Common Fixed Point
as a Contractive Fixed Point

T. Phaneendra†,‡ and M. Chandra Shekhar‡

†Applied Analysis Division, School of Advanced Sciences
VIT University, Vellore - 632 014, Tamil Nadu State, India
e-mail: drtp.indra@gmail.com
‡Department of Mathematics, Vijay Rural Engineering College
Nizamabad - 503 003, Andhra Pradesh State, India
e-mail: maisa.chandrashekharc@gmail.com

Abstract : In this paper, we prove a common fixed point theorem for a wider class of generalized contraction type mappings relative to a self-map and show that the common fixed point will be a contractive fixed point of the reference map under certain condition on contraction constant. Our result is a generalization of common fixed point theorems of first author, and of Akkouchi.

Keywords : self-map; associated Sequence; common fixed point; attracting and contractive fixed points.

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1 Introduction

In this paper \( X \) represents a metric space with metric \( d \). If \( x \) is a point of \( X \) and \( S \) a self-map on \( X \), we write \( Sx \) for the image of \( x \) under \( S \), \( S(X) \) for the range of \( S \), and \( TS \) for the composition of self-maps \( T \) and \( S \).

A point \( p \in X \) is a fixed point for a self-map \( S \) on \( X \) if \( Sp = p \). We denote by \( Fix(S) \), the set of all fixed points of \( S \), and by \( Fix(S, T) \) the set of all common fixed points of \( S \) and \( T \).

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‡Corresponding author.

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Given a reference map $S$ on $X$ and $0 < \alpha < 1$, $B(S, \alpha)$ denotes the class of self-maps $T$ on $X$ satisfying the contractive type condition:

$$d(Sx, TSy) \leq \alpha \max\{d(x, Sx), d(x, Sy), d(Sy, TSy), \frac{1}{2}[d(x, TSy) + d(Sx, Sy)]\}$$

for all $x, y \in X$ (1.1)

where $0 < \alpha < 1$. Fisher [1] proved the following result:

**Theorem 1.1.** Given a self-map $S$ on $X$ and $T \in B(S, \alpha)$, where $0 < \alpha < 1$, suppose that either $S$ or $T$ is continuous on $X$. If $X$ is complete, then there is a unique point $p$ in $X$ such that $p \in Fix(S, T)$.

Later in 2004, the author [2] obtained the conclusion of Theorem 1.1 by replacing the completeness of the space $X$ without an appeal to the continuity condition, as given below:

**Theorem 1.2.** Given a self-map $S$ on $X$ and $T \in B(S, \alpha)$, where $0 < \alpha < 1$, suppose that the associated sequence $\langle x_n \rangle_{n=1}^\infty$ at some $x_0 \in X$ with the choice

$$x_n =\begin{cases} Sx_{n-1} & \text{if } n \text{ is odd} \\ Tx_{n-1} & \text{if } n \text{ is even} \end{cases}$$

has a subsequence converging to a point $z$ in $X$. Then

(i) the sequence (1.2) will also converge to $z$.

(ii) $z$ will be a unique point such that $Fix(S) = Fix(S, T) = \{z\}$.

In this paper, we prove that a common fixed point can also be obtained by generalizing the condition (1.1), which will be a contractive fixed point for the reference map $S$, when $0 < \alpha < \frac{1}{2}$.

### 2 Notation

Let $\beta \geq 0$. Given $0 < \alpha < 1$, and self-map $S$ on $X$, we shall denote by $B_\beta(S, \alpha)$ the class of all mappings $T : X \to X$ satisfying the condition:

$$[1 + \beta d(x, Sy)] d(Sx, TSy) \leq \alpha \max\{d(x, Sx), d(x, Sy), d(Sy, TSy), \frac{1}{2}[d(x, TSy) + d(Sx, Sy)]\}$$

$$+ \beta[d(x, Sx)d(Sy, TSy) + d(x, TSy)d(Sx, Sy)]$$

for all $x, y \in X$ (2.1)

**Remark 2.1.** Writing $\beta = 0$ in (2.1), we get (1.1). Thus $B_0(S, \alpha) = B(S, \alpha)$. We note that if $B^*(S, \alpha) = \bigcup_{\beta \geq 0} B_\beta(S, \alpha)$, then $B^*(S, \alpha)$ includes $B(S, \alpha)$. 

Let $S$ be a self-map on $X$ and $x \in X$. The $S$-orbit or simply orbit at $x$ is the sequence $O_S(x)$ of iterates: $Sx, S^2x, \ldots$. If $p \in \text{Fix}(S)$ has a neighbourhood $N$ in $X$ such that the $S$-orbit at each $x$ in $N$ converges to $p$, then $p$ is called an attracting fixed point of $S$. In case $N = X$, $p$ is called a contractive fixed point.

**Example 2.2.** Consider $S : \mathbb{R} \to \mathbb{R}$ defined by $Sx = x^3$ for all $x$. Then $0, 1,$ and $-1$ are the fixed points of $S$, and $O_S(x) = \langle x^{3n} \rangle_{n=1}^{\infty}$. Note that for $|x| < 1$, the orbit converges to 0, while for $|x| > 1$ the orbit diverges to $\pm \infty$ respectively. Thus 0 is an attracting but not a contractive fixed point.

**Example 2.3.** Let $S : \mathbb{R} \to \mathbb{R}$ given by $Sx = x^2$ for all $x$. Then 0 is the only fixed point of $S$, which is also a contractive one, to which the orbit $O_S(x) = \langle x^2^n \rangle_{n=1}^{\infty}$ at each real $x$ converges.

### 3 Main Result

Our main result is

**Theorem 3.1.** Given $0 < \alpha < 1$, $\beta \geq 0$, and self-map $S$ on $X$, let $T \in B_\beta(S, \alpha)$. Suppose that the associated sequence $\langle x_n \rangle_{n=1}^{\infty}$ at some $x_0 \in X$ with the choice (1.2) has a subsequence converging to a point $z$ of $X$. Then the following assertions will be true:

(i) $\lim_{n \to \infty} x_n = z$

(ii) $z \in \text{Fix}(S) \subset \text{Fix}(T)$

(iii) $\text{Fix}(S) = \text{Fix}(T) = \text{Fix}(S, T) = \{z\}$ whenever $T(X) \subset S(X)$

(iv) $z$ is a unique common fixed point of $S$ and $T$

(v) $S$ and $TS$ are continuous at $z$.

Further if $0 < \alpha < \frac{1}{2}$, the unique common fixed point $z$ will be a contractive fixed point for the reference map $S$.

**Proof.** Let $x_0$ be arbitrary point of $X$. We write $r_n = d(x_n, x_{n+1}), n = 1, 2, 3, \ldots$. First we establish that

$$r_n \leq \alpha \max \{r_n, r_{n-1}\} \quad \text{for all } n.$$  \hspace{1cm} (3.1)

Suppose that $n$ is even. Writing $x = x_n$ and $y = x_{n-2}$ in (2.1), we get

$$[1 + \beta d(x_n, x_{n-1})] d(x_{n+1}, x_n)$$

$$\leq \alpha \max \left\{ d(x_n, x_{n+1}), d(x_n, x_{n-1}), 0, \frac{1}{2} d(x_{n+1}, x_{n-1}) \right\}$$

$$+ \beta d(x_n, x_{n+1}) d(x_{n-1}, x_n) + 0$$
or
\[(1 + \beta r_{n-1}) r_n \leq \alpha \max \left\{ r_n, \frac{1}{2} d(x_{n+1}, x_{n-1}) \right\} + \beta r_n r_{n-1} \]
from which (3.1) follows, since \( \frac{1}{2}(a + b) \leq \max\{a, b\} \). While if \( n \) is odd, we take
\( x = y = x_{n-1} \) in (2.1) and simplify as above to get (3.1).

It is remarkable to note that if \( r_m > r_{m-1} \) for some integer \( m > 1 \), then (3.1)
would give a contradiction that \( 0 < r_m \leq \alpha r_m < r_m \), since \( 0 < \alpha < 1 \).

Hence \( r_n \leq r_{n-1} \) for all \( n \), and from (3.1) we get \( r_n \leq \alpha r_{n-1} \) for all \( n \).

In other words, \( (x_n)_{n=1}^\infty \) is a contractive sequence and hence is a Cauchy sequence
in \( X \) [4, Theorem 3.1.7].

In view of the convergence of the subsequence, assertion (i) follows, and
\[ \lim_{n \to \infty} x_{2n-1} = \lim_{n \to \infty} S x_{2n} = \lim_{n \to \infty} x_{2n} = \lim_{n \to \infty} T x_{2n-1} = z \]
for some \( z \in X \). (3.2)

Writing \( x = z \) and \( y = x_{2n} \) in (2.1),
\[ [1 + \beta d(z, x_{2n+1})] d(S z, x_{2n+2}) \]
\[ \leq \alpha \max \left\{ d(z, S z), d(z, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), \frac{1}{2} [d(z, x_{2n+2}) + d(x_{2n+1}, S z)] \right\} \]
\[ + \beta d(z, S z) d(x_{2n+1}, T S x_{2n+2}) + d(z, x_{2n+2}) d(x_{2n+1}, S z) \]
Proceeding the limit as \( n \to \infty \) in this, and using (3.2), we obtain that
\[ (1 + \beta d(p, S p)) d(S p, T S p) \]
\[ \leq \alpha \max \left\{ d(p, S p), d(p, S p), d(S p, T S p), \frac{1}{2} [d(p, T S p) + d(S p, S p)] \right\} \]
\[ + \beta d(p, S p) d(S p, T S p) + d(p, T S p)d(S p, S q) \]
or \( d(S z, z) \leq \alpha d(z, S z) \) so that \( d(S z, z) = 0 \) or \( z \in \text{Fix}(S) \).

Now we observe that \( \text{Fix}(S) \subset \text{Fix}(T) \). In fact, if \( p \in \text{Fix}(S) \), then writing
\( x = y = p \) in (2.1), we get
\[ [1 + \beta d(p, S p)] d(S p, T S p) \]
\[ \leq \alpha \max \left\{ d(p, S p), d(p, S p), d(S p, T S p), \frac{1}{2} [d(p, T S p) + d(S p, S p)] \right\} \]
\[ + \beta d(p, S p) d(S p, T S p) + d(p, T S p)d(S p, S q) \]
or \( d(p, T p) \leq \alpha d(p, T p) \) so that \( d(p, T p) = 0 \) or \( p \in \text{Fix}(T) \). Thus \( p \) is a fixed
point of \( T \) whenever it is a fixed point of \( S \), and (ii) immediately follows.

To prove (iii), in view of (ii), it suffices to prove that \( \text{Fix}(T) \subset \text{Fix}(S) \). In
fact, let \( p \) be a fixed point of \( T \). Since \( T(X) \subset S(X) \), we get \( p = T p = S q \)
for some \( q \in X \). Now (2.1) with \( x = p \), and \( y = q \) gives
\[ [1 + \beta d(p, S q)] d(S p, T S q) \]
\[ \leq \alpha \max \left\{ d(p, S p), d(p, S q), d(S q, T S q), \frac{1}{2} [d(p, T S q) + d(S p, S q)] \right\} \]
\[ + \beta d(p, S p) d(S q, T S q) + d(p, T S q)d(S p, S q) \].
which on routine simplification gives \(d(Sp, p) \leq \alpha \ d(p, Sp)\) so that \(p = Sp\), that is \(p \in Fix(S)\). The other way of stating \((iii)\) is that \(z\) is a common fixed point of \(S\) and \(T\).

The uniqueness of the common fixed point follows directly from (2.1), proving \((iv)\). To get the continuity of \(S\) at \(z\), we see from (2.1) with \(y = z\) that

\[
[1 + \beta d(x, S)z)] d(Sx, TSz)
\]

\[
\leq \alpha \ max \left \{ d(x, Sx), d(x, Sz), d(Sz, TSz), \frac{1}{2}[d(x, TSz) + d(Sx, Sz)] \right \}
\]

\[
+ \beta [d(x, Sx)d(Sz, TSz) + d(x, TSz)d(Sx, Sz)],
\]

or

\[
[1 + \beta d(x, z)] d(Sx, z) \leq \alpha \ max \left \{ d(x, Sx), d(x, z), 0, \frac{1}{2}[d(x, z) + d(Sx, z)] \right \}
\]

\[
+ \beta [0 + d(x, z)d(Sx, z)].
\]

So that

\[
d(Sx, z) \leq \alpha \ max \left \{ d(x, Sx), d(x, z), 0, \frac{1}{2}[d(x, z) + d(Sx, z)] \right \},
\]

which on using the triangle inequality of the metric \(d\) gives

\[
d(Sx, z) \leq \alpha \ max \left \{ d(x, z) + d(z, Sx), d(x, z), \frac{1}{2}[d(x, z) + d(Sx, z)] \right \}
\]

\[
= \alpha \ [d(x, z) + d(Sx, z)].
\]

So that \((1 - \alpha)d(Sx, z) \leq \alpha \ d(x, z)\). Thus

\[
d(Sx, Sz) = d(Sx, z) \leq \left( \frac{\alpha}{1 - \alpha} \right) d(x, z).
\]

(3.3)

Given \(\epsilon > 0\), choose

\[
\delta = \left( \frac{1 - \alpha}{\alpha} \right) \epsilon.
\]

Then (3.3) would imply that \(d(Sx, Sz) < \epsilon\) whenever \(d(x, z) < \delta\), showing that \(S\) is continuous at \(z\).

Again writing \(x = z\) in (2.1), using \(Sz = z\) and then simplifying, we get

\[
d(z, TS) \leq \alpha \ max \left \{ d(z, Sy), \frac{1}{2}[d(z, TSy) + d(z, Sy)] \right \}
\]

\[
= \alpha \ max \{d(z, Sy), d(z, TSy)\}
\]

or

\[
d(z, TSy) \leq \alpha \ max \{d(Sy, z), d(TSy, z)\} = d\alpha(Sy, z) = \left( \frac{\alpha^2}{1 - \alpha} \right) d(y, z)
\]
for all \( y \in X \), in view of (3.3).

This reveals that \( TS \) is continuous at \( z \), proving the last part of (v).

In the remainder of the proof, we suppose that \( 0 < \alpha < \frac{1}{2} \) and \( y_0 \in X \) is arbitrary. Then the iterates:

\[
y_n = s^n y_0, n = 1, 2, 3, \ldots
\]

(3.4)

will describe the \( S \)-orbit at \( y_0 \). From again (2.1) with \( x = y_n \) and \( y = z \), we observe that

\[
[1 + \beta d(y_n, z)] d(Sy_n, z)
\]

\[
\leq \alpha \max \left\{ d(y_n, Sy_n), d(y_n, z), \frac{1}{2} \left[ d(y_n, z) + d(Sy_n, z) \right] \right\} + \beta d(y_n, z) d(Sy_n, z).
\]

Setting \( u_n = d(y_n, z) \), this can be written as

\[
u_{n+1} \leq \alpha \max \left\{ d(y_n, y_{n+1}), u_n, \frac{1}{2} [u_n + u_{n+1}] \right\} + \beta u_n u_{n+1} \text{ for all } n.
\]

(3.5)

If \( u_{m+1} > u_m \) for some positive integer \( m \), we would obtain that

\[
d(y_m, y_{m+1}) \leq d(y_n, z) + d(z, y_{m+1}) = u_m + u_{m+1} < 2u_{m+1}
\]

which together with (3.5) implies a contradiction that \( u_{m+1} \leq 2\alpha u_{m+1} < u_{m+1} \).

Hence we must have \( u_{n+1} > u_n \) for all \( n \) so that again from (3.5), we have \( u_{n+1} \leq 2\alpha u_n \) for all \( n \). Proceeding the limit as \( n \to \infty \) in this and using the choice of \( \alpha \), we find that

\[
\lim_{n \to \infty} u_n = \lim_{n \to \infty} d(y_n, z) = 0.
\]

That is, the \( S \)-orbit at \( y_0 \) converges to \( z \). Since \( y_0 \) is arbitrary point in \( X \), it follows that the common fixed point \( z \) is a contractive fixed point of \( S \).

\[\square\]

**Remark 3.2.** Writing \( \beta = 0 \) in Theorem 3.1, we get Theorem 1.2, in view of Remark 2.1.

**Remark 3.3.** When \( \beta = 0 \) and \( X \) is complete in Theorem 3.1, we get Theorem 1.1 of Akkouchi [5].

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