On the Bessel Ultra-Hyperbolic Heat Equation

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Abstract: In this article, we study the equation

\[ \frac{\partial}{\partial t} u(x, t) = c^2 \Box^k_B u(x, t) \]

with the initial condition \( u(x, 0) = f(x) \) for \( x \in \mathbb{R}^+_n \). The operator \( \Box^k_B \) is named the Bessel ultra-hyperbolic operator iterated \( k \)-times and is defined by

\[ \Box^k_B = (B_{x_1} + B_{x_2} + \ldots + B_{x_p} - B_{x_{p+1}} - \ldots - B_{x_{p+q}})^k \]

where \( k \) is a non-negative integer, \( p + q = n \), \( B_{x_i} = \frac{\partial^2}{\partial x_i^2} + 2v_i \frac{\partial}{\partial x_i} \), \( 2v_i = 2\alpha_i + 1 \), \( \alpha_i > -\frac{1}{2} \) \([3,5-10] \), \( x_i > 0 \), \( i = 1, 2, \ldots, n \), and \( n \) is the dimension of the \( \mathbb{R}^+_n \), \( u(x, t) \) is an unknown for \( (x, t) = (x_1, \ldots, x_n, t) \in \mathbb{R}^+_n \times (0, \infty) \), \( f(x) \) is a given generalized function and \( c \) is a positive constant. We obtain the solution of such equation which is related to the spectrum and the kernel which is so called the Bessel ultra-hyperbolic heat kernel. Moreover, such the Bessel ultra-hyperbolic heat kernel has interesting properties and also related to the kernel of an extension of the heat equation.

Keywords: Heat kernel, Dirac-delta distribution, Bessel ultra-hyperbolic operator, Fourier Bessel transform, \( B \)-convolution, Spectrum.

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1 Introduction

It is known that for the ultra-hyperbolic heat equation

\[ \frac{\partial}{\partial t} u(x, t) = c^2 \Box^k u(x, t) \]
with the initial condition \( u(x, 0) = f(x) \) where \( \Box^k \) is the ultra-hyperbolic operator iterated \( k \) times defined by

\[
\Box^k = \left( \partial_{x_1}^2 + \partial_{x_2}^2 + \ldots + \partial_{x_p}^2 - \partial_{x_{p+1}}^2 - \ldots - \partial_{x_{p+q}}^2 \right)^k,
\]

\( p + q = n \) is the dimension of the Euclidean space \( \mathbb{R}^n \) and \( k \) is a positive integer.

In [7] Nonlaopon and Kananthai obtained the following solution

\[
u(x, t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(y) \exp \left( c^2 t \left[ \sum_{j=p+1}^{p+q} \xi_j^2 \right]^k + i(\xi, x - y) \right) d\xi dy
\]

or the solution in the classical convolution form

\[
u(x, t) = E(x, t) * f(x) \tag{1.2}
\]

where

\[
E(x, t) = \frac{1}{(2\pi)^n} \int_{\Omega} \exp \left( c^2 t \left[ \sum_{j=p+1}^{p+q} \xi_j^2 \right]^k + i(\xi, x) \right) d\xi \tag{1.3}
\]

and \( \Omega \subset \mathbb{R}^n \) is the spectrum of \( E(x, t) \) for any fixed \( t > 0 \).

We can extend (1.1) to the equation

\[
\frac{\partial}{\partial t} u(x, t) = c^2 \Box_B u(x, t) \tag{1.4}
\]

with the initial condition

\[
u(x, 0) = f(x) \tag{1.5}
\]

where \( B_{x_i} = \partial^2_{x_i} + \sum_{i=1}^{p} \frac{2\nu_i}{x_i} \partial_{x_i} \), \( p + q = n \) is the dimension \( \mathbb{R}_n^+, \mathbb{R}_n^+ = \{ x : x = (x_1, x_2, \ldots , x_n), x_1 > 0, \ldots , x_n > 0 \} \) and \( \Box_B \) is the Bessel ultra-hyperbolic operator, defined by

\[
\Box_B = B_{x_1} + B_{x_2} + \ldots + B_{x_p} - B_{x_{p+1}} - \ldots - B_{x_{p+q}}, \quad p + q = n.
\]

Then, we obtain

\[
u(x, t) = E(x, t) * f(x) \tag{1.6}
\]

as a solution of (1.1) which satisfies (1.5) where \( E(x, t) \) is the kernel of (1.4) or the elementary solution of (1.1) and is defined by

\[
E(x, t) = C_v \frac{e^{-c^2 t V(y)}}{\prod_{i=1}^{n} J_{\nu_i - \frac{1}{2}}(x_i y_i) y_i^{2\nu_i}} \, dy \tag{1.7}
\]

where \( V(y) = \sum_{i=1}^{p} y_i^2 - \sum_{j=p+1}^{p+q} y_j^2 > 0 \).

Moreover, we obtain \( E(x, t) \to \delta \) as \( t \to 0 \) where \( \delta \) is the Dirac-delta distribution, we studied the Bessel ultra-hyperbolic heat kernel which is related to spectrum.
Now, the purpose of this work is to study the equation
\[ \frac{\partial}{\partial t}u(x, t) = c^2 \Box_B^k u(x, t) \] (1.8)
which the initial condition
\[ u(x, 0) = f(x), \quad x \in \mathbb{R}_n^+, \] (1.9)
where the operator \( \Box_B^k \) is named the Bessel ultra-hyperbolic operator iterated \( k \)–times, defined by
\[ \Box_B^k = (B_{x_1} + B_{x_2} + \ldots + B_{x_p} - B_{x_{p+1}} - \ldots - B_{x_p+q})^k, \] (1.10)
where \( k \) is a positive integer.

We obtain \( u(x, t) = E(x, t) \ast f(x) \) a solution in the \( B \)-convolution form of (??) which satisfies condition (??) where
\[ E(x, t) = C_v e^{(-1)^k c^2 t v^k(y)} \prod_{i=1}^{n} J_{v_i - \frac{1}{2}}(x_i y_i) y_i^{2 v_i} dy \] (1.11)
and \( \Omega \subset \mathbb{R}_n^+ \) is the spectrum of \( E(x, t) \) for any fixed \( t > 0 \). The function \( E(x, t) \) is called the Bessel ultra-hyperbolic heat kernel iterated \( k \)–times or the elementary solution of (??). And all properties of \( E(x, t) \) will be studied in details.

2 Preliminaries

The generalized shift operator \( T^y \) has the following form [?, ?, ?]:
\[ T^y \varphi(x) = C_v \prod_{i=1}^{n} \sin^{2 v_i - 1} \theta_i \prod_{i=1}^{n} J_{v_i - \frac{1}{2}}(x_i y_i) y_i^{2 v_i} dy \] (2.1)
where \( x, y \in \mathbb{R}_n^+ \),
\[ C_v^* = \prod_{i=1}^{n} \frac{\Gamma(v_i + 1)}{\Gamma(\frac{1}{2}) \Gamma(v_i)}. \]
We remark that this shift operator is closely connected with the Bessel differential operator \( B = (B_{x_1}, \ldots, B_{x_n}) \) [?].

The convolution operator determined by the \( T^y \) is as follows.
\[ (f \ast \varphi)(x) = \int_{\mathbb{R}_n^+} f(y) T^y \varphi(x)(\prod_{i=1}^{n} y_i^{2 v_i}) dy. \] (2.1)
Convolution (??) known as a \( B \)-convolution. We note the following properties of the \( B \)-convolution and the generalized shift operator.
a. $T^y.1 = 1$

b. $T^0.f(x) = f(x)$

c. If $f(x), g(x) \in C(\mathbb{R}_n^+) \text{, } g(x)$ is a bounded function all $x \in \mathbb{R}_n^+$ and

$$\int_{\mathbb{R}_n^+} |f(x)| \left( \prod_{i=1}^{n} x_i^{2v_i} \right) dx < \infty$$

then

$$\int_{\mathbb{R}_n^+} T^y f(x) g(y) \left( \prod_{i=1}^{n} y_i^{2v_i} \right) dy = \int_{\mathbb{R}_n^+} f(y) T^y g(x) \left( \prod_{i=1}^{n} y_i^{2v_i} \right) dy.$$

d. From c., we have the following equality for $g(x) = 1$.

$$\int_{\mathbb{R}_n^+} T^y f(x) \left( \prod_{i=1}^{n} y_i^{2v_i} \right) dy = \int_{\mathbb{R}_n^+} f(y) \left( \prod_{i=1}^{n} y_i^{2v_i} \right) dy.$$

e. $(f * g)(x) = (g * f)(x)$.

The Fourier-Bessel transformation and its inverse transformation are defined as follows (see, [?]-[?])

$$(F_B f)(x) = C_v \int_{\mathbb{R}_n^+} f(y) \left( \prod_{i=1}^{n} J_{v_i - \frac{1}{2}} (x_i y_i) y_i^{2v_i} \right) dy \tag{2.2}$$

$$(F_B^{-1} f)(x) = (F_B f)(-x), \quad C_v = \left( \prod_{i=1}^{n} 2^{v_i - \frac{1}{2}} \Gamma \left( v_i + \frac{1}{2} \right) \right)^{-1} \tag{2.3}$$

where $J_{v_i - \frac{1}{2}} (x_i y_i)$ is the normalized Bessel function which is the eigenfunction of the Bessel differential operator. The following equalities for Fourier-Bessel transformation is true (see, [?, ?, ?]),

$$F_B \delta(x) = 1$$

$$F_B (f * g)(x) = F_B f(x) \cdot F_B g(x) \tag{2.4}$$

**Definition 1.** The spectrum of the kernel $E(x,t)$ (???) is the bounded support of the Fourier Bessel transform $F_B E(x,t)$ for any fixed $t > 0$.

**Definition 2.** Let $x = (x_1, x_2, ..., x_n) \in \mathbb{R}_n^+$ and denote by

$$\Gamma_+ = \{ x \in \mathbb{R}_n^+ : x_1^2 + ... + x_p^2 - x_{p+1}^2 - ... - x_{p+q}^2 > 0 \}$$

the set of an interior of the forward cone, and $\Gamma_+$ denotes the closure of $\Gamma_+$. 

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Let $\Omega$ be spectrum of $E(x,t)$ defined by definition (??) and $\Omega \subset \Gamma_+$. Let $F_B E(x,t)$ be the Fourier Bessel transform of $E(x,t)$ and define

$$
F_B E(x,t) = \begin{cases} 
  e^{(-1)^k c^2 t(x_1^2 + \ldots + x_p^2 - x_{p+1}^2 - \ldots - x_{p+q}^2)^k} & \text{for } x_i \in \Omega \\
  0 & \text{for } x_i \notin \Omega.
\end{cases} 
$$

(2.5)

Lemma 1. (Fourier Bessel Transform of $\Box_B^k$ operator)

$$
F_B \Box_B^k u(x) = (-1)^k c^2 V^k(x) F_B u(x).
$$

Proof. We can use the mathematical induction method, for $k = 1$, we have

$$
F_B (\Box_B u) (x) = C_{vR^+} (\Box_B u(y)) \left( \prod_{i=1}^{n} J_{v_i - \frac{1}{2}} \left( x_i y_i \right) y_i^{2v_i} \right) dy
$$

$$
= C_{vR^+} \left( \sum_{i=1}^{p} \frac{\partial^2 u(y)}{\partial y_i^2} \right) \left( \prod_{i=1}^{n} J_{v_i - \frac{1}{2}} \left( x_i y_i \right) y_i^{2v_i} \right) dy
$$

$$
+ C_{vR^+} \left( \sum_{i=1}^{p} \frac{2v_i}{y_i} \frac{\partial u(y)}{\partial y_i} \right) \left( \prod_{i=1}^{n} J_{v_i - \frac{1}{2}} \left( x_i y_i \right) y_i^{2v_i} \right) dy
$$

$$
- C_{vR^+} \left( \sum_{i=p+1}^{p+q} \frac{2v_i}{y_i} \frac{\partial^2 u(y)}{\partial y_i^2} \right) \left( \prod_{i=1}^{n} J_{v_i - \frac{1}{2}} \left( x_i y_i \right) y_i^{2v_i} \right) dy
$$

$$
- C_{vR^+} \left( \sum_{i=p+1}^{p+q} \frac{2v_i}{y_i} \frac{\partial u(y)}{\partial y_i} \right) \left( \prod_{i=1}^{n} J_{v_i - \frac{1}{2}} \left( x_i y_i \right) y_i^{2v_i} \right) dy
$$

$$
= I_1 + I_2 + I_3 + I_4.
$$

If we apply partial integration to twice in the $I_1$ and $I_2$ integrals and once in the $I_3$ and $I_4$ integrals, then we have

$$
F_B (\Box_B u) (x) = C_{vR^+} u(y) \left( \sum_{i=1}^{p} B_{y_i} \prod_{i=1}^{n} J_{v_i - \frac{1}{2}} \left( x_i y_i \right) y_i^{2v_i} \right) dy.
$$

$$
- C_{vR^+} u(y) \left( \sum_{i=p+1}^{p+q} B_{y_i} \prod_{i=1}^{n} J_{v_i - \frac{1}{2}} \left( x_i y_i \right) y_i^{2v_i} \right) dy.
$$

Here, if we use the following equality [?],

$$
\int_0^\infty u(y) B_{y_i} J_{v_i - \frac{1}{2}} \left( x_i y_i \right) y_i^{2v_i} dy_i = -x_i^2 \int_0^\infty u(y) J_{v_i - \frac{1}{2}} \left( x_i y_i \right) y_i^{2v_i} dy_i
$$

then, we get

$$
F_B (\Box_B u) (x) = -V(x) C_{vR^+} u(y) \left( \prod_{i=1}^{n} J_{v_i - \frac{1}{2}} \left( x_i y_i \right) y_i^{2v_i} \right) dy
$$

$$
= -V(x) F_B u(x).
$$
Then, from inverse Fourier transform we finally obtain
\[ \Box_B u(x) = -F_B^{-1}V(x)F_B u(x). \]

Assume the statement is true for \( k - 1 \), i.e,
\[ \Box_B^{k-1} u(x) = (-1)^{k-1}F_B^{-1}V^{k-1}(x)F_B u(x). \]

Then, we must prove that it is also true for \( k \in \mathbb{N} \). Hence, we obtain
\[ \Box_B^k u(x) = \Box_B (\Box_B^{k-1} u(x)) = (-1)^k F_B^{-1}V(x) F_B (-1)^{k-1} F_B^{-1} V^{k-1}(x) F_B u(x) = (-1)^k F_B^{-1} V^k(x) F_B u(x). \]

This completes the proof. \( \Box \)

The following result can be found in [?, ?, ?], which will be applied in the sequel.

**Lemma 2.** For all \( t > 0 \), \( c \) is a positive constant and all \( x \in \mathbb{R}_+^n \) we have
\[
\int_0^\infty e^{-c^2 x^2 t} x^{2\nu} dx = \frac{\Gamma(\nu)}{2e^{2\nu+1}t^{\nu+\frac{1}{2}}} \tag{2.6}
\]
and
\[
\int_0^\infty e^{-c^2 x^2 t} J_{\nu - \frac{1}{2}}(xy)x^{2\nu} dx = \frac{\Gamma(\nu + \frac{1}{2})}{2(c^2 t)^{\nu+\frac{1}{2}}} e^{-\frac{x^2}{4ct}}. \tag{2.7}
\]

### 3 Main Results

In this section, we will state our results and give their proofs.

**Lemma 3.** Let \( L \) be the operator defined by
\[
L = \frac{\partial}{\partial t} - c^2 \Box_B^k
\]
where \( \Box_B^k \) is defined by (2.2), \( k \) is a positive integer, \( (x_1, ..., x_n) \in \mathbb{R}_+^n \), and \( c \) is a positive constant. Then we obtain
\[
E(x, t) = C_{\Omega} e^{(-1)^k c^2 t V^k(\nu)} \prod_{i=1}^{n} J_{\nu_i - \frac{1}{2}}(x_i y_i) y_i^{2\nu_i} dy \tag{3.2}
\]
as elementary solution of (2.2) in the spectrum \( \Omega \subset \mathbb{R}_+^n \) for \( t > 0 \).
Proof. Let $E(x, t)$ is the kernel or the elementary solution of operator $L$ and $\delta$ is the Dirac-delta distribution. Thus

$$\frac{\partial}{\partial t} E(x, t) - (-1)^k c^2 V^k E(x, t) = \delta(x) \delta(t).$$

Take the Fourier Bessel transform defined by (??) to both sides of the equation, using Lemma ?? and $F_B \delta(x) = 1$, we obtain

$$\frac{\partial}{\partial t} F_B E(x, t) - (-1)^k c^2 V^k(x) F_B E(x, t) = \delta(t).$$

Thus

$$F_B E(x, t) = H(t) e^{(-1)^k c^2 t V^k(x)}$$

where $H(t)$ is the Heaviside function. Since $H(t) = 1$ for $t > 0$. Therefore,

$$F_B E(x, t) = e^{(-1)^k c^2 t V^k(x)}$$

which has been already by (??). Thus, from (??), we have

$$E(x, t) = C_\nu \Omega e^{(-1)^k c^2 t V^k(y)} \prod_{i=1}^n J_{\nu_i} \left( x_i y_i \right) y_i^{2\nu_i} dy \quad \text{for } t > 0,$$

where $\Omega$ is the spectrum of $E(x, t)$.

Theorem 1. Let us consider the equation

$$\frac{\partial}{\partial t} u(x, t) - c^2 \Box_B^k u(x, t) = 0$$

with the initial condition

$$u(x, 0) = f(x)$$

where $\Box_B^k$ is defined by (??), $k$ is a positive integer, $u(x, t)$ is an unknown function for $(x, t) = (x_1, ..., x_n, t) \in \mathbb{R}^+_n \times (0, \infty)$, $f(x)$ is the given generalized function, and $c$ is a positive constant. Then we obtain

$$u(x, t) = E(x, t) * f(x)$$

as a solution of (??) which satisfies (??) where $E(x, t)$ is given by (??).

Proof. Taking the Fourier Bessel transform defined by (??) to both sides of (??) for $x \in \mathbb{R}^+_n$ and using Lemma ??, we obtain

$$\frac{\partial}{\partial t} F_B u(x, t) = (-1)^k c^2 V^k(x) F_B u(x, t).$$

Thus, we consider the initial condition (??) then we have following equality for the (6)

$$u(x, t) = f(x) * F_B^{-1} e^{(-1)^k c^2 t V^k(x)}$$
Here, if we use the (3.7), (3.8), then we have
\begin{align*}
  u(x,t) &= f(x) * F_B^{-1} e^{(-1)^k c^2 t V^k(x)} \\
  &= B e^{-1} t^{\frac{2v}{k}} f(x) \left( \prod_{i=1}^{n} y_i^{2v_i} \right) dy \\
  &= \int_{\mathbb{R}_n^+} \left[ C_v e^{(-1)^k c^2 t V^k(y)} \left( \prod_{i=1}^{n} J_{v_i - \frac{1}{2}} (x_i y_i) y_i^{2v_i} \right) dz \right] T^* f(x) \left( \prod_{i=1}^{n} y_i^{2v_i} \right) dy.
\end{align*}

Set
\begin{align*}
  E(x,t) &= C_v e^{(-1)^k c^2 t V^k(y)} \left( \prod_{i=1}^{n} J_{v_i - \frac{1}{2}} (x_i y_i) y_i^{2v_i} \right) dy. \tag{3.8}
\end{align*}

Since the integral of (3.8) is divergent, therefore we choose \( \Omega \subset \mathbb{R}_n^+ \) be the spectrum of \( E(x,t) \) and by (3.7), we have
\begin{align*}
  E(x,t) &= C_v \left( \prod_{i=1}^{n} J_{v_i - \frac{1}{2}} (x_i y_i) y_i^{2v_i} \right) dy. \tag{3.9}
\end{align*}

Thus (3.8) can be written in the \( B \)-convolution form
\begin{align*}
  u(x,t) = E(x,t) * f(x).
\end{align*}

Moreover, since \( E(x,t) \) exist, then
\begin{align*}
  \lim_{t \to 0} E(x,t) &= C_v \left( \prod_{i=1}^{n} J_{v_i - \frac{1}{2}} (x_i y_i) y_i^{2v_i} \right) dy \\
  &= C_v \left( \prod_{i=1}^{n} J_{v_i - \frac{1}{2}} (x_i y_i) y_i^{2v_i} \right) dy \tag{3.11}
\end{align*}

Thus for the solution \( u(x,t) = E(x,t) * f(x) \) of (3.7), then we have
\begin{align*}
  u(x,0) &= \lim_{t \to 0} u(x,t) \\
  &= \lim_{t \to 0} E(x,t) * f(x) \\
  &= \delta * f(x) \\
  &= f(x)
\end{align*}
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which satisfies \( (??) \).

In particular, if we put \( k = 1 \) and \( q = 0 \) in \( (??) \), then from Lemma \( ?? \) we obtain

\[
E(x, t) = C_v \exp \left( -c^2 t V(y) \prod_{i=1}^n J_{\nu_i - \frac{1}{2}}(x_i y_i) y_i^{2\nu_i} \right) dy
\]

\[
= 2^{-\frac{(n+2|\nu|)}{2}} (c^2 t)^{-\frac{n+2|\nu|}{2}} e^{-\frac{1}{4c^2 t}|x|^2}
\]

where \( |x|^2 = n \sum_{i=1}^n x_i^2. \) This completes the proof. \( \square \)

**Theorem 2.** The kernel \( E(x, t) \) defined by \( (??) \) has the following properties:

i. \( E(x, t) \in C^\infty(\mathbb{R}_+^n \times (0, \infty)) \) - the space of continuous function with infinitely differentiable.

ii. \( \left( \frac{\partial}{\partial t} - c^2 \Box_B \right) E(x, t) = 0 \) for all \( x \in \mathbb{R}_+^n, t > 0. \)

iii. \( \lim_{t \to 0} E(x, t) = \delta(x) \) for all \( x \in \mathbb{R}_+^n. \)

**Proof.**

i. From \( (??) \), and

\[
\frac{\partial^n}{\partial x^\alpha} E(x, t) = C_v \frac{\partial^n}{\partial \alpha} \nu e^{(-1)^k c^2 t (y_1^2 + \ldots + y_p^2 - y_{p+1}^2 - \ldots - y_{p+q}^2)^k} \left( \prod_{i=1}^n J_{\nu_i - \frac{1}{2}}(x_i y_i) y_i^{2\nu_i} \right) dy,
\]

\( E(x, t) \in C^\infty \) for \( x \in \mathbb{R}_+^n, t > 0. \)

ii. From \( u(x, t) = E(x, t) * f(x) \), we have following equality for \( f(x) = \delta(x) \) by Fourier Bessel Transformation

\[
u(x, t) = E(x, t).
\]

Then by direct computation, we obtain,

\[
\left( \frac{\partial}{\partial t} - c^2 \Box_B \right) E(x, t) = 0.
\]

iii. This case is obvious by \( (??) \). \( \square \)

**References**


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