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Generalized Hypersubstitutions of Many-Sorted Algebras

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Abstract : The concept of many-sorted algebras or heterogeneous algebras is useful for abstract data type specifications in Theoretical Computer Science. It is used to explain for abstract data types. Modules and vector spaces are examples of many-sorted algebras. In this paper we extend the concept of a generalized hypersubstitution from one-sorted algebras or homogeneous algebras to many-sorted algebras. We define the I-sorted set of all Σ-generalized hypersubstitutions on special type and define a binary operation on this set. We show that this set together with the binary operation forms a monoid.

Keywords : many-sorted algebra; i-sorted Σ-generalized hypersubstitution; i-sorted Σ-algebras; Σ-terms.
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1 Introduction

The concept of many-sorted algebras or heterogeneous algebras has been extended from one-sorted to many-sorted base structure of algebra by G. Birkhoff and John D. Lipson [1] in 1970.

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Let I be a nonempty set and $A := (A_i)_{i\in I}$ be an I-sorted set, an I-indexed collection of sets, where A_i is a set of element of sort i of A, for every $i \in I$. Let $I^* := \bigcup_{n\geq 1} I^n$, $\Sigma \subseteq I^* \times I$ and $\Sigma_n := I^{n+1}$. For $n \in \mathbb{N}^+$, an I-sorted n-ary operation on A is a mapping $f_\gamma^A : A_{k_1} \times ... \times A_{k_n} \to A_i$ where $\gamma = (k_1, ..., k_n, i) \in \Sigma_n$. Let K_γ be a set of indices with respect to γ. The structure of a pair $\underline{A} := (A, ((f_\gamma^A)_k)_{k\in K_\gamma, \gamma\in\Sigma})$ is called an *I-sorted Σ-algebra*.

Example 1.1. A vector space over field $\mathbb{F}$: Let V be a set of vectors and F a universe of a field $\mathbb{F}$. The structure $\underline{A} := (\{V, F\}, \{+, \cdot\})$ is an I-sorted Σ-algebra with $I = \{1, 2\}$, $A_1 = V$, $A_2 = F$ and $\Sigma = \{(1, 1, 1), (2, 1, 1)\}$, that is there are two binary operations, namely $+$ (addition) and $\cdot$ (scalar multiplication), i.e.,

$$+ := f^A_{(1,1,1)} : V \times V \to V \quad \text{and} \quad \cdot := f^A_{(2,1,1)} : F \times V \to V.$$

There are many papers study about many-sorted algebras. In 2008, K. Denecke and S. Lekkoksung [2] introduced the concept of terms for I-sorted Σ-algebras.

Definition 1.2. Let I be an indexed set and $n \in \mathbb{N}^+$. A set $X^{(n)} := (X_i^{(n)})_{i\in I}$ be an I-sorted set of n variables, $X^{(n)}$ is called an *n-element I-sorted alphabet*, where $X_i^{(n)} = \{x_{i1}, x_{i2}, ..., x_{in}\}, i \in I$. A set $X = (X_i)_{i\in I}$ is an I-sorted set of variables, X is called an *I-sorted alphabet*, where $X_i = \{x_{i1}, x_{i2}, x_{i3}, ...\}, i \in I$. Let $((f_\gamma)_k)_{k\in K_\gamma,\gamma\in\Sigma}$ be a Σ-sorted set of operation symbols. Then for each $i \in I$, an *n-ary Σ-term of sort i*, is inductively defined as follows:

1. $x_{ij} \in X_i^{(n)}$ is an n-ary Σ-term of sort i,
2. $f_\gamma(t_{k_1}, ..., t_{k_m})$ is an n-ary Σ-term of sort i where $\gamma = (k_1, ..., k_m, i) \in \Sigma$ and $t_{k_1}, \ldots, t_{k_m}$ are n-ary Σ-terms of sorts $k_1, \ldots, k_m$, respectively..

The set of all n-ary Σ-terms of sort i is denoted by $W_n(i)$ and $W(i) := \bigcup_{n\in\mathbb{N}^+} W_n(i)$ is called the set of all Σ-terms of sort i. The set $W_\Sigma(X) := (W(i))_{i\in I}$ is called an I-sorted set of all Σ-terms and its elements are called I-sorted Σ-terms.

2 Generalized Hypersubstitutions

In Universal algebra we use identities to classify algebras into collections called varieties. Hyperidentities are used to classify varieties into collections called hypervarieties. The tool used to study hyperidentities is the concept of a hypersubstitution which was introduced by K. Denecke, D. Lau, R. Pöschel, and D. Schweigert [3]. In 2000, S. Leeratanavalee and K. Denecke extended the concept of a hypersubstitution to the concept of a generalized hypersubstitution [4]. A generalized hypersubstitution of type $\tau = (n_i)_{i\in I}$ is a mapping $\sigma : \{f_i \mid i \in I\} \longrightarrow W_\tau(X)$ which assigns to every n_i-ary operation symbol f_i a term of the same type which does not necessarily preserve arity. The set of all generalized hypersubstitutions of type τ

is denoted by $Hyp_G(\tau)$. To define a binary operation on $Hyp_G(\tau)$, we define first the concept of a *generalized superposition* of terms $S^m : W_\tau(X)^{m+1} \longrightarrow W_\tau(X)$ by the following steps:

for any term $t \in W_\tau(X)$,

(i) if $t = x_j, 1 \leq j \leq m$, then $S^m(x_j, t_1, \ldots, t_m) := t_j$,

(ii) if $t = x_j, m < j \in \mathbb{N}$, then $S^m(x_j, t_1, \ldots, t_m) := x_j$,

(iii) if $t = f_i(s_1, \ldots, s_{n_i})$, then
$S^m(t, t_1, \ldots, t_m) := f_i(S^m(s_1, t_1, \ldots, t_m), \ldots, S^m(s_{n_i}, t_1, \ldots, t_m))$.

Every generalized hypersubstitution σ can be extended to a mapping $\hat{\sigma} : W_\tau(X) \longrightarrow W_\tau(X)$ by the following steps:

(i) $\hat{\sigma}[x] := x \in X$,

(ii) $\hat{\sigma}[f_i(t_1, \ldots, t_{n_i})] := S^{n_i}(\sigma(f_i), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$, for any n_i-ary operation symbol f_i.

Then we can define a binary operation $\circ_G$ on $Hyp_G(\tau)$ by $\sigma_1 \circ_G \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$ where $\circ$ denotes the usual composition of mappings and $\sigma_1, \sigma_2 \in Hyp_G(\tau)$. Let σ_{id} be the hypersubstitution which maps each n_i-ary operation symbol f_i to the term $f_i(x_1, \ldots, x_{n_i})$.

We have the following proposition.

Proposition 2.1. [4] *For arbitrary terms $t, t_1, \ldots, t_n \in W_\tau(X)$ and for arbitrary generalized hypersubstitutions $\sigma, \sigma_1, \sigma_2$ we have*

(i) $S^n(\hat{\sigma}[t], \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n]) = \hat{\sigma}[S^n(t, t_1, \ldots, t_n)]$,

(ii) $(\hat{\sigma}_1 \circ \sigma_2)\hat{} = \hat{\sigma}_1 \circ \hat{\sigma}_2$.

It turns out that $\underline{Hyp_G(\tau)} := (Hyp_G(\tau), \circ_G, \sigma_{id})$ is a monoid with σ_{id} as the identity element. For more details of generalized hypersubstitutions, see [5–8].

3 I-Sorted Σ-Generalized Hypersubstitution

In this section, we extend the concept of generalized hypersubstitutions from one-sorted algebras to many-sorted algebras. We first introduce the definition of superposition for Σ-terms and give some properties of Σ-generalized hypersubstitutions.

For $\gamma \in I^*$, let $\gamma(j)$ denote the j-th component of γ. Then for any $i \in I$, we set

$$\Lambda_n(i) := \{\alpha \in I^{n+1} \mid \alpha(n+1) = i\}$$

and $\Lambda(i) := \bigcup_{n=1}^{\infty} \Lambda_n(i)$ and $\Lambda := \bigcup_{i \in I} \Lambda(i)$. Let $\Sigma_m(i) := \{\gamma \in \Sigma_m \mid \gamma(m+1) = i\}$ and $\Sigma(i) := \bigcup_{i \in I} \Sigma_m(i)$.

Definition 3.1. The superposition operation

$$S_\beta : W(i) \times W(k_1) \times ... \times W(k_n) \to W(i),$$

for $\beta = (k_1, ..., k_n, i) \in \Lambda$, is defined inductively by the following steps:

1. If $t = x_{ij} \in X_i$, then

   (1.1) $S_\beta(x_{ij}, t_1, ..., t_n) = x_{ij}$ if $i \neq k_j, \forall j$ and,

   (1.2) $S_\beta(x_{ij}, t_1, ..., t_n) = t_j$ if $i = k_j, 1 \leq j \leq n$ and,

   (1.3) $S_\beta(x_{ij}, t_1, ..., t_n) = x_{ij}$ if $j > n$.

2. If $t = f_\gamma(s_1, ..., s_m) \in W(i)$, for $\gamma = (i_1, ..., i_m, i) \in \Sigma$ and $s_q \in W(i_q)$, $1 \leq q \leq m$, and assume that $S_{\beta_q}(s_q, t_1, ..., t_n)$ with $\beta_q = (k_1, ..., k_n, i_q) \in \Lambda(i_q)$ are already defined, then

$$S_\beta(f_\gamma(s_1, ..., s_m), t_1, ..., t_n) := f_\gamma(S_{\beta_1}(s_1, t_1, ..., t_n), ..., S_{\beta_m}(s_m, t_1, ..., t_n)),$$

for $t_j \in W(k_j)$, $1 \leq j \leq n$.

Example 3.2. Let $I = \{1, 2, 3\}$ and $f_{(1,1,1)}$ with $(1,1,1) \in \Sigma(1)$. We consider $m = 4$, $n = 2$ and calculate

$$\begin{aligned} &S_\beta(x_{14}, S_{\beta_1}(x_{12}, f_{(1,1,1)}(x_{11}, x_{12}), x_{23}, x_{31}, x_{11}),\\ &\qquad S_{\beta_2}(x_{23}, f_{(1,1,1)}(x_{11}, x_{12}), x_{23}, x_{31}, x_{11})) = x_{14},\end{aligned}$$

$$\begin{aligned} &S_\gamma(S_\beta(x_{14}, x_{12}, x_{23}), f_{(1,1,1)}(x_{11}, x_{12}), x_{23}, x_{31}, x_{11})\\ &\qquad = S_\gamma(x_{14}, f_{(1,1,1)}(x_{11}, x_{12}), x_{23}, x_{31}, x_{11}) = x_{11}.\end{aligned}$$

where $\beta = (1, 2, 1), \gamma = \beta_1 = (1, 2, 3, 1, 1)$ and $\beta_2 = (1, 2, 3, 1, 2)$. We see that

$$\begin{aligned} &S_\beta(x_{14}, S_{\beta_1}(x_{12}, f_{(1,1,1)}(x_{11}, x_{12}), x_{23}, x_{31}, x_{11}),\\ &\qquad S_{\beta_2}(x_{23}, f_{(1,1,1)}(x_{11}, x_{12}), x_{23}, x_{31}, x_{11}))\\ &\qquad \neq S_\gamma(S_\beta(x_{14}, x_{12}, x_{23}), f_{(1,1,1)}(x_{11}, x_{12}), x_{23}, x_{31}, x_{11}).\end{aligned}$$

Lemma 3.3. *Let $m, n \in \mathbb{N}^+$ with $m \leq n$, $\beta = (i_1, ..., i_n, i) \in \Lambda_n(i), \gamma = (i_1, ..., i_m, i)$ $\in \Lambda_m(i)$ and $\beta_j = (i_1, ..., i_m, i_j) \in \Lambda_m(i_j), 1 \leq j \leq n$. Then for any Σ-terms $s \in W(i)$, $l_j \in W(i_j)$, $t_q \in W(i_q)$ where $1 \leq j \leq n$ and $1 \leq q \leq m$, we have*

$$S_\beta(s, S_{\beta_1}(l_1, t_1, ..., t_m), ..., S_{\beta_n}(l_n, t_1, ..., t_m)) = S_\gamma(S_\beta(s, l_1, ..., l_n), t_1, ..., t_m).$$

Proof. We will prove by induction on the complexity of the Σ-term $s \in W(i)$.
If $s = x_{ij} \in X(i)$,
Case 1: $i \neq i_j$. Then

$$\begin{aligned} &S_\beta(s, S_{\beta_1}(l_1, t_1, ..., t_m), ..., S_{\beta_n}(l_n, t_1, ..., t_m))\\ &\qquad = S_\beta(x_{ij}, S_{\beta_1}(l_1, t_1, ..., t_m), ..., S_{\beta_n}(l_n, t_1, ..., t_m))\\ &\qquad = x_{ij}\\ &\qquad = S_\gamma(x_{ij}, t_1, ..., t_m)\\ &\qquad = S_\gamma(S_\beta(x_{ij}, l_1, ..., l_n), t_1, ..., t_m).\end{aligned}$$

Case 2: $i = i_j, 1 \leq j \leq n$. Then

$$\begin{aligned}
&S_\beta(s, S_{\beta_1}(l_1, t_1, ..., t_m), ..., S_{\beta_n}(l_n, t_1, ..., t_m)) \\
&\qquad = S_\beta(x_{ij}, S_{\beta_1}(l_1, t_1, ..., t_m), ..., S_{\beta_n}(l_n, t_1, ..., t_m)) \\
&\qquad = S_{\beta_j}(l_j, t_1, ..., t_m) \\
&\qquad = S_{\beta_j}(S_\beta(x_{ij}, l_1, ..., l_n), t_1, ..., t_m) \\
&\qquad = S_\gamma(S_\beta(x_{ij}, l_1, ..., l_n), t_1, ..., t_m).
\end{aligned}$$

Case 3: $j > n$. Then

$$\begin{aligned}
&S_\beta(s, S_{\beta_1}(l_1, t_1, ..., t_m), ..., S_{\beta_n}(l_n, t_1, ..., t_m)) \\
&\qquad = S_\beta(x_{ij}, S_{\beta_1}(l_1, t_1, ..., t_m), ..., S_{\beta_n}(l_n, t_1, ..., t_m)) \\
&\qquad = x_{ij} \\
&\qquad = S_\gamma(x_{ij}, t_1, ..., t_m) \\
&\qquad = S_\gamma(S_\beta(x_{ij}, l_1, ..., l_n), t_1, ..., t_m).
\end{aligned}$$

If $s = f_\alpha(s_1, ..., s_h) \in W(i)$ with $\alpha = (p_1, ..., p_h, i) \in \Sigma(i)$ and $s_r \in W(p_r)$, $1 \leq r \leq h$. We assume that $S_{\alpha_r}(s_r, S_{\beta_1}(l_1, t_1, ..., t_m), ..., S_{\beta_n}(l_n, t_1, ..., t_m)) = S_{\gamma_r}(S_{\alpha_r}(s_r, l_1, ..., l_n), t_1, ..., t_m)$ where $\alpha_r = (i_1, ..., i_n, p_r) \in \Lambda(p_r)$ and $\gamma_r = (i_1, ..., i_m, p_r) \in \Lambda(p_r), 1 \leq r \leq h$, then

$$\begin{aligned}
&S_\beta(s, S_{\beta_1}(l_1, t_1, ..., t_m), ..., S_{\beta_n}(l_n, t_1, ..., t_m)) \\
&\quad = S_\beta(f_\alpha(s_1, ..., s_h), S_{\beta_1}(l_1, t_1, ..., t_m), ..., S_{\beta_n}(l_n, t_1, ..., t_m)) \\
&\quad = f_\alpha(S_{\alpha_1}(s_1, S_{\beta_1}(l_1, t_1, ..., t_m), ..., \\
&\qquad S_{\beta_n}(l_n, t_1, ..., t_m)), ..., S_{\alpha_h}(s_h, S_{\beta_1}(l_1, t_1, ..., t_m), ..., S_{\beta_n}(l_n, t_1, ..., t_m))) \\
&\quad = f_\alpha(S_{\gamma_1}(S_{\alpha_1}(s_1, l_1, ..., l_n), t_1, ..., t_m), ..., S_{\gamma_h}(S_{\alpha_h}(s_h, l_1, ..., l_n), t_1, ..., t_m)) \\
&\quad = S_\gamma(f_\alpha(S_{\alpha_1}(s_1, l_1, ..., l_n), ..., S_{\alpha_h}(s_h, l_1, ..., l_n)), t_1, ..., t_m) \\
&\quad = S_\gamma(S_\beta(f_\alpha(s_1, ..., s_h), l_1, ..., l_n), t_1, ..., t_m). \qquad \square
\end{aligned}$$

Lemma 3.4. *For $t \in W(i), i \in I$, let $\beta = (k_1, ..., k_n, i) \in \Lambda_n(i)$ and $x_{k_1 1} \in X_{k_1}, ..., x_{k_n n} \in X_{k_n}$, we have*

$$S_\beta(t, x_{k_1 1}, ..., x_{k_n n}) = t.$$

Proof. We will prove by induction on the complexity of the Σ-term $t \in W(i)$.
If $t = x_{ij} \in X(i)$,
Case 1: $i \neq k_j$. Then $S_\beta(t, x_{k_1 1}, ..., x_{k_n n}) = S_\beta(x_{ij}, x_{k_1 1}, ..., x_{k_n n}) = x_{ij}$.
Case 2: $i = k_j, 1 \leq j \leq n$. Then

$$S_\beta(t, x_{k_1 1}, ..., x_{k_n n}) = S_\beta(x_{ij}, x_{k_1 1}, ..., x_{k_n n}) = x_{k_j j} = x_{ij}.$$

Case 3: $j > n$. Then $S_\beta(t, x_{k_1 1}, ..., x_{k_n n}) = S_\beta(x_{ij}, x_{k_1 1}, ..., x_{k_n n}) = x_{ij}$.
If $t = f_\gamma(s_1, ..., s_m) \in W(i)$ with $\gamma = (i_1, ..., i_m, i) \in \Sigma(i)$ and $s_r \in W(i_r)$,

$1 \leq r \leq m$. Assume that $S_{\beta_r}(s_r, x_{k_1 1}, ..., x_{k_n n}) = s_r$ where $\beta_r = (k_1, ..., k_n, i_r) \in \Lambda(i_r)$.

$$\begin{aligned} S_\beta(t, x_{k_1 1}, ..., x_{k_n n}) &= S_\beta(f_\gamma(s_1, ..., s_m), x_{k_1 1}, ..., x_{k_n n}) \\ &= f_\gamma(S_{\beta_1}(s_1, x_{k_1 1}, ..., x_{k_n n}), ..., S_{\beta_m}(s_m, x_{k_1 1}, ..., x_{k_n n})) \\ &= f_\gamma(s_1, ..., s_m) \\ &= t. \end{aligned}$$

□

For each $i \in I$, an arbitary mapping

$$\sigma_i : \{f_\gamma \mid \gamma \in \Sigma(i)\} \to W(i)$$

is called a Σ-generalized hypersubstitution of sort i. The set of all Σ-generalized hypersubstitutions of sort i is denoted by $\Sigma(i)$-Hyp_G.

The I-sorted mapping $\sigma := (\sigma_i)_{i \in I}$ is said to be an I-sorted Σ-generalized hypersubstitution and let Σ-$Hyp_G := (\Sigma(i)\text{-}Hyp_G)_{i \in I}$ be the I-sorted set of all Σ-generalized hypersubstitutions.

Any Σ-generalized hypersubstitution σ_i of sort i can be extended to a mapping $\hat{\sigma}_i : W(i) \to W(i)$ definded by

1. $\hat{\sigma}[x_{ij}] := x_{ij}$, for $x_{ij} \in X_i$,

2. $\hat{\sigma}[f_\gamma(t_1, ..., t_n)] := S_\gamma(\sigma_i(f_\gamma), \hat{\sigma}_{k_1}[t_1], ..., \hat{\sigma}_{k_n}[t_n])$ where $\gamma = (k_1, ..., k_n, i)$ and $t_j \in W(k_j), 1 \leq j \leq n$, assume that $\hat{\sigma}_{k_j}[t_j]$ are already defined.

Since the extension of a Σ-generalized hypersubstitution of sort i is unique, we can define a binary operation $\circ_G^i$ on $\Sigma(i)$-Hyp_G by

$$(\sigma_1)_i \circ_G^i (\sigma_2)_i := (\hat{\sigma_1})_i \circ (\sigma_2)_i,$$

for $(\sigma_1)_i, (\sigma_2)_i \in \Sigma(i)$-$Hyp_G$ and $\circ$ is the usual composition of mapping.
Let $(\sigma_{id})_i \in \Sigma(i)$-$Hyp_G$ which maps each operation symbol f_γ to the Σ-term $f_\gamma(x_{k_1 1}, ..., x_{k_n n})$, for $\gamma = (k_1, ..., k_n, i) \in \Sigma(i)$, i.e.,

$$(\sigma_{id})_i(f_\gamma) := f_\gamma(x_{k_1 1}, ..., x_{k_n n}).$$

Example 3.5. We consider $i = 1$ and let $\Sigma(i) = \{(1,1,1,1), (2,1,1)\}$, i.e., there are two operations f_γ, f_β with $\gamma = (1,1,1,1), \beta = (2,1,1)$. Let $\sigma_1, \sigma_2, \sigma_3 \in \Sigma(i)$-$Hyp_G$ such that $\sigma_1(f_\gamma) = x_{11}, \sigma_1(f_\beta) = f_\beta(x_{21}, x_{13}), \sigma_2(f_\gamma) = f_\beta(x_{22}, x_{12})$

and $\sigma_3(f_\gamma) = f_\gamma(x_{12}, x_{11}, x_{11})$, we have

$$\begin{aligned}
((\sigma_1 \circ^i_G \sigma_2) \circ^i_G \sigma_3)(f_\gamma) &= (\sigma_1 \circ^i_G \sigma_2)\hat{}[\sigma_3(f_\gamma)] \\
&= (\sigma_1 \circ^i_G \sigma_2)\hat{}[f_\gamma(x_{12}, x_{11}, x_{11})] \\
&= S_\gamma((\sigma_1 \circ^i_G \sigma_2)(f_\gamma), x_{12}, x_{11}, x_{11}) \\
&= S_\gamma(\hat{\sigma}_1[\sigma_2(f_\gamma)], x_{12}, x_{11}, x_{11}) \\
&= S_\gamma(\hat{\sigma}_1[f_\beta(x_{22}, x_{12})], x_{12}, x_{11}, x_{11}) \\
&= S_\gamma(S_\beta(\sigma_1(f_\beta), x_{22}, x_{12}), x_{12}, x_{11}, x_{11}) \\
&= S_\gamma(S_\beta(f_\beta(x_{21}, x_{13}), x_{22}, x_{12}), x_{12}, x_{11}, x_{11}) \\
&= S_\gamma(f_\beta(x_{22}, x_{13}), x_{12}, x_{11}, x_{11}) \\
&= f_\beta(x_{22}, x_{11}),
\end{aligned}$$

$$\begin{aligned}
(\sigma_1 \circ^i_G (\sigma_2 \circ^i_G \sigma_3))(f_\gamma) &= (\hat{\sigma}_1 \circ (\hat{\sigma}_2 \circ \sigma_3))(f_\gamma) \\
&= \hat{\sigma}_1[\hat{\sigma}_2[\sigma_3(f_\gamma)]] \\
&= \hat{\sigma}_1[\hat{\sigma}_2[f_\gamma(x_{12}, x_{11}, x_{11})]] \\
&= \hat{\sigma}_1[S_\gamma(\sigma_2(f_\gamma), x_{12}, x_{11}, x_{11})] \\
&= \hat{\sigma}_1[S_\gamma(f_\beta(x_{22}, x_{12}), x_{12}, x_{11}, x_{11})] \\
&= \hat{\sigma}_1[f_\beta(x_{22}, x_{11})] \\
&= S_\beta(\sigma_1(f_\beta), x_{22}, x_{11}) \\
&= S_\beta(f_\beta(x_{21}, x_{13}), x_{22}, x_{11}) \\
&= f_\beta(x_{22}, x_{13}).
\end{aligned}$$

That is $(\sigma_1 \circ^i_G \sigma_2) \circ^i_G \sigma_3 \neq \sigma_1 \circ^i_G (\sigma_2 \circ^i_G \sigma_3)$.

From the previous example, it follows that $(\Sigma(i)\text{-}Hyp_G, \circ^i_G, (\sigma_{id})_i)$ is non associative (with identity). That is, the set $\Sigma(i)\text{-}Hyp_G$ is closed but not associative under binary operation $\circ^i_G$. So we construct a set $H(i)$ by $\sigma \in H(i)$ if for f_γ with $\gamma = (i_1, ..., i_m, i) \in \Sigma(i)$, $\sigma_i(f_\gamma) = x_{ij} \in X(i)$ or $\sigma_i(f_\gamma) = f_\beta(s_1, ..., s_m) \in W(i)$ where $\beta = (i_1, ..., i_l, i) \in \Sigma(i)$ such that $arity(f_\gamma) \leq arity(f_\beta)$. However, we can find an example which show that $H(i) \subset \Sigma(i)\text{-}Hyp_G$ is not closed under such operation.

Example 3.6. We consider $i = 1$ and let $\Sigma(i) = \{(1, 2, 3, 1, 1), (1, 2, 1)\}$, i.e., there are two operations f_γ, f_β with $\gamma = (1, 2, 3, 1, 1), \beta = (1, 2, 1)$. Let $\sigma_1, \sigma_2 \in H(i)$ such that $\sigma_1(f_\gamma) = x_{11}, \sigma_2(f_\gamma) = f_\gamma(f_\beta(x_{11}, x_{23}), x_{22}, x_{31}, x_{13})$ and $\sigma_1(f_\beta) =$

$f_\beta(x_{11}, x_{22})$, we get

$$\begin{aligned}(\sigma_1 \circ_G^i \sigma_2)(f_\gamma) &= \hat{\sigma}_1[\sigma_2(f_\gamma)]\\ &= \hat{\sigma}_1[f_\gamma(f_\beta(x_{11}, x_{23}), x_{22}, x_{31}, x_{13})]\\ &= S_\gamma(\sigma_1(f_\gamma), \hat{\sigma}_1[f_\beta(x_{11}, x_{23})], \hat{\sigma}_2[x_{22}], \hat{\sigma}_3[x_{31}], \hat{\sigma}_1[x_{13}])\\ &= S_\gamma(x_{11}, S_\beta(\sigma_1(f_\beta), x_{11}, x_{23}), x_{22}, x_{31}, x_{13})\\ &= S_\gamma(x_{11}, S_\beta(f_\beta(x_{11}, x_{22}), x_{11}, x_{23}), x_{22}, x_{31}, x_{13})\\ &= S_\gamma(x_{11}, f_\beta(x_{11}, x_{23}), x_{22}, x_{31}, x_{13})\\ &= f_\beta(x_{11}, x_{23}).\end{aligned}$$

We see that $((\sigma_1)_1 \circ_G^i (\sigma_2)_1)$ maps the operation symbol f_γ to $f_\beta(x_{11}, x_{23})$ for which $arity(f_\beta) \leq arity(f_\gamma)$, that is $((\sigma_1)_1 \circ_G^i (\sigma_2)_1) \notin H(i)$.

However, we consider the structure of many-sorted algebra whose all operation symbols of sort i have the same arity n ($n \geq 2$) and have the same structure, i.e., for each $i \in I$, $\Sigma(i) = \{\gamma\}$ and each $k \in K_\gamma, (f_\gamma)_k$ is n-ary. We denote a type of operation symbols by $\Sigma^{|I|,n}(i)$. We will prove that $(\Sigma^{|I|,n}(i)\text{-}Hyp_G, \circ_G^i, (\sigma_{id})_i)$ is a monoid.

Lemma 3.7. *For $\sigma_i \in \Sigma^{|I|,n}(i)\text{-}Hyp_G$, let $t \in W(i), t_j \in W(k_j), 1 \leq j \leq n$ and $\alpha = (k_1, ..., k_n, i) \in \Lambda$. We have*

$$\hat{\sigma}_i[S_\alpha(t, t_1, ..., t_n)] = S_\alpha(\hat{\sigma}_i[t], \hat{\sigma}_{k_1}[t_1], ..., \hat{\sigma}_{k_n}[t_n]).$$

Proof. We will prove by induction on the complexity of the Σ-term t of sort $i \in I$. If $t = x_{ij} \in X(i)$,

Case 1: $i \neq k_j$. Then we get

$$\begin{aligned}\hat{\sigma}_i[S_\alpha(t, t_1, ..., t_n)] &= \hat{\sigma}_i[S_\alpha(x_{ij}, t_1, ..., t_n)]\\ &= \hat{\sigma}_i[x_{ij}] = x_{ij}\\ &= S_\alpha(x_{ij}, \hat{\sigma}_{k_1}[t_1], ..., \hat{\sigma}_{k_n}[t_n])\\ &= S_\alpha(\hat{\sigma}_i[x_{ij}], \hat{\sigma}_{k_1}[t_1], ..., \hat{\sigma}_{k_n}[t_n]).\end{aligned}$$

Case 2: $i = k_j, 1 \leq j \leq n$. Then

$$\begin{aligned}\hat{\sigma}_i[S_\alpha(t, t_1, ..., t_n)] &= \hat{\sigma}_i[S_\alpha(x_{ij}, t_1, ..., t_n)]\\ &= \hat{\sigma}_i[t_j]\\ &= \hat{\sigma}_{k_j}[t_j]\\ &= S_\alpha(x_{ij}, \hat{\sigma}_{k_1}[t_1], ..., \hat{\sigma}_{k_n}[t_n])\\ &= S_\alpha(\hat{\sigma}_i[x_{ij}], \hat{\sigma}_{k_1}[t_1], ..., \hat{\sigma}_{k_n}[t_n]).\end{aligned}$$

Case 3: $j > n$. Then

$$\begin{aligned}\hat{\sigma}_i[S_\alpha(t, t_1, ..., t_n)] &= \hat{\sigma}_i[S_\alpha(x_{ij}, t_1, ..., t_n)]\\ &= \hat{\sigma}_i[x_{ij}] = x_{ij}\\ &= S_\alpha(x_{ij}, \hat{\sigma}_{k_1}[t_1], ..., \hat{\sigma}_{k_n}[t_n])\\ &= S_\alpha(\hat{\sigma}_i[x_{ij}], \hat{\sigma}_{k_1}[t_1], ..., \hat{\sigma}_{k_n}[t_n]).\end{aligned}$$

If $t = f_\alpha(s_1, ..., s_n) \in W(i)$ with $s_r \in W(k_r)$.
Assume that $\hat{\sigma}_{k_j}[S_{\alpha_j}(s_j, t_1, ..., t_n)] = S_{\alpha_j}(\hat{\sigma}_{k_j}[s_j], \hat{\sigma}_{k_1}[t_1], ..., \hat{\sigma}_{k_n}[t_n])$ where $\alpha_j = (k_1, ..., k_n, k_j) \in \Lambda(i_j), 1 \le r \le n$.

$$\begin{aligned}
\hat{\sigma}_i[S_\alpha(t, t_1, ..., t_n)] &= \hat{\sigma}_i[S_\alpha(f_\alpha(s_1, ..., s_n), t_1, ..., t_n)] \\
&= \hat{\sigma}_i[f_\alpha(S_{\alpha_1}(s_1, t_1, ..., t_n), ..., S_{\alpha_n}(s_n, t_1, ..., t_n))] \\
&= S_\alpha(\sigma_i(f_\alpha), \hat{\sigma}_{k_1}[S_{\alpha_1}(s_1, t_1, ..., t_n)], ..., \hat{\sigma}_{k_n}[S_{\alpha_n}(s_n, t_1, ..., t_n)]) \\
&= S_\alpha(\sigma_i(f_\alpha), S_{\alpha_1}(\hat{\sigma}_{k_1}[s_1], \hat{\sigma}_{k_1}[t_1], ..., \hat{\sigma}_{k_n}[t_n]), ..., \\
&\quad S_{\alpha_n}(\hat{\sigma}_{k_n}[s_n], \hat{\sigma}_{k_1}[t_1], ..., \hat{\sigma}_{k_n}[t_n])) \\
&= S_\alpha(S_\alpha(\sigma_i(f_\alpha), \hat{\sigma}_{k_1}[s_1], ..., \hat{\sigma}_{k_n}[s_n]), \hat{\sigma}_{k_1}[t_1], ..., \hat{\sigma}_{k_n}[t_n]) \\
&= S_\alpha(\hat{\sigma}_i[f_\alpha(s_1, ..., s_n)], \hat{\sigma}_{k_1}[t_1], ..., \hat{\sigma}_{k_n}[t_n]) \\
&= S_\alpha(\hat{\sigma}_i[t], \hat{\sigma}_{k_1}[t_1], ..., \hat{\sigma}_{k_n}[t_n]). \qquad \square
\end{aligned}$$

Lemma 3.8. *Let $(\sigma_1)_i, (\sigma_2)_i \in \Sigma^{|I|,n}(i)$-$Hyp_G$. Then*

$$((\sigma_1)_i \circ_G^i (\sigma_2)_i)\hat{} = (\hat{\sigma_1})_i \circ (\hat{\sigma_2})_i.$$

Proof. We will give a proof by induction on the complexity of the Σ-term t.
If $t = x_{ij} \in X_i$,

$$\begin{aligned}
((\sigma_1)_i \circ_G^i (\sigma_2)_i)\hat{}[x_{ij}] = x_{ij} &= (\hat{\sigma_1})_i[x_{ij}] \\
&= (\hat{\sigma_1})_i[(\hat{\sigma_2})_i[x_{ij}]] \\
&= (\hat{\sigma_1})_i \circ (\hat{\sigma_2})_i[x_{ij}] \\
&= (\hat{\sigma_1})_i \circ (\hat{\sigma_2})_i[t].
\end{aligned}$$

If $t = f_\gamma(t_1, ..., t_n) \in W(i)$ with $\gamma = (i_1, ..., i_n, i) \in \Sigma(i)$ and $t_j \in W(i_j)$. Suppose that $((\sigma_1)_{i_j} \circ_G^{i_j} (\sigma_2)_{i_j})\hat{}[t_j] = (\hat{\sigma_1})_{i_j} \circ (\hat{\sigma_2})_{i_j}[t_j], 1 \le j \le n$.

$$\begin{aligned}
((\sigma_1)_i \circ_G^i (\sigma_2)_i)\hat{}[t] &= ((\sigma_1)_i \circ_G^i (\sigma_2)_i)\hat{}[f_\gamma(t_1, ..., t_n)] \\
&= S_\gamma(\,((\sigma_1)_i \circ_G^i (\sigma_2)_i)(f_\gamma), ((\sigma_1)_{i_1} \circ_G^{i_1} (\sigma_2)_{i_1})\hat{}[t_1], ..., \\
&\quad ((\sigma_1)_{i_n} \circ_G^{i_n} (\sigma_2)_{i_n})\hat{}[t_n]) \\
&= S_\gamma(\,((\hat{\sigma}_1)_i \circ (\sigma_2)_i)(f_\gamma), ((\hat{\sigma}_1)_{i_1} \circ (\hat{\sigma}_2)_{i_1})[t_1], ..., \\
&\quad ((\hat{\sigma}_1)_{i_n} \circ (\hat{\sigma}_2)_{i_n})[t_n]) \\
&= S_\gamma(\,(\hat{\sigma}_1)_i[(\sigma_2)_i(f_\gamma)], (\hat{\sigma}_1)_{i_1}[(\hat{\sigma}_2)_{i_1}[t_1]], ..., \\
&\quad (\hat{\sigma}_1)_{i_n}[(\hat{\sigma}_2)_{i_n}[t_n]]) \\
&= (\hat{\sigma_1})_i[S_\gamma((\sigma_2)_i(f_\gamma), (\hat{\sigma}_2)_{i_1}[t_1], ..., (\hat{\sigma}_2)_{i_n}[t_n])] \\
&= (\hat{\sigma_1})_i[(\hat{\sigma_2})_i[f_\gamma(t_1, ..., t_n)]] \\
&= (\hat{\sigma_1})_i \circ (\hat{\sigma_2})_i[f_\gamma(t_1, ..., t_n)] \\
&= (\hat{\sigma_1})_i \circ (\hat{\sigma_2})_i[t]. \qquad \square
\end{aligned}$$

Theorem 3.9. *Let $(\sigma_1)_i, (\sigma_2)_i, (\sigma_3)_i \in \Sigma^{|I|,n}(i)$-$Hyp_G$. Then,*

$$((\sigma_1)_i \circ_G^i (\sigma_2)_i) \circ_G^i (\sigma_3)_i = (\sigma_1)_i \circ_G^i ((\sigma_2)_i \circ_G^i (\sigma_3)_i).$$

Proof.

$$\begin{aligned}((\sigma_1)_i \circ_G^i (\sigma_2)_i) \circ_G^i (\sigma_3)_i &= ((\sigma_1)_i \circ_G^i (\sigma_2)_i)^{\wedge} \circ (\sigma_3)_i \\ &= ((\hat{\sigma_1})_i \circ (\hat{\sigma_2})_i) \circ (\sigma_3)_i \\ &= (\hat{\sigma_1})_i \circ ((\hat{\sigma_2})_i \circ (\sigma_3)_i) \\ &= (\hat{\sigma_1})_i \circ ((\sigma_2)_i \circ_G^i (\sigma_3)_i) \\ &= (\sigma_1)_i \circ_G^i ((\sigma_2)_i \circ_G^i (\sigma_3)_i). \qquad \square\end{aligned}$$

Lemma 3.10. *For Σ-term $t \in W(i)$, $(\hat{\sigma}_{id})_i[t] = t$.*

Proof. We will prove by induction on the complexity of the Σ-term $t \in W(i)$.
If $t = x_{ij} \in X(i)$, then $(\hat{\sigma}_{id})_i[t] = (\hat{\sigma}_{id})_i[x_{ij}] = x_{ij} = t$.
If $t = f_\gamma(s_1, ..., s_n) \in W(i)$ with $\gamma = (i_1, ..., i_n, i) \in \Sigma(i)$ and $s_r \in W(i_r), 1 \leq r \leq n$. Assume that $(\hat{\sigma}_{id})_{i_r}[s_r] = s_r$ and $\gamma_r = (i_1, ..., i_n, i_r) \in \Lambda(i_r)$.

$$\begin{aligned}(\hat{\sigma}_{id})_i[t] &= (\hat{\sigma}_{id})_i[f_\gamma(s_1, ..., s_n)] \\ &= S_\gamma((\sigma_{id})_i(f_\gamma), (\hat{\sigma}_{id})_{i_1}[s_1], ..., (\hat{\sigma}_{id})_{i_n}[s_n]) \\ &= S_\gamma(f_\gamma(x_{i_1 1}, ..., x_{i_n n}), s_1, ..., s_n) \\ &= f_\gamma(S_{\gamma_1}(x_{i_1 1}, s_1, ..., s_n), ..., S_{\gamma_n}(x_{i_n n}, s_1, ..., s_n)) \\ &= f_\gamma(s_1, ..., s_n) \\ &= t. \qquad \square\end{aligned}$$

Lemma 3.11. *For $(\sigma)_i \in \Sigma^{|I|,n}(i)$-$Hyp_G$,*

$$(\sigma)_i \circ_G^i (\sigma_{id})_i = (\sigma_{id})_i \circ_G^i (\sigma)_i.$$

Proof. Let f_γ with $\gamma = (i_1, ..., i_n, i) \in \Sigma^{|I|,n}(i)$,

$$\begin{aligned}((\sigma)_i \circ_G^i (\sigma_{id})_i)(f_\gamma) = ((\hat{\sigma})_i \circ (\sigma_{id})_i)(f_\gamma) &= (\hat{\sigma})_i[(\sigma_{id})_i(f_\gamma)] \\ &= (\hat{\sigma})_i[f_\gamma(x_{i_1 1}, ..., x_{i_n n})] \\ &= S_\gamma((\sigma)_i(f_\gamma), (\hat{\sigma})_{i_1}[x_{i_1 1}], ..., (\hat{\sigma})_{i_n}[x_{i_n n}]) \\ &= S_\gamma((\sigma)_i(f_\gamma), x_{i_1 1}, ..., x_{i_n n}) \\ &= (\sigma)_i(f_\gamma) \\ &= (\hat{\sigma}_{id})_i[(\sigma)_i(f_\gamma)] \\ &= ((\hat{\sigma}_{id})_i \circ (\sigma)_i)(f_\gamma) \\ &= ((\sigma_{id})_i \circ_G^i (\sigma)_i)(f_\gamma). \qquad \square\end{aligned}$$

Theorem 3.12. *$(\Sigma^{|I|,n}(i)$-$Hyp_G, \circ_G^i, (\sigma_{id})_i)$ forms a monoid.*

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