Graded Modules which Satisfy
the Gr-Radical Formula

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Abstract: Let $G$ be a monoid with identity $e$, and $R$ be a graded commutative ring. Here we study the graded modules which satisfy the Gr-radical formula. The main part of this work is devoted to extending some results from McCasland modules to $Gr$-McCasland modules.

Keywords: Graded rings, $Gr$-McCasland modules, Graded radical formula

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1 Introduction

Let $G$ be an arbitrary monoid with identity $e$. A ring $R$ with non-zero identity is $G$-graded if it has a direct sum decomposition (as an additive group) $R = \bigoplus_{g \in G} R_g$ such that for all $g, h \in G$, $R_g R_h \subseteq R_{gh}$. If $R$ is $G$-graded, then an $R$-module $M$ is said to be $G$-graded if it has a direct sum decomposition $M = \bigoplus_{g \in G} M_g$ such that for all $g, h \in G$, $R_g M_h \subseteq M_{gh}$. An element of $R_g$ or $M_g$ is said to be a homogeneous element. If $x \in M$, then $x$ can be written uniquely as $x = \sum_{g \in G} x_g$, where $x_g$ is the homogeneous component of $x$ in $M_g$. A submodule $N \subseteq M$, where $M$ is graded, is called $G$-graded if $N = \bigoplus_{g \in G} (N \cap M_g)$ or if, equivalently, $N$ is generated by homogeneous elements. Moreover, $M/N$ becomes a $G$-graded module with $g$-component $(M/N)_g = (M_g + N)/N$ for $g \in G$. Clearly, $0$ is a graded submodule of $M$. Also, we write $h(R) = \bigcup_{g \in G} R_g$ and $h(M) = \bigcup_{g \in G} M_g$.

Throughout this paper $R$ is a commutative $G$-graded ring with identity.

Let $R$ be a $G$-graded ring. A graded ideal $I$ of $R$ is said to be a graded prime ideal if $I \neq R$; and whenever $ab \in I$, we have $a \in I$ or $b \in I$, where $a, b \in h(R)$. The graded radical of $I$, denoted by $Gr(I)$, is the set of all $x \in R$ such that for each $g \in G$ there exists $n_g > 0$ with $x_g^{n_g} \in I$. A proper graded submodule $N$ of a graded module $M$ is called graded prime if $rm \in N$, then $m \in N$ or $r \in (N : M) = \{ r \in R : rM \subseteq N \}$, where $r \in h(R)$, $m \in h(M)$ (note that...
(N : M) is graded by [2, Lemma 2.1]). A graded submodule N of a graded R-module M is called graded maximal submodule if N ≠ M and there is no graded submodule K of M such that N ⊊ K ⊊ M. A graded R-module M is called graded finitely generated if M = \( \sum_{i=1}^{n} Rx_{g_i} \), where \( x_{g_i} \in h(M) \) (1 ≤ i ≤ n). A graded R-module M is called a graded multiplicative module (denoted by Gr-multiplicative) if for every graded submodule N of M, N = IM for some graded ideal I of R. In this case, it is clear that every graded module which is multiplicative is a Gr-multiplicative module and N = (N : M)M.

**Lemma 1.1.** (cf.[5]) Let M be a graded module over a G-graded ring R and I a graded ideal of R. Then the following hold:

(i) If N is a graded submodule of M, a ∈ h(R) and m ∈ h(M), then Rm, IN and aN are graded submodule of M and aN is a graded ideal of R.

(ii) If \( \{N_i\}_{i \in J} \) is a collection of graded submodules of M, then \( \sum_{i \in J} N_i \) and \( \bigcap_{i \in J} N_i \) are graded submodules of M.

(iii) If P is a graded prime ideal of R and M a faithful graded multiplication R-module with PM ≠ M, then PM is a graded prime submodule of M.

**2 Gr-radical formula**

**Definition 2.1.** Let R be a G-graded ring, M be a graded R-module and N be a graded submodule of M.

(i) The graded radical of N in M denoted by \( Gr_M(N) \) and is defined to be the intersection of all graded prime submodules of M containing N. Should there be no graded prime submodule of M containing N, then we put \( Gr_M(N) = M \). By Lemma 1.1, It is easy to see that \( Gr_M(N) \) is a graded submodule of M containing N. On the other hand, \( Gr(R) \) denotes the intersection of all graded prime ideals of R.

(ii) The graded envelop submodule \( RGE_M(N) \) of N in M is a graded submodule of M generated by the set \( GE_M(N) = \{ rm : r \in h(R), m \in h(M) \text{ such that } r^nm \in N \text{ for some } n \in N \} \).

(iii) We say that the graded submodule N of M satisfies Gr-radical formula (graded radical formula), if \( Gr_M(N) = RGE_M(N) \).

(iv) A graded R-module M will be called a Gr-McCasland module if every graded submodule of M satisfies Gr-radical formula.

**Lemma 2.2.** Let M be a graded module over a G-graded ring R. Then N ⊊ RGE_M(N) ⊊ Gr_M(N) for every graded R-submodule N of M.

**Proof.** Obvious. \( \Box \)
3 Gr-Multiplication Modules

In this section we list some basic properties of graded multiplicative module and we will show that every Gr-multiplication module is McCasland.

Lemma 3.1. Let $I$ be a graded ideal of a $G$-graded ring $R$ and $M$ be a graded $R$-module. Then there exists a proper graded submodule $N$ of $M$ satisfies $I = (N : M)$ if and only if $IM \neq M$, $I = (IM : M)$.

Proof. Let $N$ be a proper graded submodule of $M$ and $I = (N : M)$. Then $IM \subseteq N \subseteq M$, so $IM \neq M$. It is clear that $I \subseteq (IM : M)$. Let $r \in (IM : M)$ then $rM \subseteq IM \subseteq N$, so $r \in I$. Therefore $I = (IM : M)$. The convers is clear since $IM$ is a proper graded submodule from Lemma 1.1.

Theorem 3.2. Let $M$ be a Gr-multiplicative $R$-module, $N$ a graded submodule of $M$ and $A = (N : M)$. Then $Gr_M(N) = \sqrt{A}M = \sqrt{(N : M)}M$.

Proof. Without loss of generality we can assume that $M$ is faithful by [7, p. 155]. Let $P$ denote the collection of all graded prime ideals $P$ of $R$ such that $A \subseteq P$. If $B = \sqrt{A}$, then $B = \bigcap_{P \in P} P$. Choose $P \in P$. If $PM = M$, then $Gr_M(N) \subseteq PM$. If $PM \neq M$, then since $N$ is a graded submodule of $M$ and $M$ is Gr-multiplicative then $N = AM \subseteq PM$. Therefore by Lemma 1.1, since $PM$ is a prime submodule of $M$, then $Gr_M(N) \subseteq PM$. Thus $BM = \bigcap_{P \in P} PM$, by [7, Corollary 4.2.8]. So $Gr_M(N) \subseteq BM$.

Now let $K$ be a graded prime submodule of $M$ containing $N$. Then there exists a graded prime ideal $Q = (K : M)$ of $R$ such that $K = QM$. We show that $A \subseteq Q$. By Lemma 3.1, $Q = (QM : M)$. Let $r \in A = (N : M)$. So $rM \subseteq N \subseteq K = QM$, then $r \in (QM : M) = Q$. Thus $A \subseteq Q$. As $Q$ is a graded prime ideal containing $A$, so $B = \sqrt{A} \subseteq Q$. Therefore $BM \subseteq QM = K$. Hence, since $K$ is an arbitrary graded prime submodule of $M$ containing $N$, then $BM \subseteq Gr_M(N)$.

Theorem 3.3. Let $M$ be a Gr-multiplicative $R$-module. Then $M$ is a Gr-McCasland module.

Proof. Let $N$ be a graded submodule of $M$. Then $RGE_M(N) \subseteq Gr_M(N)$ by lemma 2.2, so it suffices to show that $Gr_M(N) \subseteq RGE_M(N)$. Let $x \in Gr_M(N)$. Since $Gr_M(N) = \sqrt{(N : M)}M$, then $x = \sum_{j=1}^{k} r_j x_j$ such that $r_j \in \sqrt{(N : M)}$, $x_j \in M$. As $\sqrt{(N : M)}M$ are graded, so without loss of generality we can assume that $x = \sum_{i=1}^{n} r_{g_i} x_{g_i}$, such that $r_{g_i} \in h(R) \cap \sqrt{(N : M)}$ and $x_{g_i} \in h(M)$ for each $i = 1, 2, ..., n$. Since $r_{g_i} \in \sqrt{(N : M)}$, so there exists $n_i \in N$ such that $r_{g_i}^{n_i} M \subseteq N$ for each $i = 1, 2, ..., n$. Therefore $r_{g_i}^{n_i} x_{g_i} \in N$ and $r_{g_i} x_{g_i} \in GE_M(N) \subseteq RGE_M(N)$ for each $i = 1, 2, ..., n$. Thus $x \in RGE_M(N)$.
4 Gr-Semisimple Modules

In this section we list some basic properties of graded Semisimple module and we will show that every Gr- Semisimple Module is McCasland.

Lemma 4.1. Let $N_1$, $N_2$ be graded submodules of a graded $R$-module $M$ and $N_1 \subseteq N_2$. Then

(i) $RGE_{M/N_1}(N_2/N_1) = RGE_M(N_2)/N_1$

(ii) $Gr_{M/N_1}(N_2/N_1) = Gr_M(N_2)/N_1$

Proof. (i) Let $y \in RGE_{M/N_1}(N_2/N_1)$. So $y = \sum_{i=1}^{k} r_{g_i}(m_{g_i'} + N_1)$ such that $r_{g_i} \in h(R)$, $m_{g_i'} \in h(M)$ and there exists $n_i \in N$ such that $r_{g_i} m_{g_i'} N_i \in N_2/N_1$ for each $i = 1, 2, ..., k$. Thus $r_{g_i} m_{g_i'} \in N_2$ and $r_{g_i} m_{g_i'} \in RGE_M(N_2)$. So $y = \sum_{i=1}^{k} r_{g_i}(m_{g_i'} + N_1) \in RGE_M(N_2)/N_1$.

Now let $x \in RGE_M(N_2)/N_1$. So $x = \sum_{i=1}^{t} s_{g_i}(m_{g_i'} + N_1)$ such that $s_{g_i} \in h(R)$, $m_{g_i'} \in h(M)$ and there exists $n_i \in N$ such that $s_{g_i} m_{g_i'} \in N_2$ for each $i = 1, 2, ..., t$. So $s_{g_i} m_{g_i'} \in N_2/N_1$ and $s_{g_i}(m_{g_i'} + N_1) \in RGE_{M/N_1}(N_2/N_1)$. Therefore $x = \sum_{i=1}^{t} s_{g_i}(m_{g_i'} + N_1) \in RGE_{M/N_1}(N_2/N_1)$.

(ii) It is clear by [2, lemma 2.8].

Corollary 4.2. Let $N$, $N'$ be graded submodules of graded $R$-modules $M$, $M'$ such that $M/N \cong M'/N'$. Then $Gr_M(N) = RGE_M(N)$ if and only if $Gr_{M'}(N') = RGE_{M'}(N')$.

Proof. By lemma 4.1, we have the following implications:

$Gr_M(N) = RGE_M(N) \Leftrightarrow Gr_M(N)/N = RGE_M(N)/N$

$\Leftrightarrow Gr_{M/N}(0) = RGE_{M/N}(0) \Leftrightarrow Gr_{M'/N'}(0) = RGE_{M'/N'}(0)$

$\Leftrightarrow Gr_{M'}(N')/N' = RGE_{M'}(N')/N' \Leftrightarrow Gr_{M'}(N') = RGE_{M'}(N')$.

Corollary 4.3. Let $N$, $L$ be graded submodule of graded $R$-module $M$ such that $M = N + L$ and $Gr_{L}(N \cap L) = RGE_{L}(N \cap L)$. Then $Gr_{M}(N) = RGE_{M}(N)$.

Proof. Note that $M/N = (N + L)/N \cong L/N \cap L$. Apply Corollary 4.2.
Lemma 4.4. For a graded $R$-module $M$, we have
\[ \text{GRad}(M) = \bigcap \{ K \subseteq M \mid K \text{ is a graded maximal submodule of } M \} = \sum \{ L \subseteq M \mid L \text{ is a } Gr - \text{small submodule of } M \}. \]

Proof. The first row is just the definition. If $L \ll_{Gr} M$ and $K$ is a graded maximal submodule of $M$ not containing $L$, then $K \subseteq L + K \subseteq M$ so $L + K = M$. As $K$ is a graded maximal submodule, then $K = M$, since $L \ll_{Gr} M$. Hence every $Gr$-small submodule of $M$ is contained in $\text{GRad}(M)$.

Now assume that $m \in \text{GRad}(M) \cap h(R)$, $U \subseteq M$ with $Rm + U = M$ and $U$ is a graded submodule of $M$. If $U \neq M$, set $A = \{ K \mid K \text{ is a graded submodule of } M \text{ with } U \subseteq K \text{ and } m \notin K \}$. Then $A \neq \emptyset$. By Zorn’s lemma there is a graded submodule $L$ of $M$ maximal with respect to $U \subseteq L$ and $m \notin L$. So $M = Rm + L$, now we show that $L$ is a graded maximal submodule of $M$. Let $L'$ be a graded submodule of $M$ and $L \subseteq L' \subseteq M$. We divide the proof into two cases:

Case 1. If $m \notin L'$, then $L' \in A$ and $L = L'$.

Case 2. If $m \in L'$, then $M = Rm + L'$ and so $L' = M$.

So $L$ is a graded maximal submodule of $M$. But $m \in \text{GRad}(M) \subseteq L$ is a contradiction. So $U = M$ and $Rm \ll_{Gr} M$. Since $\text{GRad}(M)$ is a graded submodule of $M$ and every element of $\text{GRad}(M)$ is a finite sum of homogenous elements, therefore the result holds.

Lemma 4.5. Let $M$ be a $Gr$-semisimple $R$-module. Then $\text{GRad}(M) = 0$.

Proof. Since $M$ is a $Gr$-semisimple $R$-module so $M$ has no proper $Gr$-small submodule, then by lemma 4.4, $\text{GRad}(M) = 0$.

Lemma 4.6. Let $M$ be a graded $R$-module and $M'$ be a graded submodule of $M$. If $P$ is a graded prime submodule of $M$, then $P \cap M'$ is a graded prime submodule of $M'$.

Proof. Set $L = P \cap M'$. Since $P$ and $M'$ are graded submodules of $M$ so $L$ is a graded submodule of $M$. Let $rm' \in L$ for some $r \in h(R)$ and $m \in h(M') \subseteq h(M)$. Then $rm' \in P$. So $m' \in P$ or $rM' \subseteq P$ since $P$ is a graded prime submodule of $M$. Thus $m' \in L$ or $rM' \subseteq L$. Therefore $L = P \cap M'$ is a graded prime submodule of $M'$.

Lemma 4.7. Let $M = M' \oplus M''$ be a graded $R$-module, $M'$ and $M''$ be graded submodules of $M$ such that $M''$ is a $Gr$-semisimple module. Then $\text{Gr}_M(N) = \text{Gr}_{M'}(N)$ for any graded submodule $N$ of $M'$.

Proof. Let $N$ be a graded submodule of $M'$. Since $M''$ is $Gr$-semisimple and $\text{GRad}(M'') = 0$, so there exists a collection of graded maximal submodules $P_i(i \in I)$ of $M''$ such that $\bigcap_{i \in I} P_i = 0$ and there exists a collection of graded prime submodules $Q_j(j \in J)$ of $M' \oplus P_i$ such that $\text{Gr}_{M'}(N) = \bigcap_{j \in J} Q_j$. We show that for all $i \in I$ and $j \in J$, $M' \oplus P_i$ and $Q_j \oplus M''$ are graded prime submodules of $M$ containing $N$. First we show that for each $i \in I$, $L' = M' \oplus P_i$ is a graded prime submodule of $M$ containing $N$, the proof for $Q_j \oplus M''$ is the same.
Let \( rm \in L' = M' \oplus P_i \) for some \( r \in h(R) \) and \( m \in h(M) \). So \( m = m' + m'' \) for some \( m' \in M' \) and \( m'' \in M'' \). Thus \( rm-m' = rm'' \in P_i \). If \( m'-m' = 0 \), then \( m = m' + (m - m') \in M' \oplus P_i \). If \( rM'' \subseteq P_i \), then \( rM = r(M' \oplus M'') \subseteq M' \oplus P_i \). Hence:

\[
Gr_M(N) \subseteq \left( \bigcap_{i \in I} (M' \oplus P_i) \right) \cap \left( \bigcap_{j \in J} (Q_j \oplus M'') \right) = \bigcap_{j \in J} Q_j = Gr_M(N).
\]

By lemma 4.6, \( Gr_M(N) \subseteq Gr_M(N) \).

Let \( M \) and \( M' \) be two graded \( R \)-modules. A morphism of graded \( R \)-modules \( f : M \to M' \) is a morphism of \( R \)-modules verifying \( f(M_g) \subseteq M'_g \) for every \( g \in G \).

**Lemma 4.8.** Let \( M \) and \( M' \) be two graded \( R \)-modules and \( N \) be a graded \( R \)-submodule of \( M \). If \( f : M \to M' \) is a morphism of graded \( R \)-modules, then \( f(N) \) is a graded submodule of \( M' \).

**Proof.** Let \( N = \bigoplus_{g \in G} N_g \) such that \( N_g = N \cap M_g \) for all \( g \in G \). Since \( f \) is a morphism of graded \( R \)-modules, so \( f(N_g) = f(N) \cap M'_g \). We show that \( f(N) = \bigoplus_{g \in G} f(N_g) \). Let \( g \in f(N) \). Then \( y = \sum_{g \in G} m'_g \) such that \( m'_g \in M' \) for all \( g \in G \) and \( y = f(n) \) for some \( n \in N \). Thus \( n = \sum_{g \in G} n_g \), since \( N \) is graded. Without loss of generality we can assume that \( n = \sum_{i=1}^{n} n_{g_i} \) and \( n_g = 0 \) for each \( g \notin \{g_1, ..., g_n\} \). Therefore \( f(n) = \sum_{i=1}^{n} f(n_{g_i}) = y \). But every element of \( M' \) has unique representation, so \( m'_{h_i} = f(n_{g_i}) \in f(N_{g_i}) \subseteq f(N) \) for some \( h_i \in G \) and \( m'_{h_i} = 0 \in f(N) \) for all \( h_i \notin \{h_1, ..., h_n\} \). Therefore \( m'_g \in f(N) \) for all \( g \in G \) and \( f(N) \) is a graded submodule of \( M' \).

**Theorem 4.9.** Let \( M = M' \oplus M'' \) be a graded \( R \)-module, \( M' \) a \( Gr \)-McCasland \( R \)-submodule and \( M'' \) a \( Gr \)-semisimple \( R \)-submodule of \( M \). Then \( M \) is a \( Gr \)-McCasland module.

**Proof.** Let \( N \) be a graded \( R \)-submodule of \( M \). It suffices to show that \( Gr_M(N) \subseteq RGE_M(N) \). Let \( \pi : M \to M'' \) be the natural epimorphism. It is clear that \( \pi \) is a morphism of graded modules. Thus \( \pi(N) \subseteq M'' \) is a graded submodule of \( M'' \) by lemma 4.8. So there exists a graded submodule \( N'' \) of \( M'' \) such that \( M'' = \pi(N) \oplus N'' \). Then \( M = M' \oplus M'' = M' \oplus \pi(N) \oplus N'' \). We show that \( M = N + (M' \oplus N'') \). Let \( m \in M \). So there exists \( m' \in M' \), \( x \in \pi(N) \) and \( n'' \in N'' \) such that \( m = m' + x + n'' \). Since \( x \in \pi(N) \), so \( \pi(n) = x \) for some \( n \in N \). Thus \( n = n_1 + n_2 \) for some \( n_1 \in M' \), \( n_2 \in M'' \) and \( n_2 = \pi(n) = x \). So \( m = m' + x + n'' = m' + n - n_1 + n'' = n + (m' - n_1) + n'' \), then \( m \in N + (M' \oplus N'') \).

Now consider submodule \( L = N \cap (M' \oplus N'') \) of graded \( R \)-module \( H = M' \oplus N'' \). \( L \) is a graded submodule because \( N \), \( M' \) and \( N'' \) are graded. Let \( \pi' : H \to N'' \) be the natural epimorphism. Then \( \pi'(L) \subseteq \pi(N) \cap N'' = 0 \), so \( L \subseteq M' \). As \( N'' \) is \( Gr \)-semisimple and \( M' \) is a \( Gr \)-McCasland \( R \)-module, then by lemma 4.7, \( Gr_H(L) = Gr_M(L) = RGE_M(L) \subseteq RGE_H(L) \). So \( Gr_H(L) = RGE_H(L) \). Then by corollary 4.3, \( Gr_M(N) = RGE_M(N) \) since \( M = N + H \) and \( L = N \cap H \).

**Corollary 4.10.** Let \( R \) be a graded \( R \)-module. Then every \( Gr \)-semisimple \( R \)-module is a \( Gr \)-McCasland module.
5 Gr-Divisible Modules

In this section we list some basic properties of graded divisible Module and we will show that every Gr- divisible Module is McCasland.

Let $R$ be a graded ring. We say that $R$ is a Gr-integral domain whenever $a, b \in h(R)$ with $ab = 0$ implies that either $a = 0$ or $b = 0$.

Let $R$ be a Gr-integral domain. A graded $R$-module $M$ is called Gr-divisible if $aM = M$ for all $0 \neq a \in h(R)$.

If $R$ is a graded ring and $M$ is a graded $R$-module, the subset $T(M)$ of $M$ is defined by $T(M) = \{ m \in M : rm = 0 \text{ for some } 0 \neq r \in h(R) \}$.

Clearly, if $R$ is a Gr-integral domain, then $T(M)$ is a graded submodule of $M$. $T(M)$ is called the Gr-torsion submodule of $M$. A graded $R$-module $M$ is called a Gr-torsion module if $M = T(M)$ and is called Gr-torsion free module if $T(M) = 0$.

Lemma 5.1. Let $R$ be a Gr-integral domain, $M$ be a graded $R$-module and $N$ a proper graded submodule of $M$. Then $N$ is a graded prime submodule of $M$ or if $T(M/N) = L/N$ is the Gr-torsion submodule of $M/N$, then $L = M$ or $L$ is a graded prime submodule of $M$ containing $N$.

Proof. By the definition $T(M/N) = \{ m + N \in M/N : rm \in N \text{ for some } 0 \neq r \in h(R) \}$. We divide the proof into two cases:

Case 1 Let the graded $R$-module $M/N$ be Gr-torsion free. Then $T(M/N) = 0$.

If $I = (N : M) \neq 0$, then there exists $0 \neq r \in I$ such that $rM \subseteq N$. So for every $m \in M$, $r(m + N) = rm + N = N$, then $m + N \in T(M/N) = 0$ and $m \in N$. Therefore $N = M$ is a contradiction. Thus $I = (N : M) = 0$. So $M/N$ is a Gr-torsion free $R \cong R/0$-module and $I = (N : M) = 0$ is a graded prime ideal of $R$. So by [2, Theorem 2.11], $N$ is a graded prime submodule of $M$.

Case 2 If $M/N$ isn’t Gr-torsion free $R$-module, then $T(M/N) = L/N \neq 0$. If $M/N$ is Gr-torsion, then $T(M/N) = M/N$ and $L = M$. If $M/N$ isn’t Gr-torsion, then by [6, Proposition 2.6], $T(M/N)$ is a graded prime submodule of $M$. Then by [2, Lemma 2.8], $L$ is a graded prime submodule of $M$ containing $N$.

Lemma 5.2. Let $R$ be a graded ring and $M, M'$ be two graded $R$-modules and $\varphi : M \rightarrow M'$ be an epimorphism of graded modules. Let $N$ be a graded submodule of $M$ such that $\text{Ker}\varphi \subseteq N$. Then

(i) If $P$ is a graded prime submodule of $M$ containing $N$, then $\varphi(P)$ is a graded prime submodule of $M'$ containing $\varphi(N)$.

(ii) If $P'$ is a graded prime submodule of $M'$ containing $\varphi(N)$, then $\varphi^{-1}(P')$ is a graded prime submodule of $M$ containing $N$.

Proof. The proof is a direct consequence of the definition.

Lemma 5.3. Let $M, M'$ be graded $R$-modules and $N'$ be a graded submodule of $M'$. Let $\varphi : M \rightarrow M'$ be an epimorphism of graded $R$-modules. Then

(i) $\varphi^{-1}(\text{Gr}_{M'}(N')) = \text{Gr}_M(\varphi^{-1}(N'))$

(ii) $\varphi^{-1}(\text{RGE}_M(N')) = \text{RGE}_M(\varphi^{-1}(N'))$
Proof. (i) Let $x \in \text{Gr}_{M}(\varphi^{-1}(N'))$ and $L$ be a graded prime submodule of $M'$ containing $N'$. Then by lemma 5.2, $\varphi^{-1}(L)$ is a graded prime submodule of $M$ containing $\varphi^{-1}(N')$. So $\text{Gr}_{M}(\varphi^{-1}(N')) \subseteq \varphi^{-1}(L)$, then $x \in \varphi^{-1}(L)$ so $\varphi(x) \in L$.

Therefore $\varphi(x) \in \text{Gr}_{M}(N')$, then $x \in \varphi^{-1}(\text{Gr}_{M}(N'))$.

Now suppose that $y \in \varphi^{-1}(\text{Gr}_{M}(N'))$ and $K$ be a graded prime submodule of $M$ containing $\varphi^{-1}(N')$. It is clear that $\text{Ker} \varphi \subseteq \varphi^{-1}(N')$, so by lemma 5.2, $\varphi(K)$ is a graded prime submodule of $M'$ containing $N'$. Thus $\text{Gr}_{M}(N') \subseteq \varphi(K)$, then $\varphi(y) \in \text{Gr}_{M}(N') \subseteq \varphi(K)$. So there exists $m \in K$ such that $\varphi(y) = \varphi(m)$, then $y - m \in \text{Ker} \varphi \subseteq \varphi^{-1}(N') \subseteq K$. Therefore $y \in K$, so $y \in \text{Gr}_{M}(\varphi^{-1}(N'))$.

(ii) Let $rm \in GE_{M}(\varphi^{-1}(N'))$ for some $m \in h(M)$ and $r \in h(R)$. So there exists $n \in N$ such that $rm \in \varphi^{-1}(N')$, then $rm \varphi = \varphi(rm) \in N'$, so $\varphi(rm) = r \varphi(m) \in RGE_{M}(N')$ since $r \in h(R)$ and $\varphi(m) \in h(M')$. Therefore $rm \in \varphi^{-1}(RGE_{M}(N'))$.

Now let $x \in \varphi^{-1}(RGE_{M}(N'))$. So $\varphi(x) \in RGE_{M}(N')$. Without loss of generality we can consider $\varphi(x) = \sum_{i=1}^{k} g_{i} x_{i}'$ such that $g_{i} \in h(R)$ and $x_{i}' \in h(M')$ for each $i = 1, 2, ..., k$. So there exists $n_{i} \in N$ such that $r_{g_{i}}^{n_{i}} x_{i}' \in N'$ for each $i = 1, 2, ..., k$. Since $\varphi$ is an epimorphism of graded $R$-modules and $x_{i}' \in h(M')$, then there exists $x_{i}' \in h(M)$ such that $\varphi(x_{i}') = x_{i}'$. So $\varphi(r_{g_{i}}^{n_{i}} x_{i}') = r_{g_{i}}^{n_{i}} x_{i}' \in N'$. Therefore $r_{g_{i}} x_{i}' \in GE_{M}(\varphi^{-1}(N'))$ for each $i = 1, 2, ..., k$. On the other hand, $\varphi(x - \sum_{i=1}^{k} g_{i} x_{i}') = 0$. So $x - \sum_{i=1}^{k} g_{i} x_{i}' \in \text{Ker} \varphi \subseteq \varphi^{-1}(N') \subseteq RGE_{M}(\varphi^{-1}(N'))$. Therefore $x \in RGE_{M}(\varphi^{-1}(N'))$.

Let $M$ be a graded $R$-module. Then a graded homomorphic image of $M$ is a graded $R$-module $M'$ with an epimorphism of graded modules $\varphi : M \rightarrow M'$.

**Theorem 5.4.** Let $M$ be a Gr-McCasland $R$-module. Then every graded homomorphic image of $M$ is Gr-McCasland.

Proof. Let $M'$ be a graded homomorphic image of $M$. Then there exists an epimorphism of graded modules $\varphi : M \rightarrow M'$. We show that $M'$ is an Gr-McCasland module. Let $N'$ be a graded submodule of $M'$. Then $\varphi^{-1}(N')$ is a graded submodule of $M$ and since $M$ is Gr-McCasland $R$-module, then $\text{Gr}_{M}(\varphi^{-1}(N')) = RGE_{M}(\varphi^{-1}(N'))$. So by lemma 5.3, $\varphi^{-1}(\text{Gr}_{M}(N')) = \varphi^{-1}(RGE_{M}(N'))$. Then $\text{Gr}_{M}(N') = RGE_{M}(N')$ since $\varphi$ is an epimorphism.

**Theorem 5.5.** Let $G$ be a group and $R$ be a Gr-integral domain. Let $M = M_{1} + M_{2}$ be a graded $R$-module, $M_{1}$ a Gr-McCasland submodule of $M$ and $M_{2}$ be a Gr-divisible submodule of $M$. Then $M$ is a Gr-McCasland $R$-module.

Proof. Define $\alpha : M_{1} \rightarrow M/M_{2}$ with $\alpha(s_{1}) = s_{1} + M_{2}$ for every element $s_{1} \in M_{1}$. It is easy to see that $\alpha$ is an epimorphism of graded modules. Since $M_{1}$ is Gr-McCasland, so by Theorem 5.4, $M/M_{2}$ is Gr-McCasland. Let $N$ be a graded submodule of $M$. It suffices to show that $\text{Gr}_{M}(N) \subseteq RGE_{M}(N)$. Let $m \in \text{Gr}_{M}(N)$. Then $m + M_{2} \in (\text{Gr}_{M}(N) + M_{2})/M_{2} = \text{Gr}_{M/M_{2}}(N + M_{2}/M_{2}) = RGE_{M/M_{2}}(N + M_{2}/M_{2})$, since $M/M_{2}$ is Gr-McCasland. So $m + M_{2} = \sum_{i=1}^{s} g_{i}^{r}(k_{i}^{r} + M_{2})$ and
Therefore, we divide the proof into two cases:

**Case 1** If $N$ is a graded prime submodule of $M$, then $Gr.M(N) = RGE_M(N) = N$.

**Case 2** If $T(M/N) = L/N$ is $Gr$-torsion submodule of $M/N$, then $L = M$ or $L$ is a graded prime submodule of $M$ containing $N$. Therefore $Gr.M(N) \subseteq L$, so $x \in L$ and $x+N \in T(M/N)$. Thus there exists $0 \neq c \in h(R)$ such that $cx \in N$. Since $M_2$ is $Gr$-divisible module, so there exists $y \in M_2$ such that $x = cy$ so $c^2y = cx \in N$. Since $y \in M$, without loss of generality we can assume that $y = \sum_{i=1}^{t} m_{g_i}$ and $m_{g_i} = 0$ for each $h \notin \{g_1, \ldots ,g_t\}$ Then $c^2y = \sum_{i=1}^{t} c^2 m_{g_i} \in N$. Then $c^2 m_{g_i} \in N$ for each $i = 1, \ldots , l$, since $N$ is graded. Then $c m_{g_i} \in RGE_M(N)$ for each $i = 1, \ldots , l$. Therefore $x = cy = \sum_{i=1}^{t} c m_{g_i} \in RGE_M(N)$, then $m = x + \sum_{i=1}^{t} r_{g_i} (k_{g_i} - c_i) \in RGE_M(N)$ and $Gr.M(N) \subseteq RGE_M(N)$.

**Corollary 5.6.** If $G$ is a group and $R$ is a $Gr$-integrable domain, then every $Gr$-divisible $R$-module is a $Gr$-McCausland module.

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**References**


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