\Big[\lambda_{h_1}(t) + \big(1+(1+n)^{-1}\big) \\
&\qquad \times \Big(\mu_1(t)\lambda_{N_1}(t) + \lambda_{f_1}(t)\lambda_{T_1}(t)\Big)\Big]\|x_1^{n+1}(t) - x_1(t)\|_1 \\
&\qquad + \eta_1(t)\lambda_{S_1}(t)\left(1+(1+n)^{-1}\right)\|x_2^{n+1}(t) - x_2(t)\|_2\Big],
\end{aligned} \tag{5.29}$$

and

$$\begin{aligned}
&\|g_2(x_2^{n+1})(t) - w_2(t)\|_2 \\
&\quad \le \frac{1}{[\alpha_2(t) - \beta_2(t) - \rho_2(t)m_2(t)]}\Big[\mu_2(t) + \rho_2(t)\Big[\lambda_{h_2}(t) + \big(1+(1+n)^{-1}\big) \\
&\qquad \times \Big(\eta_2(t)\lambda_{S_2}(t) + \lambda_{f_2}(t)\lambda_{T_2}(t)\Big)\Big]\|x_2^{n+1}(t) - x_2(t)\|_2 \\
&\qquad + \mu_2(t)\lambda_{N_2}(t)\left(1+(1+n)^{-1}\right)\|x_1^{n+1}(t) - x_1(t)\|_1\Big].
\end{aligned} \tag{5.30}$$

Now, it follows from (5.22), (5.29) and (5.30) that

$$\begin{aligned}
&\|(g_1(x_1^{n+1}(t)), g_2(x_1^{n+1}(t))) - (w_1(t), w_2(t))\|_* \\
&\qquad = \|g_1(x_1^{n+1})(t) - w_1(t)\|_1 + \|g_2(x_2^{n+1})(t) - w_2(t)\|_2 \\
&\qquad \le \theta_1^n\Big(\|x_1^{n+1}(t) - x_1(t)\|_1 + \|x_2^{n+1}(t) - x_2(t)\|_2\Big) \to 0, \text{ as } n \to \infty.
\end{aligned} \tag{5.31}$$

Thus

$$\begin{aligned}
g_1(t, x_1^{n+1}(t)) &= w_1(t) \\
&= J_{M_{1t}}^{\rho_1(t),P_{1t}(A_{1t},B_{1t})}\Big[P_{1t}(A_{1t},B_{1t})(g_{1t}(x_1)) - \rho_1(t) \\
&\qquad \times \Big\{F_1(t,u_1(t),v_1(t)) - \Big(f_1(t,w_1(t)) - h_1(t,x_1(t))\Big)\Big\}\Big],
\end{aligned} \tag{5.32}$$

$$\begin{aligned}
g_2(t, x_2^{n+1}(t)) &= w_2(t) \\
&= J_{M_{2t}}^{\rho_2(t),P_{2t}(A_{2t},B_{2t})}\Big[P_{2t}(A_{2t},B_{2t})(g_{2t}(x_2))) - \rho_2(t) \\
&\qquad \times \Big\{F_2(t,u_2(t),v_2(t)) - \Big(f_2(t,w_2(t)) - h_2(t,x_2(t))\Big)\Big\}\Big].
\end{aligned} \tag{5.33}$$

By Lemma 4.2, it follows that $(x_1(t), x_2(t), u_1(t), u_2(t), v_1(t), v_2(t), w_1(t), w_2(t))$ is a solution of SRVI (2.1)-(2.2). This completes the proof. □

Remark 5.2. *For each* $i = 1, 2$, *let* $\alpha_i,\ \beta_i,\ m_i,\ d_i,\ e_i,\ \xi_i,\ \mu_i,\ \eta_i,\ \lambda_{h_i},\ \lambda_{S_i},\ \lambda_{N_i},\ \lambda_{T_i}$ *be constant measurable functions. Then*

(i) It is clear that $\alpha_i(t) > \beta_i(t),\ \alpha_i(t), \beta_i(t) > 0$ *and* $\alpha_i(t) > \rho_i(t)m_i(t) + \beta_i(t)$.

(ii) If g_i *is* $(d_i(t), e_i(t))$*-relaxed cocoercive, then* $d_i(t)e_i(t) > \frac{1}{4}$ *with* $d_i(t) > e_i(t)$.

(iii) Further, $\theta(t) < 1$ *and conditions (5.1)-(5.3) holds for some suitable values of constants, for example:* $\alpha_1(t) = 5,\ \alpha_2(t) = 6,\ \beta_1(t) = 2,\ \beta_2(t) = 3,\ m_1(t) = 2,\ m_2 = 3,\ d_1(t) = 0.6,\ d_2(t) = 0.7,\ e_1(t) = 0.5,\ e_2(t) = 0.4,\ \xi_1(t) = 0.1,\ \xi_2(t) = 0.2,\ \lambda_{h_1}(t) = 0.1,\ \lambda_{h_2}(t) = 0.15,\ \lambda_{S_1}(t) = 0.15,\ \lambda_{S_2}(t) = 0.1,\ \lambda_{N_1}(t) = 0.15,\ \lambda_{N_2}(t) = 0.2,\ \mu_1(t) = 0.2,\ \eta_1(t) = 0.1,\ \mu_2(t) = 0.1,\ \eta_2(t) = 0.2,\ \lambda_{f_1}(t) = 0.2,\ \lambda_{f_2}(t) = 0.15,\ \lambda_{T_1}(t) = 0.1,\ \lambda_{T_2}(t) = 0.2,\ \rho_1(t) = 0.1,\ \rho_2(t) = 0.2.$

(iv) $0 < \rho_i(t) \in \left(0, \frac{\alpha_i(t) - \beta_i(t)}{m_i(t)}\right),\ \rho_1(t) \in (0, 1.5),\ \rho_2(t) \in (0, 1).$

Remark 5.3. *If the random mapping* $g_1 : \Omega \times H_1 \to H_1$ *is* $(d_1(t), e_1(t))$*-relaxed cocoercive mapping and* $\lambda_1(t)$*-Lipschitz continuous, then we can observe that* g_1 *is either* $(e_1(t) - d_1(t)\lambda_1(t)^2)$*-strongly monotone or* $(d_1(t)\lambda_1(t)^2 - e_1(t))$*-relaxed strongly monotone according as either* $d_1(t)\lambda_1(t)^2 < e_1(t)$ *or* $d_1(t)\lambda_1(t)^2 > e_1(t)$. *Hence, we have taken care of this argument in our main result. Thus, the method presented in this paper improves the corresponding methods developed by many authors for solving variational inclusions involving relaxed-cocoercive and Lipschitz continuous mappings.*

Acknowledgements : The authors are thankful to the referees for their valuable comments towards the improvement of first version of this paper. The first author is supported by University Grants Commission, Government of India, under the Major Research Project No. F.36-7/2008 (SR). The second author is supported by Deanship of Research Unit of University of Tabuk, Ministry of Higher Education, Kingdom of Saudia Arabia. The third author is supported by NBHM, Department of Atomic Energy, Government of India under Grants-in-aid for Post-doctoral fellowship (Reference no. 2/40(36)/2009-R&D II/2882).

References

[1] R. Ahmad, F.F. Bazan, An iterative algorithm for random generalized nonlinear mixed variational inclusions for random fuzzy mappings, Appl. Math. Comput. 167 (2005) 1440–1411.

[2] R. Ahmad, A.P. Farajzadeh, On random variational inclusions with random fuzzy mappings and random relaxed cocoercive mappings, Fuzzy Sets and Systems 160 (2009) 3166–3174.

[3] Y.J. Cho, N.-J. Huang, S.M. Kang, Random generalized set-valued strongly nonlinear implicit quasi-variational inequalities, J. Inequal. Appl. 5 (2000) 531–575.

[4] X.-P. Ding, J.-Y. Park, A new class of generalized nonlinear implicit quasi-variational inclusion with fuzzy mapping, J. Comput. Appl. Math. 138 (2002) 243–257.

[5] H.R. Feng, X.P. Ding, A new system of generalized nonlinear quasi-variational-like inclusions with A-monotone operators in Banach spaces, J. Comput. Appl. Math. 225 (2009) 365–373.

[6] N.J. Huang, A new method for a class of nonlinear variational inequalities with fuzzy mappings, Appl. Math. Lett. 10 (6) (1997) 129–133.

[7] K.R. Kazmi, Iterative algorithm for generalized quasi-variational-like inclusions with fuzzy mappings, J. Comput. Appl. Math. 188 (1) (2006) 1–11.

[8] H. Lan, (A, η)-accretive mappings and set-valued variational inclusions with relaxed cocoercive mappings in Banach spaces, Appl. Math. Lett. 20 (2007) 571–577.

[9] H.Y. Lan, Approximation solvability of nonlinear random (A, η)-resolvent operator equations with random relaxed cocoercive operator, Comput. Math. Appl. 57 (2009) 624–632.

[10] H.Y. Lan, J.H. Kim, Y.J. Cho, On a new system of nonlinear A-monotone multivalued variational inclusion, J. Math. Anal. Appl. 327 (2007) 481–493.

[11] H.Y. Lan, R.U. Verma, Iterative algorithms for nonlinear fuzzy variational inclusion systems with (A, η)accerative mappings in Banach spaces, Adv. Nonlinear Var. Inequal. 11 (1) (2008) 15–30.

[12] L.A. Zadeh, Fuzzy sets, Inform. and Control 8 (1965) 338–353.

[13] S.S. Chang, Y. Zhu, On variational inequalities for fuzzy mappings, Fuzzy Set and Systems 32 (1989) 356–367.

[14] M.A. Noor, Variational inequalities with fuzzy mappings (I), Fuzzy Sets and Systems 55 (1989) 309–314.

[15] J.-Y. Park, J.-U. Jeong, A perturbed algorithm of variational inclusions for fuzzy mappings, Fuzzy Sets and Systems 115 (2000) 419–424.

[16] Y.J. Cho, H.Y. Lan, Generalized nonlinear random (A, η)-accretive equations with random relaxed cocoercive mappings in Banach space, Comput. Math. Appl. 55 (9) (2008) 2173–2182.

[17] N.J. Huang, Y.J. Cho, Random completely generalized set-valued implicit quasi- variational inequalities, Positivity 3 (1999) 201–213.

[18] X.-K. Wu, Y.-Z. Zou, A system of random nonlinear variational inclusions involving random fuzzy mappings and $H(\cdot,\cdot)$-monotone set-valued mappings, J. Inequal. Appl., Vol. 2010 (2010), Article ID 123524, 17 pages.

[19] W.-B. Zhang, Random nonlinear variational inclusions involving $H(\cdot,\cdot)$-acceretive operator for random fuzzy mapping, Bull. Malays. Math. Sci. Soc. 34 (2) (2011) 389–402.

[20] H.-X. Dai, Generalized mixed variational-like inequality for random fuzzy mappings, J. Comput. Appl. Math. 224 (2009) 20–28.

[21] J.-S. Pang, Asymmetrical variational inequalities over product of sets: Applications and iterative methods, Math. Prog. 31 (1985) 206–219.

[22] J.P. Aubin, Mathematical Methods of Game Theory and Economics, North-Holland, Amsterdam, 1982.

[23] A. Nagurney, Network economics: A variational inequality approach, Kluwer Academic Publishers, Dordrecht, 1993.

[24] S.S. Chang, Fixed Point Theorem with Applications, Chongqing Publishing House, Chongqing, 1984.

[25] I.V. Knonov, Relatively monotone variational inequalities over product sets, Oper. Res. Lett. 28 (2001) 21–26.

[26] K.R. Kazmi, M.I. Bhat, Iterative algorithm for a system of nonlinear variational-like inclusions, Comput. Math. Appl. 48 (2004) 1929–1935.

[27] K.R. Kazmi, F.A. Khan, Iterative approximation of a unique solution of a system of variational-like inclusions in real q_i-uniformly smooth Banach space, Nonlinear Anal. 67 (2007) 917–929.

[28] R.U. Verma, Generalized system of relaxed cocoercive variational inequalities and projection methods, J. Optim. Theory Appl. 121 (1) (2004) 203–210.

[29] R.U. Verma, General system of A-monotone nonlinear variational inclusion problems with applications, J. Optim. Theory Appl. 131 (1) (2006) 151–157.

[30] S. Adly, Perturbed algorithms and sensitivity analysis for a general class of variational inclusions, J. Math. Anal. Appl. 201 (1996) 609–630.

[31] R.P. Agarwal, N.-J. Huang, Y.-J. Cho, Generalized nonlinear mixed implicit quasi-variational inclusions with set-valued mappings, J. Inequal. Appl. 7 (6) (2002) 807–828.

[32] C.E. Chidume, K.R. Kazami, H. Zegeye, Iterative approximation of a solution of a general variational-like inclusions in Banach spaces, Internat. J. Math. & Math. Sci. 22 (2004) 1159–1168.

[33] X.-P. Ding, Generalized quasi-variational-like inclusions with nonconvex functionals, Appl. Math. Comput. 122 (2001) 267–282.

[34] X.-P. Ding, C.-L. Luo, Perturbed proximal point algorithms for general quasi-variational-like inclusions, J. Comput. Appl. Math. 113 (2000) 153–165.

[35] X.-P. Ding, F.-Q. Xia, A new class of completely generalized quasi-variational inclusions in Banach spaces, J. Comput. Appl. Math. 147 (2002) 369–383.

[36] X.-P. Ding, J.-C. Yao, Existence and algorithm of solutions for mixed quasi-variational-like inclusions in Banach spaces, Comput. Math. Appl. 49 (2005) 857–869.

[37] Y.P. Fang, N.J. Huang, H-monotone operator and resolvent operator technique for variational inclusions, Appl. Math. Comput. 145 (2003) 795–803.

[38] N.-J. Huang, M.-R. Bai, Y.-J. Cho, S.-M. Kang, Generalized nonlinear mixed quasi-variational inequalities, Comput. Math. Appl. 40 (2-3) (2000) 205–215.

[39] K.R. Kazmi, M.I. Bhat, Convergence and stability of iterative algorithms of generalized set-valued variational-like inclusions in Banach spaces, Appl. Math. Comput. 166 (2005) 164–180.

[40] K.R. Kazmi, M.I. Bhat, Convergence and stability of a three-step iterative algorithm for a general quasi-variational inequality problem, Fixed Point Theory and Applications, Vol. 2006 (2006), Article ID 96012, 16 pages.

[41] K.R. Kazmi, M.I. Bhat, Iterative algorithm for a system of set-valued variational-like inclusions, Kochi J. Math. 2 (2007) 107–117.

[42] K.R. Kazmi, M.I. Bhat, Convergence and stability of iterative algorithms for some classes of general variational inclusions in Banach spaces, Southeast Asian Bull. Math. 32 (2008) 99–116.

[43] K.R. Kazmi, M.I. Bhat, N. Ahmad, an iterative algorithm based on M-proximal mappings for a system of generalized implicit variational inclusions in Banach spaces, J. Comput. Appl. Math. 233 (2009) 361–371.

[44] K.R. Kazmi, F.A. Khan, Iterative approximation of a solution of multi-valued variational-like inclusion in Banach spaces: A P-η-proximal-point mapping approach, J. Math. Anal. Appl. 325 (2007) 665–674.

[45] K.R. Kazmi, F.A. Khan, Iterative algorithm for a set-valued implicit quasi-variational inequality problem in uniformly smooth Banach space, Southeast Asian Bull. Math. 33 (2009) 65–77.

[46] Z. Liu, J.-S. Ume, S.-M. Kang, General strongly nonlinear quasivariational inequalities with relaxed Lipschitz and relaxed monotone mappings, J. Optim. Theory Appl. 114 (3) (2002) 639–656.

[47] Z. Liu, J.-S. Ume, S.-M. Kang, Stability of Noor iterations with errors for generalized nonlinear complementarity problems, Acta Mathematica Academiae Paedagogicae Nyíregyháziensis 20 (2004) 53–61.

[48] L.-C. Zeng, S.-M. Guu, J.-C. Yao, Characterization of H-monotone operators with applications to variational inclusions, Comput. Math. Appl. 50 (2005) 329–337.

[49] J.H. Sun, S.W. Zhang, L.W. Zhang, An algorithm based on resolvent operators for solving positively semi-definite variational inequalities, Fixed Point Theory and Applications, Vol. 2007 (2007), Article ID 76040, 15 pages.

[50] Y.-Z. Zou, N.-J. Huang, A new system of variational inclusions involving $H(\cdot,\cdot)$-accretive operator in Banach spaces, Appl. Math. Comput. 212 (2009) 135–144.

[51] C.J. Himmelberg, Measurable relations, Fund. Math. 87 (1975) 53–72.

(Received 1 March 2011)
(Accepted 16 January 2012)

Thai J. Math. Online @ http://thaijmath.in.cmu.ac.th