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Eulerian Glued Graphs

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Abstract : The Glued Graph is the graph that results from combining two graphs by overlapping their subgraphs with respect to some isomorphism.

In this paper we show that the glued graph of any two nontrivial connected graphs is Eulerian if and only if the following conditions hold:

1.) the clones of the two original graphs contain all odd vertices from their two original graphs,

2.) every even vertex in the clones of the two original graphs is obtained from both odd or both even vertices in the two original graphs, and every odd vertex in those clones is obtained from one odd and one even vertex in the two original graphs.

Moreover, the glued graph of two Eulerian graphs is also Eulerian if and only if the clones are Eulerian. In addition, we show that the glued graph of two connected graphs, where one of these is Eulerian and one is not, is Eulerian if and only if the following conditions hold:

1.) the clone of the non-Eulerian graph contains all odd vertices,

2.) every even vertex in the clones of the two original graphs is obtained from both even vertices in two original graphs.

Keywords : Glued graph, Eulerian graph.

1 Introduction

Eulerian graph is one of the most important topics in graph theory. This topic often appears in various situations, for example, the transportation problem, the traveler's problem, and diagram-tracing puzzles. In 1736, Leonhard Euler [1] solved a well-known puzzle: "*Seven Bridges of Königsberg,*"and published under the title "*Solutio Problematis ad Geometriam Situs Pertinentis (The solution to a problem relating to the geometry of position)*". From his observation, we use the following theorem to prove our own.

Theorem 1.1. *A nontrivial connected graph G is Eulerian if and only if every vertex of G has even degree.*

"For any two nontrivial connected graphs G_1 and G_2, the *Cartesian product* $G_1 \times G_2$ is Eulerian if and only if both G_1 and G_2 are Eulerian or every vertex of G_1 and G_2 is odd vertex". The previous statement was proved in [2]. In this paper we will make a new statement, but this time on new operation namely "Glued Graphs,"which was defined by C. Promsakon and C. Uiyyasathian[3].

Let G_1 and G_2 be any nontrivial graphs, $H_1 \subseteq G_1$ and $H_2 \subseteq G_2$ be connected, not a single vertex and such that $H_1 \cong H_2$ with an isomorphism f. The *glued graph of G_1 and G_2 at H_1 and H_2 with respect to f*, denoted by $G_1 \underset{H_1 \cong_f H_2}{\triangleleft\!\triangleright} G_2$, is the graph that results from combining G_1 with G_2 by identifying H_1 and H_2 with respect to the isomorphism f between H_1 and H_2. Let H be the copy of H_1 and H_2 in the glued graph. We refer to H, H_1 and H_2 as the *clones* of the glued graph, G_1 and G_2, respectively, and refer to G_1 and G_2 as the *original graphs*. The *glued graph of G_1 and G_2 at the clone H*, written $G_1 \underset{H}{\triangleleft\!\triangleright} G_2$, means that there exist subgraph H_1 of G_1, subgraph H_2 of G_2 and isomorphism f between H_1 and H_2 such that $G_1 \underset{H}{\triangleleft\!\triangleright} G_2 = G_1 \underset{H_1 \cong_f H_2}{\triangleleft\!\triangleright} G_2$ and H is the copy of H_1 and H_2 in the resulting graph.

For our convenience, we let G_1 and G_2 be any nontrivial graphs with vertex sets $V(G_1) = \{u_1, u_2, \ldots, u_{n_1}\}$ and $V(G_2) = \{v_1, v_2, \ldots, v_{n_2}\}$, respectively, where n_1 and n_2 are natural numbers greater than 1. Let $H_1 \subseteq G_1$ and $H_2 \subseteq G_2$ be connected, not a single vertex with $V(H_1) = \{u_1, u_2, \ldots, u_n\}$ and $V(H_2) = \{v_1, v_2, \ldots, v_n\}$ where $2 \leq n \leq \min\{n_1, n_2\}$ such that $H_1 \cong H_2$ with an isomorphism f defined by $f(u_i) = v_i$ for every $i \in \{1, 2, \ldots, n\}$. The *labeled glued graph related to $G_1 \underset{H_1 \cong_f H_2}{\triangleleft\!\triangleright} G_2$*, denoted by $G_1 \underset{H_1 \cong_f H_2}{\bar{\triangleleft\!\triangleright}} G_2$, is the glued graph $G_1 \underset{H_1 \cong_f H_2}{\triangleleft\!\triangleright} G_2$ in which each vertex is labeled by (u_i, v_0) if the vertex is obtained from only u_i of $G_1 \backslash H_1$, (u_0, v_i) if the vertex is obtained from only v_i of $G_2 \backslash H_2$, or (u_i, v_i) if the vertex is obtained from both u_i of H_1 and v_i of H_2. The *labeled clone H' related to the clone H* is the clone H in which each vertex related to (u_i, v_i) in the labeled glued graph is labeled by w_i. We refer to G_1 and G_2 as the *labeled original graphs*. The *labeled glued graph of G_1 and G_2 at the labeled clone H'*, written $G_1 \underset{H'}{\bar{\triangleleft\!\triangleright}} G_2$, means that there exist subgraph H_1 of G_1 and subgraph H_2 of G_2 and isomorphism f between H_1 and H_2 such that $G_1 \underset{H'}{\bar{\triangleleft\!\triangleright}} G_2 = G_1 \underset{H_1 \cong_f H_2}{\bar{\triangleleft\!\triangleright}} G_2$.

The glued graph of any two nontrivial connected graphs may be Eulerian or non-Eulerian. In this paper, we consider the properties of the two original graphs and the clones of them to summarize the Eulerian of resulting graph. The results are presented in the next section.

2 Eulerian Glued Graph

To apply theorem 1.1 in our theorem, we must seek the degree of each vertex of the glued graph. There are relationships between the degree of the vertex of the glued graph, the two original graphs and the clone that can be stated in the following lemma.

Lemma 2.1. *Let G_1 and G_2 be any nontrivial graphs, $H_1 \subseteq G_1$, $H_2 \subseteq G_2$. Let H' be the labeled clone of the labeled glued graph $G_1 \underset{H'}{\triangleleft\!\!\!\triangleright} G_2$ and i be a natural number. Then the degree of each vertex in the labeled glued graph is as follows:*

1.) $\deg_{G_1 \underset{H'}{\triangleleft\!\!\!\triangleright} G_2}(u_i, v_0) = \deg_{G_1} u_i$ *for the vertex obtained from only u_i of $G_1 \backslash H_1$,*

2.) $\deg_{G_1 \underset{H'}{\triangleleft\!\!\!\triangleright} G_2}(u_0, v_i) = \deg_{G_2} v_i$ *for the vertex obtained from only v_i of $G_2 \backslash H_2$,*

3.) $\deg_{G_1 \underset{H'}{\triangleleft\!\!\!\triangleright} G_2}(u_i, v_i) = \deg_{G_1} u_i + \deg_{G_2} v_i - \deg_{H'} w_i$ *for the vertex obtained from both u_i of H_1 and v_i of H_2.*

Proof. Let G_1 and G_2 be any nontrivial graphs, $H_1 \subseteq G_1$, $H_2 \subseteq G_2$. Let H' be the labeled clone of the labeled glued graph $G_1 \underset{H'}{\triangleleft\!\!\!\triangleright} G_2$ and i be a natural number. The vertices $(u_i, v_0), (u_0, v_i)$ and (u_i, v_i) are the vertices which are obtained from only u_i of $G_1 \backslash H_1$, only v_i of $G_2 \backslash H_2$ and both u_i of H_1 and v_i of H_2, respectively. Since u_i in $G_1 \backslash H_1$ becomes (u_i, v_0) and v_i in $G_2 \backslash H_2$ becomes (u_0, v_i) without combining of the two labeled original graphs, the degree is still the same. So $\deg_{G_1 \underset{H'}{\triangleleft\!\!\!\triangleright} G_2}(u_i, v_0) = \deg_{G_1} u_i$ and $\deg_{G_1 \underset{H'}{\triangleleft\!\!\!\triangleright} G_2}(u_0, v_i) = \deg_{G_2} v_i$. Finally, we see that each edge of H' contributes twice in the degree of (u_i, v_i). Thus, $\deg_{G_1 \underset{H'}{\triangleleft\!\!\!\triangleright} G_2}(u_i, v_i) = \deg_{G_1} u_i + \deg_{G_2} v_i - \deg_{H'} w_i$. ∎

Remark 2.2. *Because H_1 and H_2 are subgraphs of G_1 and G_2, respectively, therefore $\deg_{H'} w_i \leq \min\{\deg_{G_1} u_i, \deg_{G_2} v_i\}$ for the vertex (u_i, v_i) in the labeled glued graph which is related to w_i in H'. Hence $\deg_{G_1 \underset{H'}{\triangleleft\!\!\!\triangleright} G_2}(u_i, v_i) > 0$.*

Remark 2.3. *Since the glued graph is an image of the labeled glued graph related to it, the glued graph preserves the properties of its labeled glued graph.*

Now, we state of the main rusult theorem.

Theorem 2.4. *The glued graph of any two nontrivial connected graphs is Eulerian if and only if the following conditions hold:*

1.) the clones of the two original graphs contain all odd vertices in their two original graphs,

2.) every even vertex in the clones of the two original graphs is obtained from both odd or both even vertices in the two original graphs, and every odd vertex in those clones is obtained from one odd and one even vertex in the two original graphs.

Proof. To prove this theorem we first consider the labeled glued graph and apply the result to the glued graph. Let G_1 and G_2 be any nontrivial connected graphs, $H_1 \subseteq G_1$, $H_2 \subseteq G_2$. Let H' be the labeled clone of the labeled glued graph $G_1 \underset{H'}{\triangleleft\!\triangleright} G_2$ and i be a natural number. The vertices $(u_i, v_0), (u_0, v_i)$ and (u_i, v_i) are the vertices obtained from only u_i of $G_1 \backslash H_1$, only v_i of $G_2 \backslash H_2$ and both u_i of H_1 and v_i of H_2, respectively. And each vertex (u_i, v_i) in the labeled glued graph related with w_i of the labeled clone. First, suppose that $G_1 \underset{H'}{\triangleleft\!\triangleright} G_2$ is Eulerian, that is, every vertex of $G_1 \underset{H'}{\triangleleft\!\triangleright} G_2$ has even degree. If the labeled clone does not contain all odd vertices of two labeled original graphs, an odd vertex will appear in $G_1 \underset{H'}{\triangleleft\!\triangleright} G_2$, which is a contradiction. Therefore, all odd vertices of the two labeled original graphs must be contained in the labeled clone. Consider each even vertex in the labeled clone. Since $\deg_{G_1 \underset{H'}{\triangleleft\!\triangleright} G_2}(u_i, v_i)$ and $\deg_{H'} w_i$ are both even, $\deg_{G_1} u_i$ and $\deg_{G_2} v_i$ must be both odd or both even (Lemma 2.1). A similar argument holds for the odd vertices. Because $\deg_{G_1 \underset{H'}{\triangleleft\!\triangleright} G_2}(u_i, v_i)$ is even but $\deg_{H'} w_i$ is odd, $\deg_{G_1} u_i + \deg_{G_2} v_i$ must be odd by Lemma 2.1, that is one must be odd and another one must be even. Conversely, assume that the labeled clones of the two labeled original graphs contain all odd vertices in their two labeled original graphs. Also suppose that that every even vertex in the labeled clones of the two labeled original graphs is obtained from both odd or both even vertices in the two labeled original graphs, and every odd vertex in those labeled clones is obtained from one odd and one even vertex in the two original graphs. So $\deg_{G_1 \underset{H'}{\triangleleft\!\triangleright} G_2}(u_i, v_0)$ and $\deg_{G_1 \underset{H'}{\triangleleft\!\triangleright} G_2}(u_0, v_i)$ are both even. Consider each even vertex in each labeled clone which is obtained from both odd or both even vertices in the two labeled original graphs, $\deg_{G_1} u_i + \deg_{G_2} v_i$ and $\deg_{H'} w_i$ are even. By Lemma 2.1, $\deg_{G_1 \underset{H'}{\triangleleft\!\triangleright} G_2}(u_i, v_i)$ is even. For each odd vertex in each labeled clone which is obtained from one odd and one even vertex in the two labeled original graphs, the argument is the same. Since $\deg_{G_1} u_i + \deg_{G_2} v_i$ and $\deg_{H'} w_i$ are odd. Clearly, by Lemma 2.1, $\deg_{G_1 \underset{H'}{\triangleleft\!\triangleright} G_2}(u_i, v_i)$ is even. Hence each vertex of $G_1 \underset{H'}{\triangleleft\!\triangleright} G_2$ is an even vertex. This concludes that the labeled glued graph is Eulerian and so is its image. ■

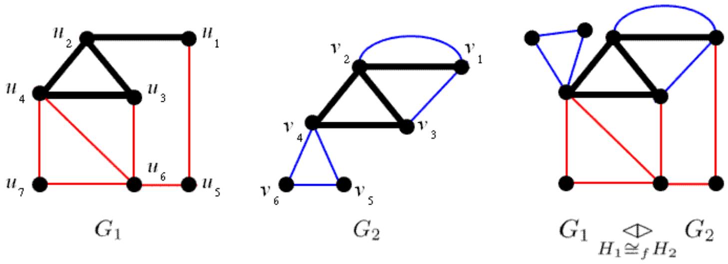

Figure 1: The glued graph of two non-Eulerian connected graphs.

Consider Figure 1, we use the bold graph to illustrate the subgraph H_1 of G_1, H_2 of G_2 and H' of $G_1 \underset{H_1 \cong_f H_2}{\lhd\!\rhd} G_2$ .Clearly, G_1 and G_2 are two non-Eulerian connected graphs, but $G_1 \underset{H_1 \cong_f H_2}{\lhd\!\rhd} G_2$ is an Eulerian graph with an isomorphism f between H_1 and H_2 defined by $f(u_i) = v_i$ for every $i \in \{1, 2, 3, 4\}$. Note that all odd vertices $u_1, u_2 \in V(H_1)$ and $v_1, v_3 \in V(H_2)$. We can see that even vertices in the clones of the two original graphs is obtained from both odd (u_3 and v_3) or both even vertices (u_4 and v_4) in the two original graphs and odd vertices in clones is obtained from one odd and one even vertex (u_1, v_1 and u_2, v_2) in the two original graphs.

A special case of Theorem 2.4 when the two original graphs are Eulerian is stated in the following.

Corollary 2.5. *The glued graph of two Eulerian graphs is also Eulerian if and only if the clones of two original graphs are Eulerian.*

Proof. Let two original graphs be Eulerian. For necessity, suppose that the glued graph is Eulerian. By Theorem 2.4, we obtain that there is no odd vertex in the clones of the two original graphs. And therefore, the clones are Eulerian. Conversely, suppose that the clones are Eulerian. Since two original graphs have only even vertices, Theorem 2.4 concludeds that the glued graph is Eulerian. ■

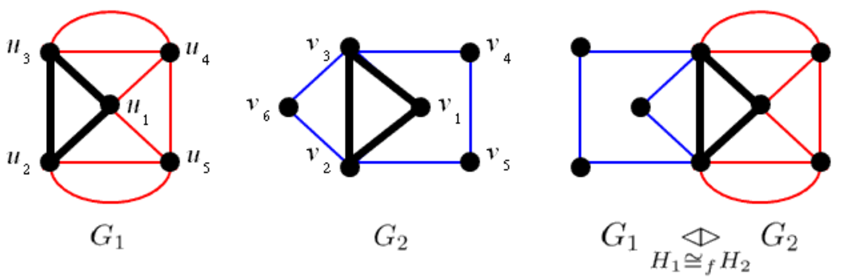

Figure 2: The glued graph of two Eulerian graphs.

We now given an example of Corollary 2.5. In Figure 2, two original graphs, the clones of two original graphs, and the glued graph are all Eulerian. The bold graph in each of them illustrates the subgraph itself and an isomorphism f from subgraph of G_1 to one of G_2 defined by $f(u_i) = v_i$ for every $i \in \{1, 2, 3\}$.

Another special case arises when one of the original graphs is Eulerian and another one is not. This case is stated in the following corollary.

Corollary 2.6. *The glued graph of two connected graphs, one of these is Eulerian and one is not, is Eulerian if and only if the following conditions hold:*

1.) the clone of the non-Eulerian graph contains all odd vertices,

2.) every even vertex in clones of two original graphs is obtained from both even vertices in two original graphs.

Proof. Let one original graph be Eulerian and the other non-Eulerian but connected. For necessity, suppose that the glued graph is Eulerian. Then by Theorem 2.4, we obtain that the clone of the non-Eulerian graph contains all odd vertices. Moreover, every even vertex in the clones of the two original graphs is obtained from both even vertices in two original graphs. Conversely, suppose that the clone of the non-Eulerian graph contains all odd vertices of the graph, every even vertex in the clones of the two original graphs is obtained from both even vertices in the two original graphs. Since odd vertices appear only in the non-Eulerian graph, Theorem 2.4 concludes that the glued graph is Eulerian. ∎

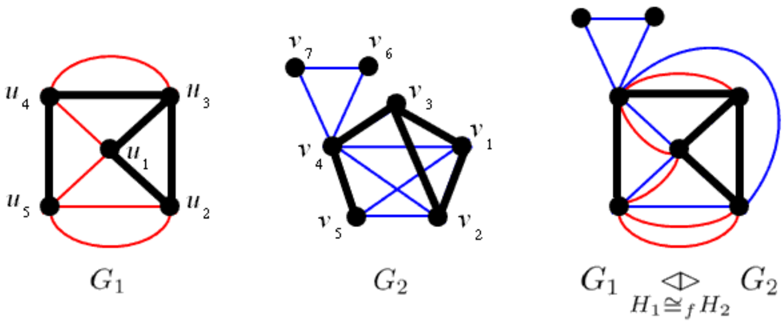

Figure 3: The glued graph of an Eulerian graph with a non-Eulerian graph.

Figure 3 indicates that G_1 is Eulerian but G_2 is not. The bold graph in G_1 and G_2 represent the subgraphs of themselves. All odd vertices of G_1 and G_2 are merely v_3 and v_5 which are also vertices of G_2. These vertices become the odd vertices in the clone of G_2 with an isomorphism f from subgraph H_1 of G_1 to subgraph H_2 of G_2 defined by $f(u_i) = v_i$ for every $i \in \{1, 2, 3, 4, 5\}$. Scrutinizing the vertices in the clones of the two original graphs with even degree, we see that they are obtained from both even vertices in their original graphs. Clearly, in Figure 3, $G_1 \underset{H_1 \cong_f H_2}{\triangleleft\!\triangleright} G_2$ is Eulerian.

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