A Generalization of Subnexuses Based on $\mathcal{N}$-Structures

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Abstract In this paper, we generalize the concepts of $\mathcal{N}$-subnexuses of types $(\in, q)$, $(\in, \in \lor q)$ and $(q, \in \lor q)$, and introduce the notions of $\mathcal{N}$-subnexuses of types $(\in, q_k)$, $(\in, \in \lor q_k)$ and $(q, \in \lor q_k)$. We investigate their basic properties, characterize subnexuses by $\mathcal{N}$-subnexuses of type $(\in, \in \lor q_k)$, and give some characterizations for $\mathcal{N}$-subnexuses of types $(\in, q_k)$ and $(q, \in \lor q_k)$. Moreover, we define $\mathcal{N}$-subnexuses of type $(\in, \in \lor q_k)$ and discuss on their different properties.

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1. Introduction

Nexuses are a type of structure algebras which defined by M. Bolourian in [1], where some properties of them such as sub-nexuses, cyclic nexuses and homomorphism of nexuses were investigated. Next, studies from algebraic view generalized on nexuses. D. Afkhami et al. [2] defined the notion of fraction over a nexus and studied its basic properties. Moreover D. Afkhami et al. [3] defined the soft nexuses over a nexus and studied the prime and maximal soft subnexuses over a nexus. H. Hedayati et al. [4] introduced normal, maximal and product fuzzy subnexuses of a nexus. Also, about applications of nexuses can see [5] and [6].

After appearance of $(\alpha, \beta)$-fuzzy substructures, based on the concepts of belongingness and quasi-coincidence for a fuzzy point of a fuzzy subset, those defined and studied on many algebraic structures which some of them can be seen in [7–14]. On the other hand, Jun et al. [15] introduced a new function which is called negative-valued function, and constructed $\mathcal{N}$-structures, as a mathematical tool for dealing with negative information.
(beside, fuzzy sets which relied on spreading positive information). They discussed $\mathcal{N}$-subalgebras and $\mathcal{N}$-ideals in BCK/BCI/BCH-algebras (see [15–19]).

By combining the above concepts, Norouzi et al. [20] introduced the notion of a subnexus based on $\mathcal{N}$-function (briefly, $\mathcal{N}$-subnexus), and investigated related properties. They discussed characterization of $\mathcal{N}$-subnexus. They also introduced the notion of $\mathcal{N}$-subnexus of type $(\alpha, \beta)$ with 

$$\quad (\alpha, \beta) \in \{(\epsilon, \epsilon), (\epsilon, q), (\epsilon, \epsilon \lor q), (q, \epsilon), (q, q), (q, \epsilon \lor q)\},$$

and investigated their basic properties. Now, in this paper, we generalize the concepts in [20] and introduce the notion of $\mathcal{N}$-subnexus of type $(\epsilon, q_k)$, $(\epsilon, \epsilon \lor q_k)$, $(q, q_k)$, $(q, \epsilon \lor q_k)$, and also investigate basic properties of them. In this way, connection of the notions is studied. Characterizations of $\mathcal{N}$-subnexus of type $(\epsilon, \epsilon \lor q_k)$ are given. Conditions for an $\mathcal{N}$-structure to be an $\mathcal{N}$-subnexus of type $(q, \epsilon \lor q_k)$ are provided. Moreover, the notion of $\mathcal{N}$-subnexus of type $(\overline{\epsilon}, \overline{\epsilon} \lor \overline{q_k})$ is defined and some characterizations of it are established, where we can see some differences with other similar $(\alpha, \beta)$-substructures.

### 2. Preliminaries

In this section we give some definitions and results which we need to develop our paper. They have been brought of [3, 4, 21], in connection with nexuses, and [18, 19] in connection with $\mathcal{N}$-structures.

An address is a sequence of $N^* = \mathbb{N} \cup \{0\}$ such that $a_k = 0$ implies that $a_i = 0$ for all $i \geq k$. The sequence of zero is called the empty address and denoted by $(\cdot)$. In other word, every nonempty address is of the form $(a_1, a_2, \ldots, a_n, 0, 0, \ldots)$ where $n \in \mathbb{N}$, and it is denoted by $(a_1, a_2, \ldots, a_n)$.

**Definition 2.1.** A set $X$ of addresses is called a nexus if

1. $(a_1, a_2, \ldots, a_n) \in X$ implies that $(a_1, \ldots, a_{n-1}, t) \in X$ for all $0 \leq t \leq a_n$.
2. $(a_i)_{i=1}^n \in X$ implies that $(a_1, a_2, \ldots, a_n) \in X$ for all $n \in \mathbb{N}$.

**Example 2.2.** A set $X = \{(\cdot), (1, 2), (2, 3), (1, 1), (1, 2), (3, 1), (3, 2)\}$ is a nexus. But, $X' = \{(\cdot), (1, 2), (2, 2)\}$ is not a nexus since $(2, 2)$ is an element of $X'$ but $(2, 1) \notin X'$.

Let $X$ be a nexus and $w \in X$. The level of $w$, denoted by $l(w)$, is said to be:

(i) 0 if $w = (\cdot)$.
(ii) $n$ if $w = (a_1, a_2, \ldots, a_n)$ for some $a_n \in \mathbb{N}$.
(iii) $\infty$ if $w$ is an infinite sequence of $\mathbb{N}$.

**Definition 2.3.** Let $v = (a_i)$ and $w = (b_i)$ be addresses where $a_i, b_i \in \mathbb{N}$. Then $v \leq w$ if $l(v) = 0$ or one of the following cases is satisfied:

(i) If $l(v) = 1$, i.e., $v = (a_1)$ for $a_1 \in \mathbb{N}$, then $l(w) \geq 1$ and $a_1 \leq b_1$.
(ii) If $1 < l(v) < \infty$, then $l(v) \leq l(w)$ and $a_i(v) \leq b_i(w)$ and for every $1 \leq i < l(v)$ we have, $a_i = b_i$.
(iii) If $l(w) = \infty$, then $v = w$.

**Definition 2.4.** A nonempty subset $S$ of a nexus $X$ is called a subnexus of $X$ if $S$ itself is a nexus. The set of all subnexuses of $X$ is denoted by $\text{SUB}(X)$.

Note that a subset $S$ of a nexus $X$ is a subnexus of $X$ if and only if it satisfies:

$$(\forall v, w \in X)(v \leq w, w \in S \Rightarrow v \in S).$$  \hspace{1cm} (2.1)
Example 2.5. Consider a nexus 

\[ X = \{(), (1), (2), (3), (1, 1), (2, 1), (3, 1), (3, 1, 1), (3, 1, 2)\}. \]

Then \( X_1 = \{(), (1), (2), (3), (2, 1)\}, X_2 = \{(), (1), (2, 1), (2, 1)\} \) and \( X_3 = \{(), (1), (2), (3), (3, 1)\} \) are subnexuses of \( X \).

For any family \( \{a_i \mid i \in \Lambda\} \) of real numbers, we define

\[
\bigvee \{a_i \mid i \in \Lambda\} := \begin{cases} 
\max \{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\
\sup \{a_i \mid i \in \Lambda\} & \text{otherwise.}
\end{cases}
\]

\[
\bigwedge \{a_i \mid i \in \Lambda\} := \begin{cases} 
\min \{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\
\inf \{a_i \mid i \in \Lambda\} & \text{otherwise.}
\end{cases}
\]

Let \( F(X, [-1, 0]) \) be the set of all functions from the set \( X \) to \([-1, 0]\) (for briefly every element of \( F(X, [-1, 0]) \) is said to be \( \mathcal{N} \)-function on \( X \)). An \( \mathcal{N} \)-structure is a pair \((X, f)\) of \( X \) and an \( \mathcal{N} \)-function \( f \) on \( X \). For any \( \mathcal{N} \)-structure \((X, f)\) and \( \alpha \in [-1, 0) \), the set \( C(f; \alpha) = \{x \in X \mid f(x) \leq \alpha\} \) is called the closed support of \((X, f)\) related to \( \alpha \), and the set \( O(f; \alpha) = \{x \in X \mid f(x) < \alpha\} \) is said to be the open support of \((X, f)\) related to \( \alpha \).

Let \( \alpha \in [-1, 0) \) and \((X, f)\) be an \( \mathcal{N} \)-structure in which \( f \) is given by

\[
f(y) = \begin{cases} 
0 & \text{if } y \neq x, \\
\alpha & \text{if } y = x.
\end{cases}
\]

In this case, \( f \) is denoted by \( x_\alpha \), and \((X, x_\alpha)\) is said to be a point \( \mathcal{N} \)-structure with support \( x \) and value \( \alpha \). For any \( \mathcal{N} \)-structure \((X, g)\), we say that a point \( \mathcal{N} \)-structure \((X, x_\alpha)\) is an \( \mathcal{N}_\varepsilon \)-subset (resp. \( \mathcal{N}_q \)-subset) of \((X, g)\) if \( g(x) \leq \alpha \) (resp. \( g(x) + \alpha + 1 < 0 \)). If a point \( \mathcal{N} \)-structure \((X, x_\alpha)\) is an \( \mathcal{N}_\varepsilon \)-subset or an \( \mathcal{N}_q \)-subset of \((X, g)\), then we say \((X, x_\alpha)\) is an \( \mathcal{N}_{\varepsilon \lor q} \)-subset of \((X, g)\).

3. A Generalization of Subnexuses by \( \mathcal{N} \)-Function

In what follows, let \( X \) and \((\alpha, k)\) be a nexus and an arbitrary element of \([-1, 0) \times (-1, 0]\), respectively, unless otherwise specified.

For any \( \mathcal{N} \)-structure \((X, g)\), we say that a point \( \mathcal{N} \)-structure \((X, x_\alpha)\) is an \( \mathcal{N}_q \)-subset of \((X, g)\) if \( g(x) + \alpha + k + 1 < 0 \). If a point \( \mathcal{N} \)-structure \((X, x_\alpha)\) is an \( \mathcal{N}_\varepsilon \)-subset or an \( \mathcal{N}_{q \lor q} \)-subset of \((X, g)\), then we say \((X, x_\alpha)\) is an \( \mathcal{N}_{\varepsilon \lor q} \)-subset of \((X, g)\).

Definition 3.1 ([20]). By a subnexus of \( X \) based on \( \mathcal{N} \)-function \( f \) (briefly, \( \mathcal{N} \)-subnexus of \( X \)), we mean an \( \mathcal{N} \)-structure \((X, f)\) in which \( f \) satisfies the following assertion:

\[
(\forall v, w \in X) (w \leq v \Rightarrow f(w) \leq f(v)). 
\]

Definition 3.2 ([20]). An \( \mathcal{N} \)-subnexus \((X, f)\) is said to be of type

(i) \((\varepsilon, \varepsilon)\) (resp., \((\varepsilon, q)\) and \((\varepsilon, \varepsilon \lor q)\)) if whenever the point \( \mathcal{N} \)-structure \((X, w_\alpha)\) is an \( \mathcal{N}_\varepsilon \)-subset of \((X, f)\) then the point \( \mathcal{N} \)-structure \((X, v_\alpha)\) is an \( \mathcal{N}_\varepsilon \)-subset (resp., \( \mathcal{N}_q \)-subset and \( \mathcal{N}_{\varepsilon \lor q} \)-subset) of \((X, f)\) for all \( v, w \in X \) with \( v \leq w \).

(ii) \((q, \varepsilon)\) (resp., \((q, q)\) and \((q, \varepsilon \lor q)\)) if whenever the point \( \mathcal{N} \)-structure \((X, w_\alpha)\) is an \( \mathcal{N}_q \)-subset of \((X, f)\) then the point \( \mathcal{N} \)-structure \((X, v_\alpha)\) is an \( \mathcal{N}_\varepsilon \)-subset (resp., \( \mathcal{N}_q \)-subset and \( \mathcal{N}_{\varepsilon \lor q} \)-subset) of \((X, f)\) for all \( v, w \in X \) with \( v \leq w \).
Definition 3.3. An \( N \)-subnexus \((X, f)\) is said to be of type
- \( (\varepsilon, q_k) \) if whenever the point \( N \)-structure \((X, w_\alpha)\) is an \( N_\varepsilon \)-subset of \((X, f)\) then the point \( N \)-structure \((X, v_\alpha)\) is an \( N_{\varepsilon q_k} \)-subset of \((X, f)\) for all \( v, w \in X \) with \( v \leq w \).
- \( (\varepsilon, \in \lor q_k) \) if whenever the point \( N \)-structure \((X, w_\alpha)\) is an \( N_\varepsilon \)-subset of \((X, f)\) then the point \( N \)-structure \((X, v_\alpha)\) is an \( N_{\varepsilon \lor q_k} \)-subset of \((X, f)\) for all \( v, w \in X \) with \( v \leq w \).
- \( (q, \in \lor q_k) \) if whenever the point \( N \)-structure \((X, w_\alpha)\) is an \( N_q \)-subset of \((X, f)\) then the point \( N \)-structure \((X, v_\alpha)\) is an \( N_{q \lor q_k} \)-subset of \((X, f)\) for all \( v, w \in X \) with \( v \leq w \).

Example 3.4. Let \((X, f)\) be an \( N \)-structure in which
\[ X = \{(), (1), (2), (1, 1), (1, 2), (1, 3), (1, 3, 1), (1, 3, 2)\} \]
is a nexus and \( f \) is defined as follows:
\[
f = \begin{pmatrix}
(1) & (2) & (1, 1) & (1, 2) & (1, 3) & (1, 3, 1) & (1, 3, 2)
\end{pmatrix}
\]
Put \( k = -0.75 \). It is easy to see that in the nexus \( X \) we have
\[
(1) \leq (2), (1, 1), (1, 2), (1, 3), (1, 3, 1), (1, 3, 2)
(1, 1) \leq (1, 2), (1, 3), (1, 3, 1), (1, 3, 2)
(1, 2) \leq (1, 3), (1, 3, 1), (1, 3, 2)
(1, 3) \leq (1, 3, 1), (1, 3, 2)
(1, 3, 1) \leq (1, 3, 2).
\]
Since, \( () \leq v \) and \( f() \leq f(v) \) for all \( v \in X \), clearly if \((X, w_\alpha)\) is an \( N_\varepsilon \)-subset of \((X, f)\) then \((X, (\alpha)_w)\) is an \( N_\varepsilon \)-subset of \((X, f)\) for all \( \alpha \in [-1, 0] \). For \( (1) \leq (2) \) we have \( f(2) = -0.93 < \beta \) and \( f(1) = -0.9 \not\leq \beta \) for all \( \beta \in (-0.93, -0.9) \), but \( f(1) + \beta + 0.75 + 1 < 0 \). This means that if \((X, (2)_{\beta})\) is an \( N_\varepsilon \)-subset of \((X, f)\) then \((X, (1)_{\beta})\) is an \( N_{q \lor 0.75} \)-subset of \((X, f)\). For \((1, 1) \leq (1, 2) \), since \( f(1, 1) \leq f(1, 2) \), if \((X, (1, 2)_{\alpha})\) is an \( N_\varepsilon \)-subset of \((X, f)\) then \((X, (1, 1)_{\alpha})\) is an \( N_\varepsilon \)-subset of \((X, f)\) for all \( \alpha \in (-0.94, 0) \). For \((1, 1) \leq (1, 3) \) and \( \beta \in (-0.96, -0.95) \), if \((X, (1, 3)_{\beta})\) is an \( N_\varepsilon \)-subset of \((X, f)\) then \((X, (1, 1)_{\beta})\) is an \( N_{q \lor 0.75} \)-subset of \((X, f)\) while is not an \( N_\varepsilon \)-subset of \((X, f)\). By a similar manner, we can see the related implication is valid for all other cases. Therefore, \((X, f)\) is an \( N \)-subnexus of type \((\varepsilon, \in \lor q_k)\) with \( k = -0.75 \).

Example 3.5. Consider the nexus \( X = \{(), (1), (1, 1), (1, 2), (1, 3)\} \) with an \( N \)-function \( f \) is defined as follows:
\[
f = \begin{pmatrix}
() & (1) & (1, 1) & (1, 2) & (1, 3)
\end{pmatrix}
\]
It is easy to see that \((X, f)\) is an \( N \)-subnexus of type \((\varepsilon, q_k)\) with \( k = -0.4 \).

Example 3.6. Define an \( N \)-function \( g \) on the set \( X = \{(), (1), (2), (2, 1), (2, 2)\} \) as:
\[
g = \begin{pmatrix}
() & (1) & (2) & (2, 1) & (2, 2)
\end{pmatrix}
\]
Then \((X, g)\) is an \( N \)-subnexus of type \((q, \in \lor q_k)\) with \( k = -0.1 \).
We note that every $\mathcal{N}$-subnexus of type $(\varepsilon, q_k)$ (resp., $(\varepsilon, \varepsilon \lor q_k)$ and $(q, \varepsilon \lor q_k)$) with $k = 0$ is an $\mathcal{N}$-subnexus of type $(\varepsilon, q)$ (resp., $(\varepsilon, \varepsilon \lor q)$ and $(q, \varepsilon \lor q)$). But the converse is not true in general as seen in the following example.

**Example 3.7.** (1) Consider the nexus $X = \{(), (1), (2), (1,1), (1,2)\}$ and the $\mathcal{N}$-function $f$ on $X$ defined as

$$f = \begin{pmatrix}
    () & (1) & (2) & (1,1) & (1,2) \\
   -1 & -0.7 & -0.73 & -0.74 & -0.75
\end{pmatrix}.$$ 

It can be seen that $(X, f)$ is an $\mathcal{N}$-subnexus of type $(\varepsilon, \varepsilon \lor q)$ which is not of type $(\varepsilon, \varepsilon \lor q_k)$ for $k = -0.75$. Indeed, we have $(1) \leq (2)$ and $(X, (2)_{-0.73})$ is an $\mathcal{N}_\varepsilon$-subset of $(X, f)$, but $f((1)) \not\leq -0.73$ and $f((1)) - 0.73 - k + 1 = 0.32 \not\leq 0$. This implies that $(X, (1)_{-0.73})$ is not an $\mathcal{N}_{\varepsilon \lor q_k}$-subset of $(X, f)$ and so $(X, f)$ is not an $\mathcal{N}$-subnexus of type $(\varepsilon, \varepsilon \lor q_k)$ for $k = -0.75$.

(2) Let $(X, f)$ be an $\mathcal{N}$-structure in which $X = \{(), (1), (1,1), (1,2)\}$ is a nexus and $g$ is defined as follows:

$$g = \begin{pmatrix}
    () & (1) & (1,1) & (1,2) \\
   -0.64 & -0.62 & -0.63 & -0.71
\end{pmatrix}.$$ 

Then $(X, g)$ is an $\mathcal{N}$-subnexus of type $(\varepsilon, q)$, but $(X, g)$ is not an $\mathcal{N}$-subnexus of type $(\varepsilon, q_k)$ for $k = -0.4$, since $(1,1) \leq (1,2)$, $g((1,2)) \leq -0.71$, $g((1,1)) \not\leq -0.71$ and $g((1,1)) - 0.71 + 0.4 + 1 \not\leq 0$.

(3) An $\mathcal{N}$-structure $(X, f)$ in which $X = \{(), (1), (2), (1,2)\}$ is a nexus and $f$ is defined as follows:

$$f = \begin{pmatrix}
    () & (1) & (2) & (1,2) \\
   -0.9 & -0.91 & -0.92 & -0.93
\end{pmatrix}.$$ 

is an $\mathcal{N}$-subnexus of type $(q, \varepsilon \lor q)$, but is not an $\mathcal{N}$-subnexus of type $(q, \varepsilon \lor q_k)$ for $k = -0.9$.

In the following, we give some characterizations for an $\mathcal{N}$-subnexus of type $(\varepsilon, \varepsilon \lor q_k)$.

**Theorem 3.8.** An $\mathcal{N}$-subnexus $(X, f)$ is of type $(\varepsilon, \varepsilon \lor q_k)$ if and only if the following assertion is valid.

$$(\forall v, w \in X) \left( v \leq w \Rightarrow f(v) \leq \bigvee \left\{ f(w), \frac{k - 1}{2} \right\} \right).$$  \hspace{1cm} (3.2)

**Proof.** Suppose that $(X, f)$ is an $\mathcal{N}$-subnexus of type $(\varepsilon, \varepsilon \lor q_k)$. For any $v, w \in X$, assume that $v \leq w$ and $f(w) > \frac{k - 1}{2}$. If $f(v) > f(w)$, then there exists $\beta \in [-1,0)$ such that $f(v) > \beta \geq f(w)$. Thus the point $\mathcal{N}$-structure $(X, w_\beta)$ is an $\mathcal{N}_\varepsilon$-subset of $(X, f)$, but the point $\mathcal{N}$-structure $(X, v_\beta)$ is not an $\mathcal{N}_\varepsilon$-subset of $(X, f)$. Also

$$f(v) + \beta - k + 1 > 2\beta - k + 1 \geq 2f(w) - k + 1 > 0,$$

and so $(X, v_\beta)$ is not an $\mathcal{N}_{q_k}$-subset of $(X, f)$. Therefore $(X, v_\beta)$ is not an $\mathcal{N}_{\varepsilon \lor q_k}$-subset of $(X, f)$, which is a contradiction. Hence $f(v) \leq f(w)$ whenever $f(w) > \frac{k - 1}{2}$. Now, suppose that $f(w) \leq \frac{k - 1}{2}$. Then the point $\mathcal{N}$-structure $(X, w_{\frac{k - 1}{2}})$ is an $\mathcal{N}_\varepsilon$-subset of $(X, f)$ and so $(X, v_{\frac{k - 1}{2}})$ is an $\mathcal{N}_{\varepsilon \lor q}$-subset of $(X, f)$ by hypothesis. If $(X, v_{\frac{k - 1}{2}})$ is an $\mathcal{N}_\varepsilon$-subset of $(X, f)$ then $f(v) \leq \frac{k - 1}{2}$ and so $f(v) \leq \bigvee \{ f(w), \frac{k - 1}{2} \}$. If $(X, v_{\frac{k - 1}{2}})$ is an
If follows from Theorem 3.8 that $f(x) \le \sqrt{f(w) + \left(\frac{k-1}{2}\right)^2} - k + 1 < 0$, that is, $f(x) < \frac{k-1}{2}$. Consequently $f(v) \le \sqrt{f(w) + \left(\frac{k-1}{2}\right)^2}$.

Conversely, assume that (3.2) is valid. Let $v, w \in X$ and $\beta \in [-1,0)$ be such that $v \le w$ and the point $\mathcal{N}$-structure $(X, w_\beta)$ is an $\mathcal{N}_c$-subset of $(X, f)$. If $f(v) \le \beta$, then the point $\mathcal{N}$-structure $(X, v_\beta)$ is an $\mathcal{N}_c$-subset of $(X, f)$. Suppose that $f(v) > \beta$. Then $f(w) \le \beta < f(v) \le \sqrt{\{f(w), \frac{k-1}{2}\}}$, and therefore $\sqrt{\{f(w), \frac{k-1}{2}\}} = \frac{k-1}{2}$. It follows that $f(v) + \beta - k + 1 < 2f(v) - k + 1 \le 2\left(\sqrt{\{f(w), \frac{k-1}{2}\}}\right) - k + 1 = 0$.

Thus $(X, v_\beta)$ is an $\mathcal{N}_{q_k}$-subset of $(X, f)$. Consequently $(X, v_\beta)$ is an $\mathcal{N}_{\mathcal{N}_c \cup q_k}$-subset of $(X, f)$ and thus $(X, f)$ is an $\mathcal{N}$-subnexus of type $(\mathcal{N}, \mathcal{N}_c \cup q_k)$.

**Corollary 3.9 ([20]).** An $\mathcal{N}$-subnexus $(X, f)$ is of type $(\mathcal{N}, \mathcal{N}_c \cup q_k)$ if and only if the following assertion is valid.

$$(\forall v, w \in X) \left(v \le w \Rightarrow f(v) \le \sqrt{\{f(w), -0.5\}}\right).$$

**Proposition 3.10.** If $(X, f)$ is an $\mathcal{N}$-subnexus of type $(\mathcal{N}, \mathcal{N}_c \cup q_k)$, then

$$(\forall v \in X) \left(f(v) \le \sqrt{\{f(v), \frac{k-1}{2}\}}\right).$$

**Proof.** Since $(\mathcal{N}, \mathcal{N}_c \cup q_k)$ is an $\mathcal{N}$-subnexus, it is straightforward.

**Corollary 3.11.** If $(X, f)$ is an $\mathcal{N}$-subnexus of type $(\mathcal{N}, \mathcal{N}_c \cup q_k)$, then

$$(\forall v \in X) (f(v)) \le \sqrt{\{f(v), -0.5\}}.$$  

**Theorem 3.12.** Let $(X, f)$ be an $\mathcal{N}$-subnexus of type $(\mathcal{N}, \mathcal{N}_c \cup q_k)$. Then

1. if there exists $x \in X$ such that $f(x) \le \frac{k-1}{2}$, then $f(x) \le \frac{k-1}{2}$.
2. if $f(\)) > \frac{k-1}{2}$, then $(X, f)$ is an $\mathcal{N}$-subnexus of type $(\mathcal{N}, \mathcal{N}_c)$.

**Proof.** (1) Assume that there exists $x \in X$ such that $f(x) \le \frac{k-1}{2}$. If $x = ()$, it is true. If $x \neq ()$, then $f(\)) \le \sqrt{\{f(x), \frac{k-1}{2}\}} = \frac{k-1}{2}$ by Theorem 3.8 and hypothesis.

(2) Suppose that $f(\)) > \frac{k-1}{2}$ and $f(v) > f(w)$ for all $v, w \in X$ with $v \le w$. It follows from Theorem 3.8 that $f(w) \le \sqrt{\{f(w), \frac{k-1}{2}\}} = \frac{k-1}{2}$. Since $(\mathcal{N}, \mathcal{N}_c)$, we have $f(\)) \ge \sqrt{\{f(w), \frac{k-1}{2}\}} = \frac{k-1}{2}$. This is a contradiction, and hence $f(v) \le f(w)$ for all $v, w \in X$ with $v \le w$. Therefore $(X, f)$ is an $\mathcal{N}$-subnexus of type $(\mathcal{N}, \mathcal{N}_c)$.

**Corollary 3.13 ([20]).** Let $(X, f)$ be an $\mathcal{N}$-subnexus of type $(\mathcal{N}, \mathcal{N}_c \cup q_k)$. Then

1. if there exists $x \in X$ such that $f(x) \le -0.5$, then $f(\)) \le -0.5$.
2. if $f(\)) + 0.5 > 0$, then $(X, f)$ is an $\mathcal{N}$-subnexus of type $(\mathcal{N}, \mathcal{N}_c)$.

**Theorem 3.14.** If $-1 < k < r \le 0$, then every $\mathcal{N}$-subnexus of type $(\mathcal{N}, \mathcal{N}_c \cup q_k)$ is an $\mathcal{N}$-subnexus of type $(\mathcal{N}, \mathcal{N}_c \cup q_r)$. 


Proof. Let \((X, f)\) be an \(N\)-subnexus of type \((\in, \in \vee q_r)\). Then
\[
f(v) \leq \bigvee \left\{ f(w), \frac{k-1}{2} \right\} \leq \bigvee \left\{ f(w), \frac{r-1}{2} \right\},
\]
for all \(v, w \in X\) with \(v \leq w\). It follows from Theorem 3.8 that \((X, f)\) is an \(N\)-subnexus of type \((\in, \in \vee q_r)\).

The following example shows that if \(-1 < k < r \leq 0\), then an \(N\)-subnexus of type \((\in, \in \vee q_r)\) may not be an \(N\)-subnexus of type \((\in, \in \vee q_k)\).

Example 3.15. The \(N\)-structure \((X, f)\) defined in Example 3.4 is an \(N\)-subnexus of type \((\in, \in \vee q_r)\) for \(r = -0.75\), but it is not an \(N\)-subnexus of type \((\in, \in \vee q_k)\) for \(k = -0.9\). Indeed, \((1) \leq (2)\), but \(f((1)) = -0.9 \not\leq -0.93 = \bigvee \{-0.93, -0.95\} = \bigvee \{f((2)), \frac{k-1}{2}\}\).

Theorem 3.16. An \(N\)-structure \((X, f)\) is an \(N\)-subnexus of type \((\in, \in \vee q_k)\) if and only if for every \(\alpha \in [\frac{k-1}{2}, 0]\) the nonempty closed support of \((X, f)\) related to \(\alpha\) is a subnexus of \(X\).

Proof. Assume that \((X, f)\) is an \(N\)-subnexus of type \((\in, \in \vee q_k)\) and let \(\alpha \in [\frac{k-1}{2}, 0]\) such that \(C(f; \alpha) \neq \emptyset\). Let \(v \leq w\) and \(w \in C(f; \alpha)\). Then \(f(v) \leq \bigvee \{f(w), \frac{k-1}{2}\}\) by Theorem 3.8. If \(\bigvee \{f(w), \frac{k-1}{2}\} = f(w)\), then \(f(v) \leq f(w) \leq \alpha\) and thus \(v \in C(f; \alpha)\). Also, if \(\bigvee \{f(w), \frac{k-1}{2}\} = \frac{k-1}{2}\), then \(f(v) \leq \frac{k-1}{2} \leq \alpha\), and thus \(v \in C(f; \alpha)\). Hence \(C(f; \alpha)\) is a subnexus of \(X\).

Conversely, let \((X, f)\) be an \(N\)-structure such that the nonempty closed support of \((X, f)\) related to \(\alpha\) is a subnexus of \(X\) for all \(\alpha \in [\frac{k-1}{2}, 0]\). If there exist \(v, w \in X\) such that \(v \leq w\) and \(f(v) > \bigvee \{f(w), \frac{k-1}{2}\}\), then we can take \(\beta \in [-1, 0]\) such that \(f(v) > \beta \geq \bigvee \{f(w), \frac{k-1}{2}\}\). Thus \(w \in C(f; \beta)\) and \(\beta \geq \frac{k-1}{2}\). Since \(C(f; \beta)\) is a subnexus of \(X\), we have \(v \in C(f; \beta)\). Hence \(f(v) \leq \beta\), a contradiction. Therefore \(f(v) \leq \bigvee \{f(w), \frac{k-1}{2}\}\) for all \(v, w \in X\). It follows from Theorem 3.8 that \((X, f)\) is an \(N\)-subnexus of type \((\in, \in \vee q_k)\).

Corollary 3.17 ([20]). An \(N\)-structure \((X, f)\) is an \(N\)-subnexus of type \((\in, \in \vee q)\) if and only if for every \(\alpha \in [-0.5, 0]\) the nonempty closed support of \((X, f)\) related to \(\alpha\) is a subnexus of \(X\).

Theorem 3.18. Let \(S\) be a subnexus of \(X\). For any \(\alpha \in (\frac{k-1}{2}, 0)\), there exists an \(N\)-subnexus of type \((\in, \in \vee q_k)\) for which \(S\) is represented by the closed support of \((X, f)\) related to \(\alpha\).

Proof. Let \((X, f)\) be an \(N\)-structure in which \(f\) is given by
\[
f(x) = \begin{cases} 
\alpha & \text{if } x \in S, \\
0 & \text{if } x \notin S,
\end{cases}
\]
for all \(x \in X\) where \(\alpha \in (\frac{k-1}{2}, 0)\). Assume that \(f(\nu) > \bigvee \{f(\omega), \frac{k-1}{2}\}\) for some \(\nu, \omega \in X\) such that \(\nu \leq \omega\). By \(|Im(f)| = 2\), it follows that \(f(\nu) = 0\) and \(\bigvee \{f(\omega), \frac{k-1}{2}\} = \alpha\). Since \(\alpha > \frac{k-1}{2}\), we have \(f(\omega) = \alpha\) and so \(\omega \in S\). Since \(S\) is a subnexus of \(X\), we obtain \(\nu \in S\) and thus \(f(\nu) = \alpha < 0\), which is a contradiction. Therefore \(f(\nu) \leq \bigvee \{f(\omega), \frac{k-1}{2}\}\) for all \(\nu, \omega \in X\). Hence, \((X, f)\) is an \(N\)-subnexus of type \((\in, \in \vee q_k)\), by Theorem 3.8. Obviously, \(S\) is represented by the closed support of \((X, f)\) related to \(\alpha\).
Corollary 3.19 ([20]). Let $S$ be a subnexus of $X$. For any $\alpha \in (0, 1)$, there exists an $\mathcal{N}$-subnexus of type $(\varepsilon, \varepsilon \lor q)$ for which $S$ is represented by the closed support of $(X, f)$ related to $\alpha$.

Note that every $\mathcal{N}$-subnexus of type $(\varepsilon, \varepsilon)$ is an $\mathcal{N}$-subnexus of type $(\varepsilon, \varepsilon \lor q)$, but the converse is not true in general as seen in the following example.

Example 3.20. The $\mathcal{N}$-structure $(X, f)$ defined in Example 3.4 is an $\mathcal{N}$-subnexus of type $(\varepsilon, \varepsilon \lor q)$, but is not an $\mathcal{N}$-subnexus of type $(\varepsilon, \varepsilon)$, since $(1) \leq (2)$ but $f((1)) = -0.9 \not\leq -0.93 = f((2))$.

Now, we give a condition for an $\mathcal{N}$-subnexus of type $(\varepsilon, \varepsilon \lor q)$ to be an $\mathcal{N}$-subnexus of type $(\varepsilon, \varepsilon)$.

Theorem 3.21. Let $(X, f)$ be an $\mathcal{N}$-subnexus of type $(\varepsilon, \varepsilon \lor q)$ such that $f(x) \geq \frac{k-1}{2}$ for all $x \in X$. Then $(X, f)$ is an $\mathcal{N}$-subnexus of type $(\varepsilon, \varepsilon)$.

Proof. Let $v, w \in X$ such that $v \leq w$ and $(X, w_\alpha)$ is an $\mathcal{N}_{\varepsilon}$-subset of $(X, f)$ for $\alpha \in [-1, 0)$. Then $f(w) \leq \alpha$. It follows from Theorem 3.8 and the hypothesis that

$$f(v) \leq \sqrt{f(w), \frac{k-1}{2}} = f(w) \leq \alpha.$$ 

Thus $(X, v_\alpha)$ is an $\mathcal{N}_{\varepsilon}$-subset of $(X, f)$. Therefore $(X, f)$ is an $\mathcal{N}$-subnexus of type $(\varepsilon, \varepsilon)$.

For any $\mathcal{N}$-structure $(X, f)$ and $\alpha \in [-1, 0)$, the $q_k$-support and the $\varepsilon \lor q_k$-support of $(X, f)$ related to $\alpha$ are defined as follow

$$\mathcal{N}_{q_k}(f; \alpha) = \{x \in X \mid (X, x_\alpha) is an \mathcal{N}_{q_k}-subset of (X, f)\},$$

and

$$\mathcal{N}_{\varepsilon \lor q_k}(f; \alpha) = \{x \in X \mid (X, x_\alpha) is an \mathcal{N}_{\varepsilon \lor q_k}-subset of (X, f)\}.$$

Note that the $\varepsilon \lor q_k$-support is the union of the closed support and the $q_k$-support, that is, $\mathcal{N}_{\varepsilon \lor q_k}(f; \alpha) = C(f; \alpha) \cup \mathcal{N}_{q_k}(f; \alpha)$.

Theorem 3.22. An $\mathcal{N}$-structure $(X, f)$ is an $\mathcal{N}$-subnexus of type $(\varepsilon, \varepsilon \lor q_k)$ if and only if the $\varepsilon \lor q_k$-support of $(X, f)$ related to $\alpha$ is a subnexus of $X$ for all $\alpha \in [-1, 0)$.

Proof. Suppose that $(X, f)$ is an $\mathcal{N}$-subnexus of type $(\varepsilon, \varepsilon \lor q_k)$. Let $v, w \in X$ such that $v \leq w$ and $w \in \mathcal{N}_{\varepsilon \lor q_k}(f; \alpha)$ for $\alpha \in [-1, 0)$, $k \in (-1, 0]$. So $(X, w_\alpha)$ is an $\mathcal{N}_{\varepsilon \lor q_k}$-subset of $(X, f)$. Thus $f(w) \leq \alpha$ or $f(w) + \alpha - k + 1 < 0$. If $f(w) \leq \alpha$, then $(X, w_\alpha)$ is an $\mathcal{N}_{\varepsilon}$-subset of $(X, f)$. By hypothesis $(X, v_\alpha)$ is an $\mathcal{N}_{\varepsilon \lor q_k}$-subset of $(X, f)$ and so $v \in \mathcal{N}_{\varepsilon \lor q_k}(f; \alpha)$. If $f(w) + \alpha - k + 1 < 0$, we consider the following two cases:

If $\alpha \geq \frac{k-1}{2}$, then by hypothesis and Theorem 3.8, $f(v) \leq \sqrt{f(w), \frac{k-1}{2}} = \frac{k-1}{2} \leq \alpha$. So $(X, v_\alpha)$ is an $\mathcal{N}_{\varepsilon \lor q_k}$-subset of $(X, f)$ and so $v \in \mathcal{N}_{\varepsilon \lor q_k}(f; \alpha)$. If $\alpha < \frac{k-1}{2}$, then we have:

i) If $\sqrt{f(w), \frac{k-1}{2}} = \frac{k-1}{2}$, then by hypothesis and Theorem 3.8,

$$f(v) + \alpha - k + 1 < f(v) + \frac{1-k}{2} \leq \sqrt{f(w), \frac{k-1}{2}} + \frac{1-k}{2} = 0.$$

ii) If $\sqrt{f(w), \frac{k-1}{2}} = f(w)$, then

$$f(v) + \alpha - k + 1 \leq \sqrt{f(w), \frac{k-1}{2}} + \alpha - k + 1 = f(w) + \alpha - k + 1 < 0.$$
Thus in each case we have \( f(v) + \alpha - k + 1 < 0 \) and so \((X, v_\alpha)\) is an \( N_{qk}\)-subset of \((X, f)\). Consequently \( v \in N_{\in \cup qk}(f; \alpha) \). Therefore \( N_{\in \cup qk}(f; \alpha) \) is a subnexus of \( X \) for all \( \alpha \in [-1,0) \), \( k \in (-1,0] \).

Conversely, let \((X, f)\) be an \( N\)-structure for which \((\in \cup qk)\)-support of \((X, f)\) related to \( \alpha \) is a subnexus of \( X \) for all \( \alpha \in [-1,0) \) and \( k \in (-1,0] \). Assume that there exists \( v, w \in X \) such that \( v \leq w \) and \( f(v) > \sqrt{\{f(w), k^{-1}_2\}} \). Then \( f(v) > \beta \geq \sqrt{\{f(w), k^{-1}_2\}} \) for some \( \beta \in \{k^{-1}_2,0\} \). It follows that \( w \in C(f; \beta) \subseteq N_{\in \cup qk}(f; \beta) \) but \( v \notin C(f; \beta) \). Also \( f(v) + \beta - k + 1 > 2\beta - k + 1 \geq 0 \), that is \( v \notin N_{qk}(f; \beta) \). Thus \( v \notin N_{\in \cup qk}(f; \beta) \) which is a contradiction. Therefore \( f(v) \leq \sqrt{\{f(w), k^{-1}_2\}} \) for all \( v, w \in X \). Using Theorem 3.8, then \((X, f)\) is an \( N\)-subnexus of type \((\in, \in \cup qk)\).

**Theorem 3.23.** If \((X, f)\) is an \( N\)-subnexus of type \((\in, qk)\), then the set

\[
O(f; k) := \{x \in X \mid f(x) < k\}
\]

is a subnexus of \(X\).

**Proof.** Let \((X, f)\) be an \( N\)-subnexus of type \((\in, qk)\) and \( v, w \in X \) such that \( v \leq w \) and \( w \in O(f; k) \). Note that \((X, w_{f(w)})\) is an \( N_{\in}\)-subset of \((X, f)\). If \( f(v) \geq k \), then \( f(v) + f(w) - k + 1 \geq k + f(w) - k + 1 \geq f(w) + 1 \geq 0 \). Thus \((X, v_{f(w)})\) is not an \( N_{qk}\)-subset of \((X, f)\), a contradiction. Hence \( f(v) < k \), that is, \( v \in O(f; k) \). Hence \( O(f; k) \) is a subnexus of \(X\).

**Corollary 3.24 ([20]).** If \((X, f)\) is an \( N\)-subnexus of type \((\in, q)\), then the open support of \((X, f)\) relative \(0\) is a subnexus of \(X\).

Now, we provide conditions for an \( N\)-structure to be an \( N\)-subnexus of type \((q, \in \cup qk)\).

**Theorem 3.25.** Let \( S \) be a subnexus of \( X \) and let \((X, f)\) be an \( N\)-structure such that

\[
\begin{align*}
(1) & \quad (\forall x \in X)(x \in S \Rightarrow f(x) \leq \frac{k-1}{2}), \\
(2) & \quad (\forall x \in X)(x \notin S \Rightarrow f(x) = 0).
\end{align*}
\]

Then \((X, f)\) is an \( N\)-subnexus of type \((q, \in \cup qk)\).

**Proof.** Let \( v, w \in X \) with \( v \leq w \) and \((\alpha, k) \in [-1,0) \times (-1,0] \) be such that the point \( N\)-structure \((X, w_\alpha)\) is an \( N_q\)-subset of \((X, f)\). Then \( f(w) + \alpha + 1 < 0 \). It implies that \( v \in S \) since if \( v \notin S \), then \( w \notin S \). Thus \( f(w) = 0 \) and so \( \alpha + 1 = f(w) + \alpha + 1 < 0 \), that is, \( \alpha < -1 \), this is a contradiction. Therefore \( f(v) \leq \frac{k-1}{2} \). If \( \alpha < \frac{k-1}{2} \), then \( f(v) + \alpha - k + 1 < \frac{k-1}{2} + \frac{k-1}{2} - k + 1 = 0 \) and thus the point \( N\)-structure \((X, v_\alpha)\) is an \( N_{qk}\)-subset of \((X, f)\). If \( \alpha \geq \frac{k-1}{2} \), then \( f(v) \leq \frac{k-1}{2} \leq \alpha \) and so the point \( N\)-structure \((X, v_\alpha)\) is an \( N_{\in \cup qk}\)-subset of \((X, f)\). Thus the point \( N\)-structure \((X, v_\alpha)\) is an \( N_{\in \cup qk}\)-subset of \((X, f)\), and therefore \((X, f)\) is an \( N\)-subnexus of type \((q, \in \cup qk)\).

**Corollary 3.26 ([20]).** Let \( S \) be a subnexus of \( X \) and let \((X, f)\) be an \( N\)-structure such that

\[
\begin{align*}
(1) & \quad (\forall x \in X)(x \in S \Rightarrow f(x) \leq -0.5), \\
(2) & \quad (\forall x \in X)(x \notin S \Rightarrow f(x) = 0).
\end{align*}
\]

Then \((X, f)\) is an \( N\)-subnexus of type \((q, \in q)\).
Theorem 3.27. Let \((X, f)\) be an \(N\)-subnexus of type \((q, \in \forall q_k)\). If \(f\) is not constant on \(O(f; k)\) and \(f() \geq f(x)\) for all \(x \in X\), then there exists \(y \in X\) such that \(f(y) \leq \frac{k-1}{2}\). In particular, \(f() \leq \frac{k-1}{2}\).

Proof. Assume that \(f(x) > \frac{k-1}{2}\) for all \(x \in X\). Since \(f\) is not constant on \(O(f; k)\), there exists \(y \in O(f; k)\) such that \(f(y) \neq f() = \alpha_0\). Then \(\alpha_0 > \alpha_y\). Choose \(\beta < \frac{k-1}{2}\) such that \(\alpha_0 + \beta - k + 1 > 0 > \alpha_y + \beta + 1\). Then the point \(N\)-structure \((X, y_\beta)\) is an \(N_q\)-subset of \((X, f)\). Since \((\alpha) \leq y\), it follows that \((X, (\beta))\) is an \(N_\forall q_k\)-subset of \((X, f)\). But \(f() > \frac{k-1}{2} > \beta\) implies that the point \(N\)-structure \((X, y_\beta)\) is not an \(N\)-subset of \((X, f)\). Also \(f() + \beta - k + 1 = \alpha_0 + \beta - k + 1 > 0\) implies that \((X, (\beta))\) is not an \(N_{q_k}\)-subset of \((X, f)\). This is a contradiction, and thus \(f(y) \leq \frac{k-1}{2}\) for some \(y \in X\).

We now prove that \(f() \leq \frac{k-1}{2}\). Assume that \(\alpha_0 := f() > \frac{k-1}{2}\). Note that there exists \(y \in X\) such that \(\alpha_y := f(y) \leq \frac{k-1}{2}\) and so \(\alpha_y < \alpha_0\). Choose \(\alpha_1 < \alpha_0\) such that \(\alpha_y + \alpha_1 - k + 1 < 0 < \alpha_0 + \alpha_1 - k + 1\). Then \(f(y) + \alpha_1 - k + 1 = \alpha_y + \alpha_1 - k + 1 < 0\), and thus the point \(N\)-structure \((X, y_\alpha_1)\) is an \(N_q\)-subset of \((X, f)\). Since \((\alpha) \leq y\), we know that \((X, (\alpha))\) is an \(N_\forall q_k\)-subset of \((X, f)\). But \(f() + \alpha_1 - k + 1 = \alpha_0 + \alpha_1 - k + 1 > 0\) and also \(f() = \alpha_0 > \alpha_1\) which is a contradiction. Therefore \(f() \leq \frac{k-1}{2}\).

Corollary 3.28 ([20]). Let \((X, f)\) be an \(N\)-subnexus of type \((q, \in \forall q)\). If \(f\) is not constant on the open support of \((X, f)\) related to 0 and \(f() \geq f(x)\) for all \(x \in X\), then there exists \(y \in X\) such that \(f(y) \leq -0.5\). In particular, \(f() \leq -0.5\).

Theorem 3.29. If \((X, f)\) is an \(N\)-subnexus of type \((q, q_k)\) such that \(f() \geq f(x)\) for all \(x \in X\), then \(f\) is constant on \((O(f; k))\).

Proof. Assume that \(f\) is not constant on \((O(f; k))\). Then there exists \(x \in O(f; k)\) such that \(\alpha_x = f(x) \neq f() = \alpha_0\). Then \(\alpha_0 > \alpha_x\), and so \(f(x) + (-\alpha_0) + 1 = \alpha_x - \alpha_0 < 0\). Hence \((X, x_{-1-\alpha})\) is an \(N_q\)-subset of \((X, f)\). Note that \((\alpha) \leq x\) and \(f() + (-\alpha_0 - k + 1 = -k > 0\), which implies that \((X, (\alpha_{-1-\alpha}))\) is not an \(N_{q_k}\)-subset of \((X, f)\). This is a contradiction, and therefore \(f\) is constant on \((O(f; k))\).

Corollary 3.30 ([20]). If \((X, f)\) is an \(N\)-subnexus of type \((q, q)\) such that \(f() \geq f(x)\) for all \(x \in X\), then \(f\) is constant on the open support of \((X, f)\) related to 0.

4. \(N\)-Subnexus of Type \((\Xi, \Xi \lor q_k)\)

Let \((X, g)\) be an \(N\)-structure. A point \(N\)-structure \((X, x_\alpha)\) is said to be an \(N_{\forall} \subseteq (\Xi, \Xi \lor q_k)\)-subset if \(g(x) > \alpha\) (resp., \(g(x) + \alpha - k + 1 \geq 0\)). If a point \(N\)-structure \((X, x_\alpha)\) is an \(N_{\forall}\)-subset or an \(N_{\forall\forall}\)-subset of \((X, g)\), then \((X, x_\alpha)\) is said to be an \(N_{\forall\forall}\)-subset of \((X, g)\).

Definition 4.1. Let \(v, w \in X\) such that \(v \leq w\) and \(\alpha \in [-1, 0]\). We say an \(N\)-structure \((X, f)\) is an \(N\)-subnexus of type \((\Xi, \Xi \lor q_k)\), if \((X, v_\alpha)\) is an \(N_{\forall}\)-subset of \((X, f)\) then \((X, w_\alpha)\) is an \(N_{\forall\forall}\)-subset of \((X, f)\).

Similarly, we can define \(N\)-subnexus of type \((\Xi, \Xi)\). According to [20], we have \((X, f)\) is an \(N\)-subnexus of type \((\Xi, \Xi)\) if and only if it is an \(N\)-subnexus of type \((\Xi, \Xi)\).

Moreover, it is important to note that every \(N\)-subnexus of type \((\Xi, \Xi \lor q_k)\) is an \(N\)-subnexus of type \((\Xi, \Xi \lor q)\) for \(k = 0\). Also, every \(N\)-subnexus of type \((\Xi, \Xi \lor q)\)
is an $\mathcal{N}$-subnexus of type $(\Xi, \Xi \vee \eta k)$, for all $k \in (-1,0]$, which is a different property with respect to other types of $\mathcal{N}$-subnuxes and their generalizations by $k \in (-1,0]$. Therefore, according to [20], we can obtain the following corollaries:

**Corollary 4.2.** Let $(X, f)$ be an $\mathcal{N}$-structure such that for all $v, w \in X$ with $v \leq w$ we have $\bigwedge\{f(v), -0.5\} \leq f(w)$. Then, $(X, f)$ is an $\mathcal{N}$-subnexus of type $(\Xi, \Xi \vee \eta k)$.

**Corollary 4.3.** If the nonempty closed support of $(X, f)$ related to $\alpha$ is a subnexus of $X$ for every $\alpha \in [-1, -0.5)$, then $(X, f)$ is an $\mathcal{N}$-subnexus of type $(\Xi, \Xi \vee \eta k)$.

Moreover, see the following example:

**Example 4.4.** Let $(X, f)$ be an $\mathcal{N}$-structure in which $X = \{(0), (1), (2), (2, 1), (2, 2)\}$ is a nexus and $f$ is given as

$$f = \left(\begin{array}{cccc}
(0) & (1) & (2) & (2, 1) \\
-0.93 & -0.95 & -0.92 & -0.91 & -0.9
\end{array}\right).$$

We have $(0) \leq (1)$ and $f((0)) = -0.93 > -0.94 = \alpha$. But $f((1)) < -0.94$ and $f(1) - 0.94 + 1 < 0$, and so $(X, f)$ is not an $\mathcal{N}$-subnexus of type $(\Xi, \Xi \vee \eta q)$. While, for $k = -0.9$ we have $f(1) - 0.94 - k + 1 \geq 0$ and therefore $(X, f)$ is an $\mathcal{N}$-subnexus of type $(\Xi, \Xi \vee \eta -0.9)$.

**Theorem 4.5.** An $\mathcal{N}$-structure $(X, f)$ is an $\mathcal{N}$-subnexus of type $(\Xi, \Xi \vee \eta k)$ if and only if $\bigwedge\{f(v), k-1/2\} \leq f(w)$, for all $v, w \in X$ such that $v \leq w$.

**Proof.** Let $(X, f)$ be an $\mathcal{N}$-subnexus of type $(\Xi, \Xi \vee \eta k)$ and there exist $v, w \in X$ such that $v \leq w$ and $\bigwedge\{f(v), k-1/2\} > f(w) = \alpha$. So, $\alpha \in [-1, k-1/2)$. It follows that $(X, w_\alpha)$ is an $\mathcal{N}_\Xi$-subset of $(X, f)$ and $(X, v_\alpha)$ is an $\mathcal{N}_\Xi$-subset of $(X, f)$. Hence $(X, w_\alpha)$ is an $\mathcal{N}_{k-1/2}$-subset of $(X, f)$. Therefore $2\alpha - k + 1 = f(w) + \alpha - k + 1 \geq 0$, which implies that $\alpha \geq k-1/2$. This is contradiction, and so $\bigwedge\{f(v), k-1/2\} \leq f(w)$ for all $v, w \in X$ with $v \leq w$. Conversely, let for all $v, w \in X$ such that $v \leq w$ we have $\bigwedge\{f(v), k-1/2\} \leq f(w)$. Let $v \leq w$ for $v, w \in X$ and $\alpha \in [-1, 0)$ such that a point $\mathcal{N}$-structure $(X, v_\alpha)$ is an $\mathcal{N}_\Xi$-subset of $(X, f)$. Then $f(v) > \alpha$. If $f(v) \leq f(w)$, then $\alpha < f(w)$ and thus $(X, w_\alpha)$ is an $\mathcal{N}_\Xi$-subset of $(X, f)$. If $f(v) > f(w)$, then we have $f(w) \geq \bigwedge\{f(v), k-1/2\} = k-1/2$. Suppose that $(X, w_\alpha)$ is not an $\mathcal{N}_\Xi$-subset of $(X, f)$. Then $\alpha \geq f(w) \geq k-1/2$. Hence, we have $f(w) + \alpha - k + 1 \geq k-1/2 + k-1/2 - k + 1 = 0$ and so $(X, w_\alpha)$ is an $\mathcal{N}_{k-1/2}$-subset of $(X, f)$. Therefore $(X, f)$ is an $\mathcal{N}$-subnexus of type $(\Xi, \Xi \vee \eta k)$.

**Proposition 4.6.** If $(X, f)$ is an $\mathcal{N}$-structure of type $(\Xi, \Xi \vee \eta k)$, then $f(w) \geq f((0))$ or $f(w) \geq k-1/2$ for all $w \in X$.

**Proof.** Using Theorem 4.5, the proof is straightforward.

**Theorem 4.7.** An $\mathcal{N}$-structure $(X, f)$ is an $\mathcal{N}$-subnexus of type $(\Xi, \Xi \vee \eta k)$ if and only if $\emptyset \neq C(f; \alpha)$ is a subnexus of $X$ for every $\alpha \in [-1, k-1/2)$.

**Proof.** For an $\mathcal{N}$-subnexus $(X, f)$ of type $(\Xi, \Xi \vee \eta k)$, let $v, w \in X$ with $v \leq w$ and $w \in C(f; \alpha)$ for $\alpha \in [-1, k-1/2)$. By Theorem 4.5, we have $\bigwedge\{f(v), k-1/2\} \leq f(w) \leq \alpha$. Since $\alpha < k-1/2$, then $f(v) \leq \alpha$ and so $v \in C(f; \alpha)$. Conversely, let $(X, f)$ be an $\mathcal{N}$-structure such that $\emptyset \neq C(f; \alpha)$ is a subnexus of $X$ for all $\alpha \in [-1, k-1/2)$. Let for $v, w \in X$ such that $v \leq w$ we have $\bigwedge\{f(v), k-1/2\} > f(w)$. Put, $\beta := \frac{1}{2} \left(\bigwedge\{f(v), k-1/2\}\right) + f(w)$, then
β ∈ [−1, k−1/2] and \( \land \{f(v), \frac{k-1}{2}\} > \beta \geq f(w) \). Thus \( w \in C(f; \beta) \) but \( v \notin C(f; \beta) \) which is a contradiction. Therefore \( \land \{f(v), \frac{k-1}{2}\} \leq f(w) \) for all \( v, w \in X \) with \( v \leq w \). Using Theorem 4.5, then \((X, f)\) is an \( \mathcal{N}\)-subnexus of type \((\overline{\epsilon}, \overline{\epsilon} \lor \overline{\alpha})\). □

It is easy to see that every \( \mathcal{N}\)-subnexus of type \((\overline{\epsilon}, \overline{\epsilon})\) is an \( \mathcal{N}\)-subnexus of type \((\overline{\epsilon}, \overline{\epsilon} \lor \overline{\alpha})\), but the converse is not generally valid. See the following example:

**Example 4.8.** Let \( X = \{(\cdot), (1), (2), (1, 1), (1, 2), (1, 2, 1)\} \) be a nexus and define \( h \) on \( X \) as

\[
h = \begin{pmatrix}
-0.4 & 1 & 2 & 1, 1 & 1, 2 & 1, 2, 1 \\
\end{pmatrix}
\]

It is routine to verify that \((X, h)\) is an \( \mathcal{N}\)-subnexus of type \((\overline{\epsilon}, \overline{\epsilon} \lor \overline{\alpha})\) and so an \( \mathcal{N}\)-subnexus of type \((\overline{\epsilon}, \overline{\epsilon})\) for all \( \alpha \in (-1, 0] \). But it is not an \( \mathcal{N}\)-subnexus of type \((\overline{\epsilon}, \overline{\epsilon})\) since \((\cdot) \leq (1)\) and \((X, (\cdot))_{0.42}\) is an \( \mathcal{N}\)-subset of \((X, h)\), but \((X, (1))_{0.42}\) is not an \( \mathcal{N}\)-subset of \((X, h)\).

**Theorem 4.9.** Let \((X, f)\) be an \( \mathcal{N}\)-structure such that \( f(x) \leq \frac{k-1}{2} \) for all \( x \in X \). Then, every \( \mathcal{N}\)-subnexus of type \((\overline{\epsilon}, \overline{\epsilon} \lor \overline{\alpha})\) is an \( \mathcal{N}\)-subnexus of type \((\overline{\epsilon}, \overline{\epsilon})\).

**Proof.** Let \( v, w \in X \) and \( \alpha \in [-1, 0) \) be such that \( v \leq w \) and \((X, v)\) is an \( \mathcal{N}\)-subset of \((X, f)\). Then \( f(v) > \alpha \). Since \( f(x) \leq \frac{k-1}{2} \) for all \( x \in X \), by Theorem 4.5, we have \( \alpha < f(v) = \land \{f(v), \frac{k-1}{2}\} \leq f(w) \). Thus \((X, w)\) is an \( \mathcal{N}\)-subset of \((X, f)\). Therefore \((X, f)\) is an \( \mathcal{N}\)-subnexus of type \((\overline{\epsilon}, \overline{\epsilon})\). □

**References**


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