Starlikeness and Subordination of Two Integral Operators

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Abstract: In this paper, we consider some sufficient conditions for two integral operators to be starlike in the open unit disk.

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1 Introduction and definitions

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$. A function $f$ belonging to $\mathcal{A}$ is said to be starlike of order $\alpha$ if it satisfies

$$\text{Re} \left( \frac{zf'(z)}{f(z)} \right) > \alpha \quad (z \in \mathcal{U})$$

for some $\alpha(0 \leq \alpha < 1)$. We denote by $S^*(\alpha)$ the subclass of $\mathcal{A}$ consisting of functions which are starlike of order $\alpha$ in $\mathcal{U}$. Clearly $S^*(0) = S^*$ the class of all starlike functions with respect the origin.

Recently, Breaz and Breaz in [3] and Breaz et al. [7] introduced and studied the integral operators

$$F_n(z) = \int_0^{\frac{z}{t}} \left( \frac{f_1(t)}{t} \right)^{\alpha_1} \ldots \left( \frac{f_n(t)}{t} \right)^{\alpha_n} \, dt \quad (1.1)$$

and

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\[ F_{\alpha_1,\ldots,\alpha_n}(z) = \int_0^1 (f'_1(t))^\alpha_1 \cdots (f'_n(t))^\alpha_n \, dt \quad (1.2) \]

where \( f_i \in A \) and for \( \alpha_i > 0 \), for all \( i = 1, \ldots, n \) (see also [1, 4, 6]).

Breaz and G"uney [5] considered the above integral operators and they obtained their properties on the classes \( S^*_\alpha(b), C_\alpha(b) \) of starlike and convex functions of complex order \( b \) and type \( \alpha \) introduced and studied by Frasin [8] (see [2]).

Very recently, Frasin [9] obtained some sufficient conditions for the above integral operators to be in the classes \( S^*_\alpha \) and \( C_\alpha \), where \( C_\alpha \) and \( UCV \) denote the subclasses of \( A \) consisting of functions which are, respectively, close-to-convex of order \( \alpha(0 \leq \alpha < 1) \) in \( U \) and uniformly convex functions.

In the present paper, we obtain some sufficient conditions for starlikeness of the above integral operators \( F_n \) and \( F_{\alpha_1,\ldots,\alpha_n} \).

In order to derive our main results, we have to recall here the following results:

\textbf{Lemma 1.1.} (\cite{10}) If \( f \in A \) satisfies

\[ \text{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} < \frac{\beta + 1}{2(\beta - 1)} \quad (z \in U) \quad (1.3) \]

for some \( 2 \leq \beta < 3 \), or

\[ \text{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} < \frac{5\beta - 1}{2(\beta + 1)} \quad (z \in U) \quad (1.4) \]

for some \( 1 < \beta \leq 2 \), then \( f \in S^* \).

\textbf{Lemma 1.2.} (\cite{10}) If \( f \in A \) satisfies

\[ \text{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > -\frac{\beta + 1}{2\beta(\beta - 1)} \quad (z \in U) \quad (1.5) \]

for some \( \beta \leq -1 \), or

\[ \text{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \frac{3\beta + 1}{2\beta(\beta + 1)} \quad (z \in U) \quad (1.6) \]

for some \( \beta > 1 \), then \( f \in S^* \left( \frac{3\beta + 1}{2\beta} \right) \).

\section{Starlikeness for the integral operator \( F_n \)}

Applying Lemma 1.1, we derive

\textbf{Theorem 2.1.} Let \( \alpha_i > 0 \) be real numbers for all \( i = 1, \ldots, n \). If \( f_i \in A \) for all \( i = 1, \ldots, n \) satisfies
Re \( \left( \frac{zf_i'(z)}{f_i(z)} \right) \) < 1 + \( \frac{3 - \beta}{2(\beta - 1)\alpha_i} \) \( (z \in \mathcal{U}) \) \hspace{1cm} (2.1)

for some 2 ≤ \( \beta < 3 \), or

Re \( \left( \frac{zf_i'(z)}{f_i(z)} \right) \) < 1 + \( \frac{3(\beta - 1)}{2(\beta + 1)\alpha_i} \) \( (z \in \mathcal{U}) \) \hspace{1cm} (2.2)

for some 1 < \( \beta \leq 2 \), then \( F_n \in S^* \).

**Proof.** It follows from (1.1) that

\[
F_n'(z) = \left( \frac{f_1(z)}{z} \right)^{\alpha_1} \cdots \left( \frac{f_n(z)}{z} \right)^{\alpha_n}.
\]

Thus we have

\[
F_n''(z) = \left[ \alpha_1 \left( \frac{f_1'(z)}{f_1(z)} - \frac{1}{z} \right) + \cdots + \alpha_n \left( \frac{f_n'(z)}{f_n(z)} - \frac{1}{z} \right) \right] F_n'(z). \hspace{1cm} (2.4)
\]

Then from (2.4), we obtain

\[
z F_n''(z) = \sum_{i=1}^{n} \alpha_i \left( \frac{zf_i'(z)}{f_i(z)} - 1 \right)
\]

or, equivalently,

\[
1 + \frac{z F_n''(z)}{F_n'(z)} = \sum_{i=1}^{n} \alpha_i \left( \frac{zf_i'(z)}{f_i(z)} \right) + 1 - \sum_{i=1}^{n} \alpha_i. \hspace{1cm} (2.6)
\]

Taking the real part of both terms of (2.6), we have

\[
\text{Re} \left( 1 + \frac{z F_n''(z)}{F_n'(z)} \right) = \sum_{i=1}^{n} \alpha_i \text{Re} \left( \frac{zf_i'(z)}{f_i(z)} \right) + 1 - \sum_{i=1}^{n} \alpha_i
\]

\[
= \alpha_1 \text{Re} \left( \frac{zf_1'(z)}{f_1(z)} \right) + \alpha_2 \text{Re} \left( \frac{zf_2'(z)}{f_2(z)} \right) + \cdots
\]

\[
+ \alpha_n \text{Re} \left( \frac{zf_n'(z)}{f_n(z)} \right) + 1 - [\alpha_1 + \alpha_2 + \cdots + \alpha_n], \hspace{1cm} (2.7)
\]

using the hypothesis (2.1) it follows from (2.7) that

\[
\text{Re} \left( 1 + \frac{z F_n''(z)}{F_n'(z)} \right) < \alpha_1 \left( 1 + \frac{3 - \beta}{2(\beta - 1)\alpha_1} \right) + \alpha_2 \left( 1 + \frac{3 - \beta}{2(\beta - 1)\alpha_2} \right) + \cdots
\]

\[
+ \alpha_n \left( 1 + \frac{3 - \beta}{2(\beta - 1)\alpha_n} \right) + 1 - [\alpha_1 + \alpha_2 + \cdots + \alpha_n]
\]

\[
< \frac{\beta + 1}{2(\beta - 1)} \hspace{1cm} (z \in \mathcal{U}),
\]
for some $2 \leq \beta < 3$. Also from the hypothesis (2.2) and (2.7), we get

$$\Re \left( 1 + \frac{zf''(z)}{f(z)} \right) < \frac{5\beta - 1}{2(\beta + 1)} \quad (z \in U),$$

for some $1 < \beta \leq 2$. Hence by Lemma 1.1 we get $F \in S^*$. This completes the proof.

Letting $n = 1$, $\alpha_1 = \alpha$ and $f_1 = f$ in Theorem 2.1, we have

**Corollary 2.2.** Let $\alpha > 0$. If $f \in A$ satisfies

$$\Re \left( \frac{zf''(z)}{f(z)} \right) < 1 + \frac{3 - \beta}{2(\beta - 1)\alpha} \quad (z \in U),$$

for some $2 \leq \beta < 3$, or

$$\Re \left( \frac{zf''(z)}{f(z)} \right) < 1 + \frac{3(\beta - 1)}{2(\beta + 1)\alpha} \quad (z \in U),$$

for some $1 < \beta \leq 2$, then $\int_0^z \left( \frac{f(t)}{t} \right)^\alpha dt \in S^*$. Applying Lemma 1.2, we derive

**Theorem 2.3.** Let $\alpha_i > 0$ be real numbers for all $i = 1, \ldots, n$. If $f_i \in A$ for all $i = 1, \ldots, n$ satisfies

$$\Re \left( \frac{zf''(z)}{f(z)} \right) > 1 + \frac{\beta - 2\beta^2 - 1}{2\beta(\beta - 1)\alpha_i} \quad (z \in U),$$

for some $\beta \leq -1$, or

$$\Re \left( \frac{zf''(z)}{f(z)} \right) > 1 + \frac{\beta - 2\beta^2 + 1}{2\beta(\beta + 1)\alpha_i} \quad (z \in U),$$

for some $\beta > 1$, then $F_n \in S^* \left( \frac{\beta + 1}{2\beta} \right)$.

**Proof.** Using (2.7), (2.8), (2.9) and applying Lemma 1.2, we have $F_n \in S^* \left( \frac{\beta + 1}{2\beta} \right)$.

Letting $n = 1$, $\alpha_1 = \alpha$ and $f_1 = f$ in Theorem 2.3, we have

**Corollary 2.4.** Let $\alpha > 0$. If $f \in A$ satisfies

$$\Re \left( \frac{zf''(z)}{f(z)} \right) > 1 + \frac{\beta - 2\beta^2 - 1}{2\beta(\beta - 1)\alpha} \quad (z \in U),$$

for some $\beta \leq -1$, or

$$\Re \left( \frac{zf''(z)}{f(z)} \right) > 1 + \frac{\beta - 2\beta^2 + 1}{2\beta(\beta + 1)\alpha} \quad (z \in U),$$

for some $\beta > 1$, then $\int_0^z \left( \frac{f(t)}{t} \right)^\alpha dt \in S^* \left( \frac{\beta + 1}{2\beta} \right)$. 


3 Starlikeness for the integral operator $F_{\alpha_1, \ldots, \alpha_n}$

Applying Lemma [1.1], we derive

**Theorem 3.1.** Let $\alpha_i > 0$ be real numbers for all $i = 1, \ldots, n$. If $f_i \in A$ for all \( i = 1, \ldots, n \) satisfies

$$\text{Re} \left( \frac{zf''_i(z)}{f'_i(z)} \right) < \frac{3 - \beta}{2(\beta - 1)n\alpha_i}$$

(3.1)

for some $2 \leq \beta < 3$, or

$$\text{Re} \left( \frac{zf''_i(z)}{f'_i(z)} \right) < \frac{3(\beta - 1)}{2(\beta + 1)n\alpha_i}$$

(3.2)

for some $1 < \beta \leq 2$, then $F_{\alpha_1, \ldots, \alpha_n} \in S^*$.  

**Proof.** From (1.2), we easily get

$$zF''_{\alpha_1, \ldots, \alpha_n}(z) = \sum_{i=1}^{n} \alpha_i \left( \frac{zf''_i(z)}{f'_i(z)} \right). \quad (3.3)$$

It follows from (3.3) that

$$\text{Re} \left( 1 + \frac{zF''_{\alpha_1, \ldots, \alpha_n}(z)}{F''_{\alpha_1, \ldots, \alpha_n}(z)} \right) = 1 + \sum_{i=1}^{n} \alpha_i \text{Re} \left( \frac{zf''_i(z)}{f'_i(z)} \right)$$

$$= 1 + \alpha_1 \text{Re} \left( \frac{zf''_1(z)}{f'_1(z)} \right) + \alpha_2 \text{Re} \left( \frac{zf''_2(z)}{f'_2(z)} \right) + \cdots$$

$$+ \alpha_n \text{Re} \left( \frac{zf''_n(z)}{f'_n(z)} \right), \quad (3.4)$$

which, in the light of the hypothesis (3.1), yields

$$\text{Re} \left( 1 + \frac{zF''_{\alpha_1, \ldots, \alpha_n}(z)}{F''_{\alpha_1, \ldots, \alpha_n}(z)} \right) < 1 + \alpha_1 \left( \frac{3 - \beta}{2(\beta - 1)n\alpha_1} \right) + \alpha_1 \left( \frac{3 - \beta}{2(\beta - 1)n\alpha_2} \right) + \cdots$$

$$+ \alpha_n \left( \frac{3 - \beta}{2(\beta - 1)n\alpha_n} \right)$$

$$< \frac{\beta + 1}{2(\beta - 1)} \quad (z \in U),$$

for some $2 \leq \beta < 3$. On the other hand, by the hypothesis (3.2) and (3.4), we have

$$\text{Re} \left( 1 + \frac{zF''_{\alpha_1, \ldots, \alpha_n}(z)}{F''_{\alpha_1, \ldots, \alpha_n}(z)} \right) < \frac{5\beta - 1}{2(\beta + 1)} \quad (z \in U),$$

for some $1 < \beta \leq 2$. Hence by Lemma [1.1] we get $F_{\alpha_1, \ldots, \alpha_n} \in S^*$.
Letting \( n = 1, \alpha_1 = \alpha \) and \( f_1 = f \) in Theorem 3.1, we have

**Corollary 3.2.** Let \( \alpha > 0 \). If \( f \in \mathcal{A} \) satisfies

\[
\text{Re} \left( \frac{zf''(z)}{f'(z)} \right) < \frac{3 - \beta}{2(\beta - 1)\alpha}
\]

for some \( 2 \leq \beta < 3 \), or

\[
\text{Re} \left( \frac{zf''(z)}{f'(z)} \right) < \frac{3(\beta - 1)}{2(\beta + 1)\alpha}
\]

for some \( 1 < \beta \leq 2 \), then \( \int_0^z (f'(t))^\alpha \, dt \in \mathcal{S}^* \).

Finally, we have

**Theorem 3.3.** Let \( \alpha_i > 0 \) be real numbers for all \( i = 1, \ldots, n \). If \( f_i \in \mathcal{A} \) for all \( i = 1, \ldots, n \) satisfies

\begin{align*}
\text{Re} \left( \frac{zf''(z)}{f'(z)} \right) &> \frac{\beta - 2\beta^2 - 1}{2\beta(\beta - 1)n\alpha_i} \quad (3.5) \\
\text{Re} \left( \frac{zf''(z)}{f'(z)} \right) &> \frac{\beta - 2\beta^2 + 1}{2\beta(\beta + 1)n\alpha_i} \quad (3.6)
\end{align*}

for some \( \beta \leq -1 \), or

for some \( \beta > 1 \), then \( F_{\alpha_1, \ldots, \alpha_n} \in \mathcal{S}^* \left( \frac{\beta + 1}{2\beta} \right) \).

**Proof.** The theorem follows easily by using (3.4), (3.5), (3.6) and applying Lemma 1.2. \( \square \)

Letting \( n = 1, \alpha_1 = \alpha \) and \( f_1 = f \) in Theorem 3.3, we have

**Corollary 3.4.** Let \( \alpha > 0 \). If \( f \in \mathcal{A} \) satisfies

\[
\text{Re} \left( \frac{zf''(z)}{f'(z)} \right) > \frac{\beta - 2\beta^2 - 1}{2\beta(\beta - 1)\alpha}
\]

for some \( \beta \leq -1 \), or

\[
\text{Re} \left( \frac{zf''(z)}{f'(z)} \right) > \frac{\beta - 2\beta^2 + 1}{2\beta(\beta + 1)\alpha}
\]

for some \( \beta > 1 \), then \( \int_0^z (f'(t))^\alpha \, dt \in \mathcal{S}^* \left( \frac{\beta + 1}{2\beta} \right) \).

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References


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