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Iterative Algorithm for a System of Equilibrium Problems of Bregman Strongly Nonexpansive Mapping

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Abstract : We prove a strong convergence theorem for fixed point of a Bregman strongly nonexpansive operator in real reflexive Banach spaces. This point is also a solution to a system of equilibrium problems. Finally, we provide examples to illustrate the main result.

Keywords : iterative algorithm; system of equilibrium problems; Bregman strongly nonexpansive mapping.
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1 Introduction

Let $\mathcal{C}$ be a nonempty closed convex subset of a real reflexive Banach space E with the dual space E^*. The norm and the dual pairing between E and E^* are denoted by $||.||$ and $\langle ., . \rangle$ respectively.

Definition 1.1. Let $f : E \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lower semi-continuous function. The domain of a convex function $f : E \to \mathbb{R}$ is defined to be

$$\text{dom} f := \{x \in E : f(x) < +\infty\}.$$

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The Fenchel conjugate of f is the function $f^*: E^* \to (-\infty, +\infty]$ defined by

$$f^*(\xi) := \sup\{\langle \xi, x\rangle - f(x) : x \in E\}.$$

A mapping $T : \mathcal{C} \to \mathcal{C}$ is said to be nonexpansive if

$$||T(x) - T(y)|| \leq ||x - y||, \quad \forall x, y \in \mathcal{C}.$$

For a mapping $T : \mathcal{C} \to \mathcal{C}$ define

$$Fix(T) := \{x \in \mathcal{C} : x = Tx\}$$

to denote the fixed points of T. Let H be a bifunction from $\mathcal{C} \times \mathcal{C}$ into $\mathbb{R}$. The equilibrium problem for $H : \mathcal{C} \times \mathcal{C} \to \mathbb{R}$ is to find $x \in \mathcal{C}$ such that

$$H(x, y) \geq 0, \quad \forall y \in \mathcal{C}. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $EP(H)$. It is well known that equilibrium problems and their generalizations have been important tools for solving problems arising in the fields of linear or nonlinear programming, variational inequalities, complementary problems, optimization problems, fixed point problems and have been widely applied to physics, structural analysis, management science and economics (see, for example [1–7]). One of the most important and interesting topics in the theory of equilibria is to develop efficient and implementable algorithms for solving equilibrium problems and their generalizations (see, e.g., [5–8] and the references therein). Since the equilibrium problems are closely related to both the fixed point problems and the variational inequalities problems, finding the common elements of these problems has drawn several people's attention and has become one of the most important topics in the past few years (see, e.g., [9–18] and the references therein). In 1967, Bregman [19] discovered an elegant and effective technique of using the so-called Bregman distance function D_f (see, Section 2, Definition 2.1) in the process of designing and analyzing feasibility and optimization algorithms. This opened a growing area of research in which Bregman's technique has been applied in various ways in order to design and analyze not only iterative algorithms for solving feasibility and optimization problems, but also algorithms for solving variational inequalities, for approximating equilibria, for computing fixed points of nonlinear mappings and so on (see, e.g., [20, 21]) and the references therein.

In 2015, Kumama, Witthayaratb, Kumam, Suantaie and Wattanawitoon [22] introduced the following algorithm:

$$\begin{cases} x_1 = x \in \mathcal{C}, \\ z_n = Res_H^f(x_n), \\ y_n = \nabla f^*(\beta_n \nabla f(x_n) + (1 - \beta_n)\nabla f(T_n(z_n))), \\ x_{n+1} = \nabla f^*(\alpha_n \nabla f(x_n) + (1 - \alpha_n)\nabla f(T_n(y_n))), \end{cases} \tag{1.2}$$

where $T_n, n \in \mathbb{N}$, is a Bregman strongly nonexpansive mapping.

In 2016, Yekini Shehu [23] studied the approximation of a fixed point of a left Bregman strongly nonexpansive mapping which is also solution to a finite system of equilibrium problems in reflexive real Banach spaces:

$$\begin{cases} y_n = \nabla f^*(\alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(x_n)), \\ x_{n+1} = \nabla f^*(\beta_n \nabla f(y_n) + (1-\beta_n)\nabla f(T(y_n))). \end{cases} \tag{1.3}$$

In this paper, inspired by (1.2) and (1.3) we introduce the following new algorithm for a system of equilibrium problems:

$$\begin{cases} x_1 = x \in \mathcal{C}, \\ z_n = Res^f_{g_N} \circ Res^f_{g_{N-1}} \circ \cdots \circ Res^f_{g_2} \circ Res^f_{g_1}(x_n), \\ y_n = \nabla f^*(\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(T(z_n))), \\ x_{n+1} = \nabla f^*(\beta_n \nabla f(x) + \gamma_n \nabla f(z_n) + \eta_n \nabla f(T(y_n))), \end{cases} \tag{1.4}$$

where T is a Bregman strongly nonexpansive mapping. We will prove that the sequence x_n converges strongly to a point of $\Omega := Fix(T) \cap \bigcap_{k=1}^N EP(g_k)$.

The purpose of this paper is to prove a strong convergence theorem for approximating a fixed point of a left Bregman strongly relatively nonexpansive mapping which is also a solution to a finite system of equilibrium problems in the framework of reflexive real Banach spaces. Our results complement many known recent results in the literature. Numerical examples are also provided.

2 Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}_{n=1}^\infty$ converges weakly to x, and $x_n \to x$ to indicate that the sequence $\{x_n\}_{n=1}^\infty$ converges strongly to x. For any $x \in \text{int}(\text{dom} f)$, the right-hand derivative of f at x in the direction $y \in E$ is defined by

$$f^{'}(x,y) := \lim_{t \searrow 0} \frac{f(x+ty)-f(x)}{t}.$$

The function f is called Gâteaux differentiable at x if $\lim_{t \searrow 0} \frac{f(x+ty)-f(x)}{t}$ exists for all $y \in E$. In this case, $f^{'}(x,y)$ coincides with ∇f, the value of the gradient (∇f) of f at x. The function f is called Gâteaux differentiable if it is Gâteaux differentiable for any $x \in \text{int}(\text{dom} f)$ and f is called Fréchet differentiable at x if this limit uniformly for all y satisfying $||y|| = 1$. The function f is uniformly Fréchet differentiable on a subset $\mathcal{C}$ of E if the limit is attained uniformly for any $x \in \mathcal{C}$ and $||y|| = 1$. It is known that if f is Gâteaux differentiable (resp. Fréchet differentiable) on $\text{int}(\text{dom} f)$, then f is continuous and its Gâteaux derivative ∇f is norm-to-weak* continuous (resp. continuous) on $\text{int}(\text{dom} f)$(see [24]).

The function f is said to be Legendre if it satisfies the following two conditions:

(**L1**) $\text{int}(\text{dom}f) \neq \emptyset$ and the subdifferential ∂f is single-valued on its domain.

(**L2**) $\text{int}(\text{dom}f^*) \neq \emptyset$ and the subdifferential ∂f^* is single-valued on its domain.

The class of Legendre functions in infinite dimensional Banach spaces was first introduced and studied by Bauschke, Borwein and Combettes in [20]. Their definition is equivalent to conditions (**L1**) and (**L2**) because the space E is assumed to be reflexive (see [20], Theorems 5.4 and 5.6, page 634). It is well known that in reflexive spaces, $\nabla f = (\nabla f^*)^{-1}$ (see [24], page 83). When this fact is combined with conditions (**L1**) and (**L2**), we obtain

$$\text{ran}(\nabla f) = \text{dom}(\nabla f^*) = \text{int}(\text{dom}f^*) \quad \textit{and} \quad \text{ran}(\nabla f^*) = \text{dom}(\nabla f) = \text{int}(\text{dom}f).$$

It is also known that f is Legendre if and only if f^* is Legendre (see [20], Corollary 5.5, page 634) and that the functions f and f^* are Gâteaux differentiable and strictly convex in the interior of their respective domains. When the Banach space E is smooth and strictly convex, in particular, in a Hilbert space, the function $(\frac{1}{p})||.||^p$ with $p \in (1, \infty)$ is Legendre (cf. [20], Lemma 6.2, page 639). To see some examples and to get more information regarding Legendre functions, see, for instance, [20, 25].

Definition 2.1. The bifunction $D_f : \text{dom}f \times \text{int}(\text{dom}f) \to [0, +\infty)$ defined by

$$D_f(y, x) := f(y) - f(x) - \langle \nabla f(x), y - x\rangle \tag{2.1}$$

is called the Bregman distance with respect to f (cf. [19, 26]).

The Bregman distance does not satisfy the well-known properties of a metric, but it does have the following important property which is called the three point identity: for any $x \in \text{dom}f$ and $y, z \in \text{int}(\text{dom}f)$

$$D_f(x, y) + D_f(y, z) - D_f(x, z) = \langle \nabla f(z) - \nabla f(y), x - y\rangle. \tag{2.2}$$

Definition 2.2. Let $f : E \to (-\infty, -\infty]$ be a convex and Gâteaux differentiable function. The Bregman projection with respect to f on $x \in \text{int}(\text{dom}f)$ onto a nonempty, closed and convex subset $\mathcal{C} \subset \text{int}(\text{dom}f)$ is defined as the necessarily unique vector $Proj^f_{\mathcal{C}}(x) \in \mathcal{C}$ which satisfies

$$D_f(Proj^f_{\mathcal{C}}(x), x) = \inf\{D_f(y, x) : y \in \mathcal{C}\}. \tag{2.3}$$

Remark. In the following we provide two examples for the Bregman projection $Proj^f_{\mathcal{C}}(x)$:

1. If E is a Hilbert space and $f(x) = \frac{1}{2}||x||^2$, then the Bregman projection $Proj^f_{\mathcal{C}}(x)$ is reduced to the metric projection of x onto $\mathcal{C}$.
2. If $\mathcal{C}$ is a smooth Banach space and $f(x) = \frac{1}{2}||x||^2$, then the Bregman projection $Proj^f_{\mathcal{C}}(x)$ is reduced to the generalized projection $\Pi_{\mathcal{C}}(x)$, which is defined by
$$\phi(\Pi_{\mathcal{C}}(x), x) = \min_{y\in\mathcal{C}} \phi(y, x),$$
where $\phi(y, x) = ||y||^2 - 2\langle y, J(x)\rangle + ||x||^2$ and J is the normalized duality mapping from $E \to 2^{E^*}$.

Definition 2.3. Let $f : E \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. Then

(i) the function f is called totally convex at x if its modulus of total convexity at $x \in \text{int}(\text{dom}f)$, that is, the bifunction $\nu_f : \text{int}(\text{dom}f) \times [0, +\infty) \to [0, +\infty]$ defined by

$$\nu_f(x, t) := \inf\{D_f(y, x) : y \in \text{dom}f, ||y - x|| = t\}$$

is positive whenever $t > 0$;

(ii) the function f is called totally convex if it is totally convex at every point $x \in \text{int}(\text{dom}f)$;

(iii) the function f is called totally convex on bounded subsets if $\nu_f(B, t)$ is positive for any nonempty bounded subset B is the function $\nu_f : \text{int}(\text{dom}f) \times [0, +\infty) \to [0, +\infty]$ defined by

$$\nu_f(B, t) := \inf\{\nu_f(x, t) : x \in B \cap \text{dom}f\}.$$

Examples of totally convex functions can be found, for instance, in [27–29]. We remark in passing that f is totally convex on bounded subsets if and only if f is uniformly convex on bounded subsets (see [28], Theorem 2.10, page 9).

Definition 2.4. Let $\mathcal{C}$ be a nonempty closed convex subset of $\text{int}(\text{dom}f)$ and let $T : \mathcal{C} \to \mathcal{C}$ be a mapping. A point p in $\mathcal{C}$ is said to be an asymptotic fixed point of T (see [30]) if $\mathcal{C}$ contains a sequence $\{x_n\}_{n=1}^{\infty}$ which converges weakly to p such that $\lim_{n\to\infty} \|x_n - T(x_n)\| = 0$. The set of asymptotically fixed points of T is denoted by $\widehat{Fix}(T)$.

Definition 2.5. A mapping T with a nonempty asymptotic fixed point set is said to be:

(i) left Bregman strongly nonexpansive with respect to a nonempty $\widehat{Fix}(T)$ (see [31]) if

$$D_f(p, T(x)) \leq D_f(p, x), \quad \forall x \in \mathcal{C}, \ p \in \widehat{Fix}(T),$$

and if whenever $\{x_n\}_{n=1}^{\infty} \subset \mathcal{C}$ is bounded, $p \in \widehat{Fix}(T)$ and

$$\lim_{n\to\infty} (D_f(p, x_n) - D_f(p, T(x_n))) = 0,$$

it follows that $\lim_{n\to\infty} D_f(T(x_n), x_n) = 0$.

According to Martin-Marquez et al. [32], a left Bregman strongly nonexpansive mapping T with respect to a nonempty $\widehat{Fix}(T)$ is called strictly left Bregman strongly nonexpansive mapping.

(ii) The mapping $T : \mathcal{C} \to int(domf)$ is said to be left Bregman firmly nonexpansive (L-BFNE) if for all $x, y \in \mathcal{C}$

$$\langle \nabla f(T(x)) - \nabla f(T(y)), T(x) - T(y)\rangle \leq \langle \nabla f(x) - \nabla f(y), T(x) - T(y)\rangle,$$

or equivalently,

$$\begin{aligned} D_f(T(x), T(y)) + D_f(T(y), T(x)) + D_f(T(x), x) + D_f(T(y), y) \\ \leq D_f(T(x), y) + D_f(T(y), x). \end{aligned}$$

For more information and examples of L-BFNE operators, see [27, 31] (operators in this class are also called D_f-firm and BFNE).

If T is a Bregman firmly nonexpansive mapping and f is a Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E, then $Fix(T) = \widehat{Fix}(T)$ and $Fix(T)$ is closed and convex (see [33]). It is known that every Bregman firmly nonexpansive mapping is Bregman strongly nonexpansive with respect to $Fix(T) = \widehat{Fix}(T)$.

Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function and let $x \in E$, it is known from [28] that $z = Proj^f_{\mathcal{C}}(x)$ if and only if

$$\langle \nabla f(x) - \nabla f(z), y - z\rangle \leq 0, \quad \forall y \in \mathcal{C}.$$

We also know that

$$D_f(y, Proj^f_{\mathcal{C}}(x)) + D_f(Proj^f_{\mathcal{C}}(x), x) \leq D_f(y, x), \quad \forall x, y \in \mathcal{C}. \tag{2.4}$$

Following [34] and [26], we make use of the function $V_f : E \times E^* \to [0, +\infty)$ associated with f, which is defined by

$$V_f(x, x^*) = f(x) - \langle x^*, x\rangle + f^*(x^*), \quad \forall x \in E,\ x^* \in E^*.$$

Then V_f is nonnegative and $V_f(x, x^*) = D_f(x, \nabla f^*(x^*))$ for all $x \in E$ and $x^* \in E^*$. Moreover in [35], by the subdifferential inequality,

$$V_f(x, x^*) + \langle y^*, \nabla f^*(x^*) - x\rangle \leq V_f(x, x^* + y^*), \quad \forall x \in E,\ x^*, y^* \in E^*.$$

In addition, if $f : E \to (-\infty, +\infty]$ is a proper lower semi-continuous function, then $f^* : E^* \to (-\infty, +\infty]$ is a proper weak* lower semi-continuous and convex function (see [36]). Hence V_f is convex in the second variable. Thus, for all $z \in E$,

$$D_f\left(z, \nabla f^*\left(\sum_{i=1}^N t_i \nabla f(x_i)\right)\right) \leq \sum_{i=1}^N t_i D_f(z, x_i),$$

where $\{x_i\}_{i=1}^N \subset E$ and $\{t_i\}_{i=1}^N \subset (0, 1)$ with $\sum_{i=1}^N t_i = 1$.

Definition 2.6. ([29, 37]) A function $f : E \to (-\infty, +\infty)$ is called:

(i) cofinite if $domf^* = E^*$;

(ii) coercive if $\lim_{||x||\to+\infty}\left(\dfrac{f(x)}{||x||}\right)=0$;

(iii) sequentially consistent if for any two sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ in E such that $\{x_n\}_{n=1}^{\infty}$ is bounded,

$$\lim_{n\to\infty} D_f(y_n, x_n) = 0 \Rightarrow \lim_{n\to\infty} \|y_n - x_n\| = 0.$$

Lemma 2.1. ([37]) *Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function. If $x_0 \in E$ and the sequence $\{D_f(x_n, x_0)\}_{n=1}^{\infty}$ is bounded, then the sequence $\{x_n\}_{n=1}^{\infty}$ is also bounded.*

Lemma 2.2. ([29]) *A function f is totally convex on bounded subsets if and only if it is sequentially consistent.*

For solving the equilibrium problem, we shall make the following assumptions on the bifunction $g : \mathcal{C} \times \mathcal{C} \to \mathbb{R}$:

(C1) $g(x, x) = 0$ for all $x \in \mathcal{C}$,

(C2) g is monotone, that is

$$g(x, y) + g(y, x) \le 0, \quad \forall x, y \in \mathcal{C},$$

(C3) g is upper-hemicontinuous, that is

$$\limsup_{h\to 0^+} g(hz + (1-h)x, y) \le g(x, y), \quad \forall x, y, z \in \mathcal{C},$$

(C4) $g(x, 0)$ is convex and lower semicontinuous for each $x \in \mathcal{C}$.

The resolvent of a bifunction $g : \mathcal{C}\times\mathcal{C} \to \mathbb{R}$ (see [38]) is the operator $Res_g^f : E \to 2^{\mathcal{C}}$ defined by

$$Res_g^f(x) = \{z \in \mathcal{C} : g(z, y) + \langle \nabla f(z) - \nabla f(x), y - z\rangle \ge 0, \quad \forall y \in \mathcal{C}\}. \tag{2.5}$$

For any $x \in E$, there exists $z \in \mathcal{C}$ such that $z = Res_g^f(x)$; (see [39]).

Lemma 2.3. ([39]) *Let $f : E \to (-\infty, +\infty)$ be a coercive Legendre function. Let $\mathcal{C}$ be a closed and convex subset of E. If the bifunction $g : \mathcal{C} \times \mathcal{C} \to \mathbb{R}$ satisfies the conditions (**C1**) − (**C4**), then we have*

1. *Res_g^f is single-valued;*
2. *Res_g^f is a Bregman firmly nonexpansive mapping;*
3. *$Fix(Res_g^f) = EP(g)$;*
4. *$EP(g)$ is a closed and convex subset of $\mathcal{C}$;*
5. *for all $x \in E$ and $q \in Fix(Res_g^f)$,*

$$D_f(q, Res_g^f(x)) + D_f(Res_g^f(x), x) \le D_f(q, x).$$

Lemma 2.4. ([40]) *If $f : E \to \mathbb{R}$ is uniformly Fréchet differentiable and bounded on bounded subsets of E, then ∇f is uniformly continuous on bounded subsets of E.*

Lemma 2.5. ([41]) *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\alpha_n)a_n + b_n, \ \forall n \geq 0,$$

where

1. $\{\alpha_n\} \subseteq (0,1)$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
2. $\limsup_{n\to\infty} \dfrac{b_n}{a_n} \leq 0$, *and* $\sum_{n=1}^{\infty} |b_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.6. ([42]) *Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} \leq a_{n_{i+1}}$ for all $i \geq 0$. Then there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:*

$$a_{m_k} \leq a_{m_k+1} \quad \text{and} \quad a_k \leq a_{m_k+1}.$$

In fact $m_k = \max\{j \leq k : a_j < a_{j+1}\}$.

3 Main Result

This section is devoted to the main results of this paper.

Theorem 3.1. *Let E be a real reflexive Banach space and $\mathcal{C}$ be a nonempty, closed convex subset of E. For each $k = 1, 2, \cdots, N$, let g_k be a bifunction from $\mathcal{C} \times \mathcal{C}$ to $\mathbb{R}$ satisfying (**C1**) − (**C4**). Let $f : E \to \mathbb{R}$ be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E and let ∇f^* be bounded on bounded subsets of E^* and T be a Bregman strongly nonexpansive mapping on E such that $Fix(T) = \widehat{Fix}(T)$. Assume that T is uniformly continuous and $\Omega := Fix(T) \cap \bigcap_{k=1}^{N} EP(g_k)$ is nonempty and bounded. Let $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\eta_n\}$ be sequences in $(0,1)$ such that $\beta_n + \gamma_n + \eta_n = 1$, $(1-\beta_n)a \leq \eta_n$ for $a > 0$ and $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=1}^{\infty} \beta_n = \infty$. Suppose $\{x_n\}_{n=1}^{\infty}$ is generated by the following algorithm:*

$$\begin{cases} x_1 = x \in \mathcal{C}, \\ z_n = Res^f_{g_N} \circ Res^f_{g_{N-1}} \circ \cdots \circ Res^f_{g_2} \circ Res^f_{g_1}(x_n), \\ y_n = \nabla f^*(\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(T(z_n))), \\ x_{n+1} = \nabla f^*(\beta_n \nabla f(x_n) + \gamma_n \nabla f(z_n) + \eta_n \nabla f(T(y_n))). \end{cases} \tag{3.1}$$

Then the sequence $\{x_n\}$ converges strongly to a point in

$$\Omega := Fix(T) \cap \bigcap_{k=1}^{N} EP(g_k).$$

Proof. Let $x^* \in \Omega$. By taking $\theta_k^f = Res_{g_k}^f \circ Res_{g_{k-1}}^f \circ \cdots \circ Res_{g_2}^f \circ Res_{g_1}^f$ for $k = 1, 2, \cdots, N$ and $\theta_0^f = I$, we obtain $z_n = \theta_N^f x_n$. Using the fact that $Res_{g_k}^f, k = 1, 2, \cdots, N$, is a strictly left quasi-Bregmen nonexpansive mapping, we obtain from (3.1) that

$$\begin{aligned} D_f(x^*, y_n) &= D_f(x^*, \nabla f^* (\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(T(z_n)))) \\ &\leq \alpha_n D_f(x^*, x_n) + (1-\alpha_n) D_f(x^*, T(z_n)) \\ &\leq \alpha_n D_f(x^*, x_n) + (1-\alpha_n) D_f(x^*, z_n) \\ &\leq \alpha_n D_f(x^*, x_n) + (1-\alpha_n) D_f(x^*, x_n) \\ &= D_f(x^*, x_n). \end{aligned} \tag{3.2}$$

By (3.1) and (3.2) we obtain

$$\begin{aligned} D_f(x^*, x_{n+1}) &= D_f(x^*, \nabla f^*(\beta_n \nabla f(x_n) + \gamma_n \nabla f(z_n) + \eta_n \nabla f(T(y_n)))) \\ &\leq \beta_n D_f(x^*, x_n) + \gamma_n D_f(x^*, z_n) + \eta_n D_f(x^*, T(y_n)) \\ &\leq \beta_n D_f(x^*, x_n) + \gamma_n D_f(x^*, z_n) + \eta_n D_f(x^*, y_n) \\ &\leq \beta_n D_f(x^*, x_n) + \gamma_n D_f(x^*, x_n) + \eta_n D_f(x^*, x_n) \\ &\leq \beta_n D_f(x^*, x_n) + (1-\beta_n) D_f(x^*, x_n) \\ &= D_f(x^*, x_n) \\ &\vdots \\ &\leq D_f(x^*, x_1). \end{aligned} \tag{3.3}$$

Also, by (3.2) and (3.3) we conclude that

$$\begin{aligned} D_f(x^*, z_{n+1}) &= D_f(x^*, \theta_N^f x_{n+1}) \\ &\leq D_f(x^*, x_{n+1}) \\ &\leq D_f(x^*, \nabla f^*(\beta_n \nabla f(x_n) + \gamma_n \nabla f(z_n) + \eta_n \nabla f(T(y_n)))) \\ &\leq \beta_n D_f(x^*, x_n) + \gamma_n D_f(x^*, z_n) + \eta_n D_f(x^*, T(y_n)) \\ &\leq \beta_n D_f(x^*, x_n) + \gamma_n D_f(x^*, z_n) + \eta_n D_f(x^*, y_n) \\ &\leq \beta_n D_f(x^*, x_n) + \gamma_n D_f(x^*, z_n) + \eta_n D_f(x^*, x_n) \\ &\leq \max\{D_f(x^*, z_n), D_f(x^*, x_n)\} \\ &\leq \max\{D_f(x^*, z_n), D_f(x^*, x_1)\} \\ &\vdots \\ &\leq \max\{D_f(x^*, z_1), D_f(x^*, x_1)\}. \end{aligned} \tag{3.4}$$

Hence, $\{D_f(x^*, z_n)\}_{n=1}^{\infty}$ and $\{D_f(x^*, x_n)\}_{n=1}^{\infty}$ and $\{D_f(x^*, y_n)\}_{n=1}^{\infty}$ are bounded. Now, we show that the sequence $\{x_n\}_{n=1}^{\infty}$ is bounded too. Since $\{D_f(x^*, x_n)\}_{n=1}^{\infty}$ is bounded, there exists $M > 0$ such that

$$f(x^*) - \langle \nabla f(x_n), x^* \rangle + f^*(\nabla f(x_n)) = V_f(x^*, \nabla f(x_n)) = D_f(x^*, x_n) \leq M.$$

Hence, $\{\nabla f(x_n)\}_{n=1}^{\infty}$ is contained in the sublevel set $lev_{\leq}^{\psi}(M - f(x^*))$, where $\psi = f^* - \langle ., x^*\rangle$. Since f is lower-semicontinuous, f^* is $weak^*$ lower-semicontinuous. Hence, the function ψ is coercive by Moreau-Rockafellar Theorem (see [43], Theorem 7A and [44]). This shows that $\{\nabla f(x_n)\}$ is bounded. Since f is strongly coercive, f^* is bounded on bounded subsets (see [45], Lemma 3.6.1 and [20], Theorem 3.3). Hence, ∇f^* is also bounded on bounded subsets of E^*, (see [29], Proposition 1.1.11). Since f is a Legendre function, it follows that $x_n = \nabla f^*(\nabla f(x_n))$ is bounded. Therefore $\{x_n\}_{n=1}^{\infty}$ is bounded. Let $w_n := \nabla f^*(\beta_n \nabla f(x_n) + \gamma_n \nabla f(z_n) + \eta_n \nabla f(T(y_n))), n \geq 1$. Furthermore, by (3.1) and (3.2) we have

$$\begin{aligned}
&D_f(x^*, x_{n+1}) = V_f(x^*, \beta_n \nabla f(x_n) + \gamma_n \nabla f(z_n) + \eta_n \nabla f(T(y_n)))\\
&\leq V_f(x^*, \beta_n \nabla f(x_n) + \gamma_n \nabla f(z_n) + \eta_n \nabla f(T(y_n)) - \beta_n(\nabla f(x_n) - \nabla f(x^*)))\\
&\quad - \langle \nabla f^*(\beta_n \nabla f(x_n) + \gamma_n \nabla f(z_n) + \eta_n \nabla f(T(y_n)) - x^*, -\beta_n(\nabla f(x_n) - \nabla f(x^*))\rangle\\
&= V_f(x^*, \beta_n \nabla f(x^*) + \gamma_n \nabla f(z_n) + \eta_n \nabla f(T(y_n))) + \beta_n \langle w_n - x^*, \nabla f(x_n) - \nabla f(x^*)\rangle\\
&= D_f(x^*, \nabla f^*(\beta_n \nabla f(x^*) + \gamma_n \nabla f(z_n) + \eta_n \nabla f(T(y_n)))\\
&\quad + \beta_n \langle w_n - x^*, \nabla f(x_n) - \nabla f(x^*)\rangle\\
&\leq \beta_n D_f(x^*, x^*) + \gamma_n D_f(x^*, z_n) + \eta_n D_f(x^*, T(y_n)) + \beta_n \langle w_n - x^*, \nabla f(x_n) - \nabla f(x^*)\rangle\\
&\leq \gamma_n D_f(x^*, x_n) + \eta_n D_f(x^*, y_n) + \beta_n \langle w_n - x^*, \nabla f(x_n) - \nabla f(x^*)\rangle\\
&\leq \gamma_n D_f(x^*, x_n) + \eta_n D_f(x^*, x_n) + \beta_n \langle w_n - x^*, \nabla f(x_n) - \nabla f(x^*)\rangle\\
&\leq (1 - \beta_n) D_f(x^*, x_n) + \beta_n \langle w_n - x^*, \nabla f(x_n) - \nabla f(x^*)\rangle. \qquad (3.5)
\end{aligned}$$

The rest of the proof will be divided into two cases:

Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{D_f(x^*, x_n)\}_{n=1}^{\infty}$ is nonincreasing. Then $\{D_f(x^*, x_n)\}_{n=1}^{\infty}$ converges and

$$D_f(x^*, x_{n+1}) - D_f(x^*, x_n) \to 0, \quad n \to \infty.$$

Let $S_n := \nabla f^*(\frac{\gamma_n}{1-\beta_n} \nabla f(I) + \frac{\eta_n}{1-\beta_n} \nabla f(T))$ for all $n \geqslant 1$. By Propositions 13-15 of [46], we know that S_n is a left Bregman strongly nonexpansive mapping for each $n \geq 1$. Observe that

$$D_f(x^*, x_{n+1}) \leq \beta_n D_f(x^*, x) + (1 - \beta_n) D_f(x^*, S_n x_n).$$

It then follows that

$$\begin{aligned}
&D_f(x^*, x_n) - D_f(x^*, S_n x_n)\\
&= D_f(x^*, x_n) - D_f(x^*, x_{n+1}) + D_f(x^*, x_{n+1}) - D_f(x^*, S_n x_n)\\
&\leq D_f(x^*, x_n) - D_f(x^*, x_{n+1}) + \beta_n(D_f(x^*, x) - D_f(x^*, S_n x_n)) \to 0, \qquad (3.6)
\end{aligned}$$

as $n \to \infty$. Therefor $\lim_{n\to\infty} D_f(S_n x_n, x_n) = 0$. Furthermore,

$$\begin{aligned} D_f(x^*, S_n x_n) &\leq \frac{\eta_n}{1-\beta_n} D_f(x^*, T(y_n)) + \frac{\gamma_n}{1-\beta_n} D_f(x^*, x_n) \\ &= \frac{\eta_n}{1-\beta_n} D_f(x^*, T(y_n)) + (1 - \frac{\eta_n}{1-\beta_n}) D_f(x^*, x_n) \\ &= \frac{\eta_n}{1-\beta_n} (D_f(x^*, T(y_n)) - D_f(x^*, x_n)) + D_f(x^*, x_n). \end{aligned}$$

Thus

$$\frac{\eta_n}{1-\beta_n}(D_f(x^*, x_n) - D_f(x^*, T(y_n)) \leq D_f(x^*, x_n) - D_f(x^*, S_n x_n) \to 0,$$

as $n \to \infty$. Thus,

$$\lim_{n\to\infty} D_f(x^*, x_n) - D_f(x^*, T(y_n)) = 0.$$

Therefore

$$\lim_{n\to\infty} D_f(T(y_n), x_n) = 0.$$

By Lemma 2.2, we now conclude that

$$\lim_{n\to\infty} \parallel T(y_n) - x_n \parallel = 0. \tag{3.7}$$

From (3.1), Lemma 2.3 and (2.4), we have

$$\begin{aligned} \lim_{n\to\infty} D_f(x_n, z_n) &= \lim_{n\to\infty} D_f(x_n, \theta_N^f x_n) \\ &\leq \lim_{n\to\infty} [D_f(x^*, \theta_N^f x_n) - D_f(x^*, x_n)] \\ &\leq \lim_{n\to\infty} [D_f(x^*, x_n) - D_f(x^*, x_n)] = 0. \end{aligned}$$

By Lemma 2.2, we also have

$$\lim_{n\to\infty} \parallel x_n - z_n \parallel = 0. \tag{3.8}$$

Since f is uniformly Fréchet differentiable on bounded subsets of E, and by Lemma 2.4, ∇f is norm-to-norm uniformly continuous on bounded subsets of E, we have

$$\lim_{n\to\infty} \parallel \nabla f(x_n) - \nabla f(z_n) \parallel_* = 0. \tag{3.9}$$

Since f is uniformly Fréchet differentiable, it is also uniformly continuous, so that

$$\lim_{n\to\infty} \parallel f(x_n) - f(z_n) \parallel_* = 0. \tag{3.10}$$

By the definition of Bregman distance we obtain

$$\begin{aligned} &D_f(x^*, x_n) - D_f(x^*, z_n) \\ &= f(x^*) - f(x_n) - \langle \nabla f(x_n), x^* - x_n \rangle - f(x^*) + f(z_n) + \langle \nabla f(z_n), x^* - z_n \rangle \\ &= f(z_n) - f(x_n) + \langle \nabla f(z_n), x^* - z_n \rangle - \langle \nabla f(x_n), x^* - x_n \rangle \\ &= f(z_n) - f(x_n) + \langle \nabla f(z_n), x_n - z_n \rangle - \langle \nabla f(z_n) - \nabla f(x_n), x^* - x_n \rangle, \end{aligned}$$

for every $x^* \in \Omega$. From (3.8)-(3.10), we have

$$\lim_{n\to\infty}[D_f(x^*, x_n) - D_f(x^*, z_n)] = 0. \tag{3.11}$$

We now consider the following inequalities

$$\begin{aligned}
&\lim_{n\to\infty} D_f(z_n, y_n)\\
&= \lim_{n\to\infty}[D_f(x^*, y_n) - D_f(x^*, z_n)]\\
&= \lim_{n\to\infty}[D_f(x^*, \nabla f^*(\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(T(z_n))) - D_f(x^*, z_n)]\\
&\leq \lim_{n\to\infty}[\alpha_n D_f(x^*, x_n) + (1-\alpha_n) D_f(x^*, T(z_n)) - D_f(x^*, z_n)]\\
&\leq \lim_{n\to\infty}[\alpha_n D_f(x^*, x_n) + (1-\alpha_n) D_f(x^*, z_n) - D_f(x^*, z_n)]\\
&= \lim_{n\to\infty} \beta_n [D_f(x^*, x_n) - D_f(x^*, z_n)].
\end{aligned}$$

From (3.11), we have

$$\lim_{n\to\infty} D_f(z_n, y_n) = 0. \tag{3.12}$$

By Lemma 2.2, we also have

$$\lim_{n\to\infty} \| z_n - y_n \| = 0. \tag{3.13}$$

Now, by the triangle inequality, we have

$$\| x_n - y_n \| \leq \| x_n - z_n \| + \| z_n - y_n \| .$$

Combining (3.8) with (3.13), we conclude that

$$\lim_{n\to\infty} \| x_n - y_n \| = 0,$$

and since T is uniformly continuous,

$$\lim_{n\to\infty} \| T(y_n) - T(x_n) \| = 0. \tag{3.14}$$

By using again the triangle inequality, we get

$$\| T(x_n) - x_n \| \leq \| T(x_n) - T(y_n) \| + \| T(y_n) - x_n \| .$$

From (3.7) and (3.14), we obtain

$$\lim_{n\to\infty} \| T(x_n) - x_n \| = 0.$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ that converges weakly to p. Since $Fix(T) = \widehat{Fix}(T)$ we have $p \in Fix(T)$.

Next, we show that $p \in \bigcap_{k=1}^N EP(g_k)$. Now, using the fact that $Res^f_{g_k}, k = 1, 2, \cdots, N$, is a strictly quasi-Bregman nonexpansive mapping, we obtain

$$\begin{aligned} D_f(x^*, z_n) &= D_f(x^*, \theta^f_N x_n) \\ &= D_f(x^*, Res^f_{g_N} \theta^f_{N-1} x_n) \\ &\le D_f(x^*, \theta^f_{N-1} x_n) \\ &\le \cdots \le D_f(x^*, x_n). \end{aligned} \tag{3.15}$$

Since $x^* \in EP(g_N) = Fix(Res^f_{g_N})$, it follows from Lemma 2.3, (3.11) and (3.15) that

$$\begin{aligned} D_f(z_n, \theta^f_{N-1} x_n) &= D_f(Res^f_{g_N} \theta^f_{N-1} x_n, \theta^f_{N-1} x_n) \\ &\le D_f(x^*, \theta^f_{N-1} x_n) - D_f(x^*, z_n) \\ &\le D_f(x^*, x_n) - D_f(x^*, z_n) \to 0, \quad n \to \infty. \end{aligned}$$

Thus, we obtain

$$\lim_{n\to\infty} D_f(\theta^f_N x_n, \theta^f_{N-1} x_n) = \lim_{n\to\infty} D_f(z_n, \theta^f_{N-1} x_n) = 0.$$

From Lemma 2.2, we have

$$\lim_{n\to\infty} \| \theta^f_N x_n - \theta^f_{N-1} x_n \| = 0. \tag{3.16}$$

Since f is uniformly Fréchet differentiable, it follows from Lemma 2.4 and (3.16) that

$$\lim_{n\to\infty} \| \nabla f(\theta^f_N x_n) - \nabla f(\theta^f_{N-1} x_n) \|_* = 0.$$

Again, since $x^* \in EP(g_{N-1}) = Fix(Res^f_{g_{N-1}})$, it follows from Lemma 2.3, (3.11) and (3.15) that

$$\begin{aligned} D_f(\theta^f_{N-1} x_n, \theta^f_{N-2} x_n) &= D_f(Res^f_{g_{N-1}} \theta^f_{N-2} x_n, \theta^f_{N-2} x_n) \\ &\le D_f(x^*, \theta^f_{N-2} x_n) - D_f(x^*, \theta^f_{N-1} x_n) \\ &\le D_f(x^*, x_n) - D_f(x^*, z_n) \to 0, \quad n \to \infty. \end{aligned}$$

Again, we obtain

$$\lim_{n\to\infty} D_f(\theta^f_{N-1} x_n, \theta^f_{N-2} x_n) = 0.$$

From Lemma 2.2, we have

$$\lim_{n\to\infty} \| \theta^f_{N-1} x_n - \theta^f_{N-2} x_n \| = 0, \tag{3.17}$$

and hence

$$\lim_{n\to\infty} \| \nabla f(\theta^f_{N-1} x_n) - \nabla f(\theta^f_{N-2} x_n) \|_* = 0. \tag{3.18}$$

Similarly, we can verify that

$$\lim_{n\to\infty} \| \theta^f_{N-2}x_n - \theta^f_{N-3}x_n \| = \cdots = \lim_{n\to\infty} \| \theta^f_1 x_n - x_n \| = 0. \tag{3.19}$$

Now, using (3.16), (3.17) and (3.19), we conclude that

$$\lim_{n\to\infty} \| \theta^f_k x_n - \theta^f_{k-1}x_n \| = 0, \quad k = 1, 2, \cdots, N. \tag{3.20}$$

Since $x_{n_j} \rightharpoonup p$ and $\lim_{n\to\infty} \| x_n - z_n \| = 0$, from (3.16), (3.17) and (3.19), $\theta^f_k x_{n_j} \rightharpoonup p, \quad j \to \infty$, for each $k = 1, 2, \cdots, N$. Also, using (3.20), we obtain

$$\lim_{j\to\infty} \| \nabla f(\theta^f_k x_{n_j}) - \nabla f(\theta^f_{k-1}x_{n_j}) \|_* = 0, \quad k = 1, 2, \cdots, N. \tag{3.21}$$

It follows from (2.5) that for each $k = 1, 2, \cdots, N$

$$g_k(\theta^f_k x_{n_j}, y) + \langle y - \theta^f_k x_{n_j}, \nabla f(\theta^f_k x_{n_j}) - \nabla f(\theta^f_{k-1}x_{n_j})\rangle \geq 0, \quad \forall y \in \mathcal{C},$$

and by using (**C2**) we obtain

$$\langle y - \theta^f_k x_{n_j}, \nabla f(\theta^f_k x_{n_j}) - \nabla f(\theta^f_{k-1}x_{n_j})\rangle \geq g_k(y, \theta^f_k x_{n_j}). \tag{3.22}$$

By (**C4**), (3.21), (3.22) and the fact that $\theta^f_k x_{n_j} \rightharpoonup p$, we have for each $k = 1, 2, \cdots, N$

$$g_k(y, p) \leq 0, \quad \forall y \in \mathcal{C}.$$

For fixed $y \in \mathcal{C}$, let $z_t := ty + (1-t)p$ for all $t \in (0, 1]$. Since $\mathcal{C}$ is convex, it follows that $z_t \in \mathcal{C}$. This yields that $g_k(z_t, p) \leq 0$. It follows from (**C1**) and (**C4**) that

$$0 = g_k(z_t, z_t) \leq tg_k(z_t, y) + (1-t)g_k(z_t, p) \leq tg_k(z_t, y), \quad \forall y \in \mathcal{C},$$

and hence $0 \leq g_k(z_t, y)$. Now, from (**C3**), we conclude that

$$g_k(p, y) \geq 0, \quad \forall y \in \mathcal{C}.$$

Therefore $p \in EP(g_k)$, for each $k = 1, 2, \cdots, N$. Thus $p \in \bigcap_{k=1}^N EP(g_k)$. Therefore

$$p \in \Omega := Fix(T) \cap \bigcap_{k=1}^N EP(g_k).$$

Since E is a reflexive Banach space and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\} \rightharpoonup q \in \mathcal{C}$ and

$$\limsup_{n\to\infty} \langle \nabla f(x_n) - \nabla f(p), x_n - p\rangle = \langle \nabla f(x_n) - \nabla f(p), q - p\rangle.$$

On the other hand, since $\| x_{n_j} - T(x_{n_j}) \| \to 0$ as $j \to \infty$, we have $q \in \widehat{Fix}(T) = Fix(T)$. It follows from the definition of the Bregman projection that

$$\limsup_{n\to\infty} \langle \nabla f(x_n) - \nabla f(p), x_n - p\rangle = \langle \nabla f(x_n) - \nabla f(p), q - p\rangle \leq 0. \tag{3.23}$$

Now, by Lemma 2.5, (3.5) and (3.23), we conclude that $\lim_{n\to\infty} D_f(p, x_n) = 0$. Therefore, by Lemma 2.2, we have $x_n \to p$, as $n \to \infty$.

Case 2. Suppose that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $D_f(x^*, x_{n_i}) < D_f(x^*, x_{n_{i+1}})$ for all $i \in \mathbb{N}$. Then by Lemma 2.6, there exists a nondecreasing sequence $\{m_k\} \subseteq \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all $k \in \mathbb{N}$:

$$D_f(x^*, x_{m_k}) \leq D_f(x^*, x_{m_k+1}) \quad \textit{and} \quad D_f(x^*, x_k) \leq D_f(x^*, x_{m_k+1}).$$

Furthermore, we obtain

$$\begin{aligned}
&D_f(x^*, x_{m_k}) - D_f(x^*, T(x_{m_k})) \\
&= D_f(x^*, x_{m_k}) - D_f(x^*, x_{m_k+1}) + D_f(x^*, x_{m_k+1}) - D_f(x^*, T(x_{m_k})) \\
&\leq D_f(x^*, x_{m_k}) - D_f(x^*, x_{m_k+1}) + \beta_{m_k}(D_f(x^*, x) - D_f(x^*, x_{m_k})) \to 0,
\end{aligned}$$

as $k \to \infty$. Therefore $\lim_{n\to\infty} D_f(T(x_{m_k}), x_{m_k}) = 0$. By the same arguments as in Case 1, we obtain that

$$\limsup_{k\to\infty} \langle \nabla f(x_{m_k}) - \nabla f(p), x_{m_k} - p\rangle \leq 0, \tag{3.24}$$

and

$$D_f(p, x_{m_k+1}) \leq (1 - \beta_{m_k}) D_f(p, x_{m_k}) + \beta_{m_k} \langle \nabla f(x_{m_k}) - \nabla f(p), x_{m_k} - p\rangle. \tag{3.25}$$

Since $D_f(p, x_{m_k}) \leq D_f(p, x_{m_k+1})$, we conclude that

$$\begin{aligned}
\beta_{m_k} D_f(p, x_{m_k}) &\leq D_f(p, x_{m_k}) - D_f(p, x_{m_k+1}) + \beta_{m_k} \langle \nabla f(x_{m_k}) - \nabla f(p), x_{m_k} - p\rangle \\
&\leq \beta_{m_k} \langle \nabla f(x_{m_k}) - \nabla f(p), x_{m_k} - p\rangle,
\end{aligned}$$

and since $\beta_{m_k} > 0$, we get

$$D_f(p, x_{m_k}) \leq \langle \nabla f(x_{m_k}) - \nabla f(p), x_{m_k} - p\rangle. \tag{3.26}$$

Eventually, from (3.24) and (3.26), we have $D_f(p, x_{m_k}) \to 0, \quad k \to \infty$, and from (3.25) and (3.26), we have $D_f(p, x_{m_k+1}) \to 0, \quad k \to \infty$. Since $D_f(p, x_k) \leq D_f(p, x_{m_k+1})$ for all $k \in \mathbb{N}$, we conclude that $x_k \to p$, as $k \to \infty$. This completes the proof. □

4 A Numerical Experiment

In the following, we provide a numerical example of a Bregman strongly nonexpansive mapping satisfying the conditions of Theorem 3.1 and some numerical experiment results to clarify the conclusion of our algorithm (3.1). Let $E = \mathbb{R}$ be equipped with the usual absolute value norm. For $\mathcal{C} = [-\frac{2}{\pi}, 0]$ and $\theta \in [0, \frac{\pi}{2}]$, we consider the rotation mapping $\mathcal{A}_\theta : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by

$$\mathcal{A}_\theta(x, y) := \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

For $\theta = \frac{\pi}{2}$, we get

$$g_1(x,y) = g_2(x,y) = \cdots = g_N(x,y) = \mathcal{A}_{\frac{\pi}{2}}(x,y).$$

It is easy to see that g_k for each $k = 1, \cdots, N$ satisfies the conditions $(\mathbf{C1}) - (\mathbf{C4})$ and $0 \in EP(g_k)$, since $\mathcal{A}_{\frac{\pi}{2}}(0,y) = -y \geq 0$ for all $y \in \mathcal{C}$. Let $f(x) = ||x||^2$ then $\nabla f(x) = 2x$ and $\nabla f^*(x) = \frac{1}{2}x$ and the Bregman projection $P^f_{\mathcal{C}}$ reduces to the metric projection $P_{\mathcal{C}}$ from $\mathbb{R}$ into $\mathcal{C}$. Now, let $T : \mathcal{C} \longrightarrow \mathcal{C}$ be defined by

$$T(x) = \begin{cases} 0, & x = 0, \\ x|\sin(\frac{1}{x})|, & x \in [-\frac{2}{\pi}, 0). \end{cases}$$

Clearly, $Fix(T) \neq \emptyset$ (see Figure 1). Thus, for each $x \in [-\frac{2}{\pi}, 0]$ and $p \in Fix(T)$ we have

$$\begin{aligned} D_f(p, T(x)) &= f(p) - f(T(x)) - \langle \nabla f(T(x)), p - T(x) \rangle \\ &= ||p||^2 - ||T(x)||^2 - \langle 2T(x), p - T(x) \rangle \\ &= ||p||^2 + ||T(x)||^2 - 2\langle T(x), p \rangle \\ &= ||p - T(x)||^2 = \left\| 0 - x|\sin(\frac{1}{x})| \right\|^2 \\ &\leq ||0 - x||^2 = ||p - x||^2 \\ &= D_f(p, x). \end{aligned}$$

Moreover, T is uniformly continuous and we have $Fix(T) = \widehat{Fix}(T)$. Now, we put $\alpha_n = \frac{1}{n+100}$, $\beta_n = \frac{1}{10n+100} + 0.01$, $\gamma_n = \frac{1}{10n+100} + 0.01$, $\eta_n = 1 - \frac{2}{10n+100} - 0.02$. Let $x_1 = -\frac{1}{2\pi} \in [-\frac{2}{\pi}, 0]$ and fix $y = 0$, then the algorithm (3.1) reduces to:

$$\begin{cases} z_n = Res^f_{g_N} \circ Res^f_{g_{N-1}} \circ \cdots \circ Res^f_{g_2} \circ Res^f_{g_1}(x_n) = (\frac{2}{3})^N x_n, \\ \\ y_n = \frac{1}{2}(2\alpha_n x_n + 2(1-\alpha_n)T(z_n)) = \alpha_n x_n + (1-\alpha_n)T(z_n), \\ \\ x_{n+1} = \frac{1}{2}(2\beta_n x_n + 2\gamma_n z_n + 2\eta_n T(y_n)) = \beta_n x_n + \gamma_n z_n + \eta_n T(y_n). \end{cases}$$

Here comes the table of numerical results for the first step $x_1 = -\frac{1}{2\pi}$ (see Table 1). Note that x_n converges to zero. On the other hand

$$Fix(T) \cap \bigcap_{k=1}^{N} EP(g_k) = \{0\}.$$

The list plot of our algorithm is shown in Figure 1.

Numerical Results			
Iteration	x_n (our algorithm)	x_n (Kumama)	x_n (Shehu)
1	-0.159155	-0.159155	-0.159155
2	-0.003458	-0.003622	0.000030
3	-0.000108	-0.000995	0.000028
4	-3.79900×10^{-6}	-0.000630	0.000027
5	-9.41535×10^{-8}	-0.000158	0.000026
6	-2.75527×10^{-9}	-9.37770×10^{-6}	0.000025
7	-1.13377×10^{-10}	-2.13992×10^{-6}	0.000024
8	-2.36551×10^{-12}	-9.97458×10^{-7}	0.000023
9	-8.12392×10^{-14}	-1.72109×10^{-7}	0.000022
10	-2.38078×10^{-15}	-1.13249×10^{-7}	0.000022
11	-6.43541×10^{-17}	-2.71047×10^{-8}	0.000021
12	-1.67915×10^{-18}	-1.35376×10^{-8}	0.000021
13	-3.38989×10^{-20}	-4.22246×10^{-9}	0.000020
14	-1.09766×10^{-21}	-3.82420×10^{-10}	0.000020
15	-3.48769×10^{-23}	-2.08948×10^{-11}	0.000019
16	-8.28869×10^{-25}	-4.14498×10^{-12}	0.000019
17	-2.36274×10^{-26}	-1.93473×10^{-12}	0.000018
18	-6.83535×10^{-28}	-1.19831×10^{-13}	0.000018
19	-9.88195×10^{-30}	-1.29570×10^{-15}	0.000018
20	-1.90481×10^{-31}	-4.09951×10^{-17}	0.000017
21	-3.31514×10^{-33}	-8.69304×10^{-18}	0.000017
22	-1.08697×10^{-34}	-2.79989×10^{-18}	0.000017
23	-2.71051×10^{-36}	-3.12619×10^{-19}	0.000017
24	-6.51009×10^{-38}	-1.24651×10^{-19}	0.000016
25	-1.35089×10^{-39}	-1.08602×10^{-20}	0.000016
26	-3.20757×10^{-41}	-1.44533×10^{-21}	0.000016
27	-4.49109×10^{-43}	-5.53612×10^{-23}	0.000016
28	-7.81907×10^{-45}	-1.93916×10^{-23}	0.000015
29	-1.10054×10^{-46}	-4.56055×10^{-24}	0.000015
30	-2.44983×10^{-48}	-1.94134×10^{-24}	0.000015
31	-7.81145×10^{-50}	-6.56617×10^{-25}	0.000015
32	-1.36587×10^{-51}	-1.37873×10^{-25}	0.000015
33	-4.40603×10^{-53}	-7.80601×10^{-26}	0.000014
34	-1.31616×10^{-54}	-4.32801×10^{-26}	0.000014
35	-2.99150×10^{-56}	-2.45481×10^{-26}	0.000014
36	-9.88461×10^{-58}	-7.29338×10^{-27}	0.000014
37	-3.13348×10^{-59}	-7.52123×10^{-28}	0.000014
38	-4.34489×10^{-61}	-5.11110×10^{-29}	0.000014
39	-1.14032×10^{-62}	-2.26530×10^{-29}	0.000013
40	-3.84576×10^{-64}	-3.93617×10^{-30}	0.000013

Table 1: Numerical results corresponding to $x_1 = -\dfrac{1}{2\pi}$ for 40 steps.

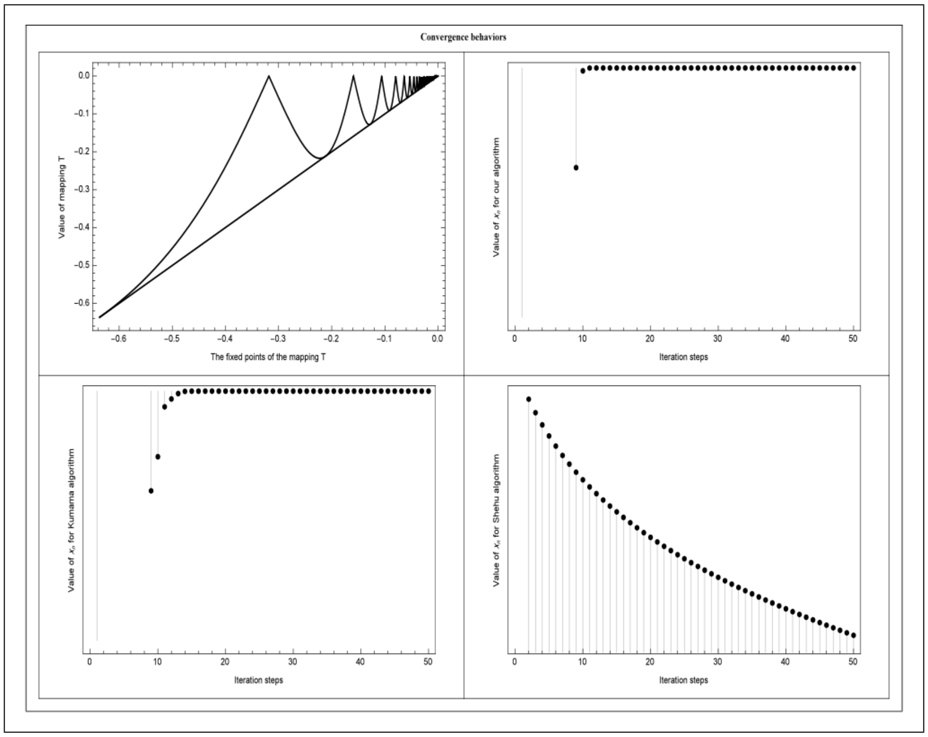

Figure 1: Convergence behaviors of the introduced algorithms

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