Some Conditions on Non-Normal Operators which Imply Normality

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Abstract: In this paper, we prove the following assertions:

(i) Let \( A, B, X \in \mathcal{B}(\mathcal{H}) \) be such that \( A^* \) is \( p \)-hyponormal or log-hyponormal, \( B \) is a dominant and \( X \) is invertible. If \( AX = BX \), then there is a unitary operator \( U \) such that \( AU = UB \) and hence \( A \) and \( B \) are normal.

(ii) Let \( T = A + iB \in \mathcal{B}(\mathcal{H}) \) be the cartesian decomposition of \( T \) with \( AB \) is \( p \)-hyponormal. If \( A \) or \( B \) is positive, then \( T \) is normal.

(iii) Let \( A, V, X \in \mathcal{B}(\mathcal{H}) \) be such that \( V, X \) are isometries and \( A^* \) is \( p \)-hyponormal. If \( VX = XA \), then \( A \) is unitary.

(iv) Let \( A, B \in \mathcal{B}(\mathcal{H}) \) be such that \( A + B \geq \pm X \). Then for every paranormal operator \( X \in \mathcal{B}(\mathcal{H}) \) we have

\[
\|AX + XB\| \geq \|X\|^2.
\]

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1 Introduction

Let \( \mathcal{H} \) be infinite dimensional complex Hilbert, and let \( \mathcal{B}(\mathcal{H}) \) denote the algebra of all bounded linear operators acts on \( \mathcal{H} \). Let \( \|\cdot\| \) denote the spectral norm, and \( \langle \cdot, \cdot \rangle \) be an inner product in \( \mathcal{H} \). For \( T \in \mathcal{B}(\mathcal{H}) \), we denote the spectrum and the point spectrum of \( T \) by \( \sigma(T) \), \( \sigma_p(T) \).

An operator \( A \in \mathcal{B}(\mathcal{H}) \) is called positive if \( \langle Ax, x \rangle \geq 0 \) for all non-zero vectors \( x \in \mathcal{H} \), isometry if \( \|Ax\| = \|x\| \) for all a non-zero vector \( x \in \mathcal{H} \), unitary if

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Let $A^*A = AA^* = I$, where $I$ is the identity operator, normal if $AA^* = A^*A$, hyponormal if $Q_A \geq 0$, where $Q_A = A^*A - AA^*$. We say that $A$ is $M$-hyponormal for $M > 0$ if $(A - \lambda I)(A - \lambda I)^* \leq M(A - \lambda I)^*(A - \lambda I)$ for all $\lambda \in \mathbb{C}$, dominant if $\text{ran}(A - \lambda I) \subset \text{ran}(A - \lambda I)^*$ for all $\lambda \in \mathbb{C}$, where $\text{ran}(T)$ is the range of $T$ and normaloid if $\|T\| = r(T)$, where $r(T)$ is the spectral radius of $T$.

In [1], an operator $T$ is called $p$-hyponormal if $|T|^p \geq |T^*|^p$ for $0 < p \leq 1$, where $|T|$ is the square roots of $T^*T$, that is, $|T| = (T^*T)^{\frac{1}{2}}$. We also say that $T$ is co-hyponormal, co-$p$-hyponormal, co-$M$-hyponormal and co-dominant if $T^*$ is hyponormal, $p$-hyponormal, $M$-hyponormal and dominant, respectively.

The well-known Fuglede-Putnam Theorem asserts that if $A$ and $B$ are normal and $AX = XB$ for some operator $X \in \mathcal{B}(\mathcal{H})$, then $A^*X = XB^*$. (See [3]). In past years several authors have extended this theorem for non-normal operators, Yoshino [15], proved that if $A^*$ is $M$-hyponormal, $B$ is dominant and $CA = BC$ for some $C \in \mathcal{B}(\mathcal{H})$, then $CA^* = B^*C$.

Recently, Uchiyama and Tanahashi [14] proved that if $A, B^*$ are $p$-hyponormal (resp. log-hyponormal) and $AX = XB$, then $A^*X = XB^*$.

## 2 Main Results

The next theorems explain what conditions imply normality for $p$-hyponormal operators.

**Theorem 2.1.** Let $A, B, X \in \mathcal{B}(\mathcal{H})$ be such that $A^*$ is a $p$-hyponormal or a log-hyponormal, $B$ is a dominant and $X$ is an invertible. If $XA = BX$, then there is a unitary $U$ such that $AU = UB$ and hence $A$ and $B$ are normal.

**Proof.** Since $XA = BX$, it follows from Fuglede-Putnam theorem for $p$-hyponormal [14] theorem 3] that $B^*X = XA^*$ and so $X^*B = AX^*$.

Now

$$AX^*X = X^*BX = X^*XA.$$  

Let $X = UP$ be the polar decomposition of $X$. Since $X$ is an invertible, it follows that $P$ is invertible and $U$ is unitary. Since $AP^2 = P^2A$ and $P$ is positive, it follows that $AP = PA$. Thus $BUP = UPA$ implies $BUP = UAP$. But $P$ is invertible, we have $BU = UA$. Therefore $A, B$ are unitary equivalent. So, $A$ is dominant and $B^*$ is $p$-hyponormal. Hence $A, B$ are normal.\[\square\]

As a consequence of theorem 2.1 we have immediately

**Corollary 2.2.** Let $A, B, X \in \mathcal{B}(\mathcal{H})$ be such that $A^*$ is a $p$-hyponormal or a log-hyponormal, $B$ is a dominant. If $X$ is an invertible positive operator, then $XA = BX$ implies $A = B$.

**Theorem 2.3.** Let $T = A + iB \in \mathcal{B}(\mathcal{H})$ be the cartesian decomposition of $T$ with $AB$ is $p$-hyponormal. If $A$ or $B$ is positive, then $T$ is normal.
Proof. Assume first that $A$ is positive. Let $S = AB$, then $SA = AS^*$. Then it follows from Fuglede-Putnam theorem for $p$-hyponormal [14] corollary 2] that $S^*A = AS$, that is, $BA^2 = A^2B$. But $A$ is positive, then $AB = BA$, i.e., $T$ is normal.  

Now, if $B$ is positive, then apply the same argument to $-iT = B - iA$. 

Theorem 2.4. Let $T = A + iB$ be the cartesian decomposition of $T$. If $T^*$ is hyponormal operator and $AB$ is $p$-hyponormal operator, then $T$ is normal operator.

Proof. Let $Q = AB$, then $QA = AQ^* = ABA$. Then by Fuglede-Putnam’s theorem for $p$-hyponormal operators, we have $Q^*A = AQ$, i.e., $BA^2 = A^2B$.

Now

$$ (Q + Q^*)A = A(Q + Q^*) $$

and

$$ (Q - Q^*)A = A(Q^* - Q). $$

Since $T^*$ is hyponormal, we have

$$ TT^* - T^*T = 2i(BA - AB) = 2i(Q^* - Q) \geq 0. $$

Let $Y = 2i(BA - AB)$ then $Y \geq 0$ and $YA = -AY$. Now

$$ Y^2A = Y(YA) $$

$$ = Y(-AY) $$

$$ = -YA^2Y $$

$$ = -(AY)Y $$

$$ = AY^2. $$

But $Y$ is positive, then $YA = AY = 0$. Hence, $A(AB - BA) = (AB - BA)A = 0$ implies that $\sigma(AB - BA) = \{0\}$. Therefore $AB - BA$ is quasinilpotent skew-hermitian. Thus $AB - BA = 0$. So $T$ is normal.

Theorem 2.5. Let $A, V, X \in \mathcal{B}(\mathcal{H})$ be such that $V, X$ are are isometries and $A^*$ is $p$-hyponormal. If $VX = XA$, then $A$ is unitary.

Proof. Since $VX = XA$, then by Fuglede-Putnam theorem [14] corollary 2], we have $V^*X =XA^*$. Now multiplying the first equation by $V^*$, we get $X = V^*XA$, then $X(I - A^*A) = 0$ implies that $X^*X(I - A^*A) = 0$. Hence $A^*A = I$, so $A$ is an isometry. Therefore $A$ and $A^*$ are $p$-hyponormal. So $A$ is normal isometry. Hence $A$ is unitary.

The following theorem show that if $A, B \in \mathcal{B}(\mathcal{H})$ are hyponormal and $A^*B = BA^*$, the sum and product of $A$ and $B$ are hyponormal.
Theorem 2.6. Let $A, B \in \mathcal{B}(H)$ be such that $A, B$ are hyponormal and $A^*B = BA^*$. Then
(a) $A + B$ is hyponormal.
(b) $AB$ is hyponormal.

Proof. Since $A^*B = BA^*$, then $B^*A = AB^*$. Now

(a) $(A + B)^*(A + B) - (A + B)(A + B)^* = (A^*A + A^*B + B^*A + B^*B) - (AA^* + AB^* + BA^* + BB^*)$

$$= (A^*A - AA^*) + (B^*B - BB^*).$$

Using the fact that, the sum of two positive operators is positive operator. The result follows.

(b) $(AB)^*(AB) - (AB)(AB)^* = B^*A^*AB - ABB^*A^*$

$$= B^*A^*AB - B^*AA^*B + B^*AA^*B - ABB^*A^*$$

$$= B^*(A^*A - AA^*)B + A(B^*B - BB^*)A^*$$

$$= B^*Q_AB + AQB^*,$$

where $Q_A = A^*A - AA^* \geq 0$ and $Q_B = B^*B_BB^* \geq 0$. The result holds by using the fact that if $X \geq 0$, then $E^*XE \geq 0$ and $EXE^* \geq 0$.

Recall that [9], an operator $T$ is paranormal operator if $\|T^2x\| \geq \|Tx\|^2$ for every unit vector $x \in H$.

Lemma 2.7. ([5, 6]) If $T$ is paranormal operator, then $T$ is normaloid.

Theorem 2.8. Let $P, Q \in \mathcal{B}(H)$. Let $C = PQ - QP$. If $P$ is normaloid, then $\|I - C\| \geq 1$.

Proof. Since $P$ is normaloid, it follows that $r(P) = \|P\|$. So there is a $\lambda \in \sigma(P)$ such that $|\lambda| = \|P\|$. Hence there is a sequence of unit vectors $\{x_n\}$ in $H$ such that $(P - \lambda I)x_n \to 0$, the normaloidity of $P$ implies $(P - \lambda I)^*x_n \to 0$. Now

$$\|I - C\| \geq \|(I - C)x_n, x_n\| = 1 - \langle Cx_n, x_n \rangle \geq 1 - \|Cx_n, x_n\|. $$

The result follows if we show that $\langle Cx_n, x_n \rangle \to 0$.

But

$$\langle Cx_n, x_n \rangle = \langle (P - \lambda I)Q - Q(P - \lambda I)x_n, x_n \rangle$$

$$= \langle Qx_n, (P - \lambda I)^*x_n \rangle - \langle (P - \lambda I)x_n, Q^*x_n \rangle$$

So

$$\|\langle Cx_n, x_n \rangle\| \leq \|Q\|\|\|(P - \lambda I)^*x_n\| + \|(P - \lambda I)x_n\|\| \to 0.$$
Theorem 2.9. Let $A, B \in \mathcal{B}(\mathcal{H})$ be self-adjoint such that $A + B \geq a \geq 0$. Then for every normaloid $X \in \mathcal{B}(\mathcal{H})$ we have

$$\|AX + XB\| \geq a\|X\|.$$  

Proof. Since $X$ is normaloid, it follows that $r(X) = \|X\|$. So there is a $\lambda \in \sigma(X)$ such that $|\lambda| = \|X\|$. Hence there is a sequence of unit vectors $\{x_n\}$ in $\mathcal{H}$ such that $(X - \lambda I)x_n \to 0$, the normaloidity of $X$ implies $(X^* - \lambda I)x_n \to 0$.

Now

$$\|AX + XB\| \geq |(AX + XB)x_n, x_n|$$

$$= |(A(X - \lambda I)x_n, x_n) + ((X - \lambda I)Bx_n, x_n) + \lambda((A + B)x_n, x_n)|$$

$$= |((X - \lambda I)x_n, Ax_n) + (Bx_n, (X^* - \lambda I)x_n) + \lambda((A + B)x_n, x_n)|$$

$$\geq \lambda|((A + B)x_n, x_n)| - \text{terms which goes to zero as } n \to \infty$$

$$\geq \lambda\alpha - \text{terms which goes to zero as } n \to \infty.$$ 

Hence

$$\|AX + XB\| \geq a\|X\|.$$ 

Theorem 2.10. \cite{7}Let $A, B \in \mathcal{B}(\mathcal{H})$ be self-adjoint such that $A + B \geq \pm X$. Then for every self-adjoint $X \in \mathcal{B}(\mathcal{H})$ we have

$$\|AX + XB\| \geq \|X\|^2.$$ 

Lemma 2.11. If $A \in \mathcal{B}(\mathcal{H})$ is self-adjoint then $\pm A \leq |A|$

Proof. Let $A = U|A|$ be the polar decomposition of $A$. Since $A$ is self-adjoint then $A = U|A| = |A|U^*$ and

$$\langle U|A|U^*\rangle^2 = U|A|^2U|A|^2 = U|A|^2U^* = A^2 = \langle |A| \rangle^2,$$

and so $U|A|U^* = |A|$.

Now for any $x \in \mathcal{H}$ we have

$$|\langle Ax, x \rangle|^2 = |\langle U|A|x, x \rangle|^2$$

$$= |\langle |A|x, U^*x \rangle|^2$$

$$\leq |\langle |A|x, x \rangle| \langle |A|U^*x, U^*x \rangle$$

(by the Generalized Cauchy Schwartz inequality)

$$= |\langle |A|x, x \rangle| \langle U|A|U^*x, x \rangle$$

$$= |\langle |A|x, x \rangle| \langle |A|x, x \rangle$$

$$= |\langle |A|x, x \rangle|^2.$$ 

Hence $|\langle Ax, x \rangle| \leq |\langle |A|x, x \rangle|$. 

$\square$
Corollary 2.12. ([7]) Let \( A, B \in \mathcal{B}(\mathcal{H}) \) be self-adjoint such that \( A + B \geq |X| \) and \( A + B \geq |X^*| \). Then
\[
\max(\|AX + XB\|, \|AX^* + X^*B\|) \geq \|X\|^2.
\]

Proof. On \( \mathcal{H} \oplus \mathcal{H} \), let \( T = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \), \( S = \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix} \) and \( Y = \begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix} \). Then \( Y \) is self-adjoint and \( |Y| = \begin{pmatrix} |X^*| & 0 \\ 0 & |X| \end{pmatrix} \). From \( A + B \geq |X| \) and \( A + B \geq |X^*| \), we obtain that \( T + S \geq |Y| \) and hence \( T + S \geq \pm Y \) by Lemma 2.11. Now by applying Theorem 2.10 to \( T, S \) and \( Y \) to get
\[
\|TY + YS\| = \max(\|AX + XB\|, \|AX^* + X^*B\|) \geq \|Y\|^2 = \|X\|^2.
\]

Theorem 2.13. Let \( A, B \in \mathcal{B}(\mathcal{H}) \) be self-adjoint such that \( A + B \geq a \geq 0 \). Then for every normaloid \( X \in \mathcal{B}(\mathcal{H}) \) we have
\[
\|XAX^* + X^*BX\| \geq a\|X\|^2.
\]

Proof. Since \( X \) is normaloid, it follows from lemma 2.7 that \( r(X) = \|X\| \). So there is a sequence of unit vectors \( \{x_n\} \) in \( \mathcal{H} \) such that \( (X - t)x_n \to 0 \), where \( |t| = \|X\| \), and so \( (X - t)^*x_n \to 0 \). Now
\[
\|XAX^* + X^*BX\| \geq |\langle XAX^* + X^*BXx_n, x_n \rangle|
\]
\[
= |\langle AX^*x_n, (X - t)^*x_n \rangle + t\langle A(X - t)^*x_n, x_n \rangle + |t|^2 \langle Ax_n, x_n \rangle + \langle BXx_n, (X - t)x_n \rangle + t\langle B(X - t)x_n, x_n \rangle + |t|^2 \langle Bx_n, x_n \rangle |
\]
\[
\geq a|t|^2 - \text{terms which goes to zero as } n \to \infty.
\]
Letting \( n \to \infty \), we get
\[
\|XAX^* + X^*BX\| \geq a\|X\|^2.
\]

We point out here that Theorem 2.13 is not true if the assumption on \( X \) that is normaloid is removed. For example, consider
\[
X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},
\]
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which act on a two-dimensional Hilbert space.

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