Green’s Relations and Regularity for the Self-E-Preserving Transformation Semigroups

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Abstract: Let $T_X$ be the full transformation semigroup on a set $X$ and $E$ an arbitrary equivalence relation on $X$. We define a subsemigroup of $T_X$ as follows:

$$T_{SE}(X) = \{ \alpha \in T_X : \forall x \in X, (x, x\alpha) \in E \}$$

which is called the self-$E$-preserving transformation semigroup on $X$. Then $T_{SE}(X)$ becomes a regular semigroup. The purpose of this paper is to investigate Green’s relations for $T_{SE}(X)$. Moreover, we characterize when certain elements of $T_{SE}(X)$ are left regular, right regular and completely regular.

Keywords: transformation semigroup; Green's relations; regular; left (right) regular; completely regular.

2010 Mathematics Subject Classification: 20M20.

1 Introduction

An element $a$ of a semigroup $S$ is called regular if $a = axa$ for some $x \in S$, left regular if $a = xa^2$ for some $x \in S$, right regular if $a = a^2x$ for some $x \in S$ and completely regular if $a = axa$ and $ax = xa$ for some $x \in S$. Evidently every completely regular element is regular, left regular and right regular. If all its elements of $S$ are regular we called $S$ a regular semigroup.

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Let $T_X$ be the full transformation semigroup on a set $X$ under usual composition of mappings. It is well known that $T_X$ is a regular semigroup. Over the last decades, notions of regularity and Green’s relations of subsemigroups of $T_X$ have been widely considered see [1–6]. In [1] has introduced a family of subsemigroups of $T_X$ defined by
\[
T_E(X) = \{\alpha \in T_X : \forall a, b \in X, (a, b) \in E \Rightarrow (a\alpha, b\alpha) \in E\}
\]
where $E$ is an arbitrary equivalence relation on $X$. [1] has investigated regularity and Green’s relations for $T_E(X)$.

In the rest of the paper, let $E$ be an arbitrary equivalence relation on $X$. The following a subsemigroup of $T_X$ is considered:
\[
T_{SE}(X) = \{\alpha \in T_X : \forall x \in X, (x, x\alpha) \in E\}.
\]

In [7], $T_{SE}(X)$ is said to be the self-$E$-preserving transformation semigroup on $X$ and $T_{SE}(X) \subseteq T_E(X)$.

The paper is organized as follows. In section 2, we investigate Green’s relations of $T_{SE}(X)$. In section 3, we show that $T_{SE}(X)$ is a regular semigroup and give necessary and sufficient conditions for each element of $T_{SE}(X)$ when it is left regular, right regular and completely regular.

In this introductory section, we present a number of notations and propositions most of which will be indispensable for our research. For a set $X$ and $\alpha \in T_X$, we denote by $\pi(\alpha)$ the partition of $X$ induced by $\alpha$, namely,
\[
\pi(\alpha) = \{y^{\alpha^{-1}} : y \in X\alpha\},
\]
and $\alpha_*$ the natural bijection corresponding to $\alpha$ from $\pi(\alpha)$ onto $X\alpha$ defined by
\[
P_{\alpha_*} = x\alpha \quad \text{for all } P \in \pi(\alpha) \text{ and all } x \in P.
\]

For collections of subsets $A$ and $B$ of $X$, we say that $B$ is a refinement of $A$ or $B$ refines $A$ if $\cup A = \cup B$ and for every $B \in B$, there exists an element $A \in A$ such that $B \subseteq A$.

**Proposition 1.1.** Let $\alpha \in T_{SE}(X)$. If $y \in X\alpha$, then there exists a unique $A \in X/E$ such that $y\alpha^{-1} \subseteq A$. Hence $\pi(\alpha)$ refines $X/E$.

**Proof.** Let $y \in X\alpha$. Then $y = x\alpha$ for some $x \in X$. By $X/E$ is a partition of $X$, there exists a unique $A \in X/E$ such that $x \in A$. Let $z \in y\alpha^{-1}$. Then $za = y$. Since $\alpha \in T_{SE}(X)$, we have that $(x, y) = (x, x\alpha) \in E$ and $(z, y) = (z, z\alpha) \in E$. By transitive of $E$, we deduce that $(z, x) \in E$, so $z \in A$. Hence $y\alpha^{-1} \subseteq A$ for some $A \in X/E$.

**Proposition 1.2.** Let $\alpha \in T_{SE}(X)$. Then for every $A \in X/E$, $A\alpha \subseteq A$.

**Proof.** Let $A \in X/E$ and $x \in A$. By $\alpha \in T_{SE}(X)$, we have that $(x, x\alpha) \in E$. This means that $x\alpha \in A$. 

\[\square\]


Let \( \alpha \in T_{SE}(X) \) and \( A \in X/E \). We denote
\[
\pi_A(\alpha) = \{ P \in \pi(\alpha) : P \cap A \neq \emptyset \}.
\]

**Proposition 1.3.** Let \( \alpha \in T_{SE}(X) \) and \( A \in X/E \). Then \( A = \bigcup \pi_A(\alpha) \).

**Proof.** Let \( x \in \bigcup \pi_A(\alpha) \). Then \( x \in P \) for some \( P \in \pi(\alpha) \) such that \( P \cap A \neq \emptyset \). By Proposition 1.2, we deduce that \( P \subseteq A \). Thereby \( \bigcup \pi_A(\alpha) \subseteq A \). For the reverse inclusion, let \( x \in A \). By \( \pi(\alpha) \) is a partition of \( X \), we have \( x \in P \) for some \( P \in \pi(\alpha) \). This implies that \( P \in \pi_A(\alpha) \), hence \( x \in \bigcup \pi_A(\alpha) \). Therefore \( A = \bigcup \pi_A(\alpha) \). \( \square \)

## 2 Green’s Relations for the Self-E-Preserving Transformation Semigroups

We refer to [8, Chapter 2] for the definitions and notations of Green’s relations. In this section, we discuss Green’s relations of \( T_{SE}(X) \).

**Theorem 2.1.** Let \( \alpha, \beta \in T_{SE}(X) \). Then \( \alpha \in T_{SE}(X)\beta \) if and only if for every \( A \in X/E, A\alpha \subseteq A\beta \).

**Proof.** Suppose that \( \alpha \in T_{SE}(X)\beta \). Then \( \alpha = \delta \beta \) for some \( \delta \in T_{SE}(X) \). Let \( A \in X/E \). By Proposition 1.2, we then have \( A\delta \subseteq A\beta \). Hence \( A\alpha = A\delta \beta \subseteq A\beta \).

Conversely, assume that \( A\alpha \subseteq A\beta \) for all \( A \in X/E \). For each \( x \in X \), there exists a unique \( A \in X/E \) such that \( x \in A \). By assumption, we choose and fix an element \( x' \in A \) such that \( x\alpha = x'\beta \) for all \( x \in X \). Define \( \delta : X \to X \) by \( x\delta = x' \) for all \( x \in X \). Let \( x \in X \). Since \( x, x' \in A \), \( (x, x\delta) = (x, x') \in E \) and \( x\delta \beta = (x\delta)\beta = x'\beta = x\alpha \). These verify that \( \delta \in T_{SE}(X) \) and \( \alpha = \delta \beta \). Hence \( \alpha \in T_{SE}(X)\beta \), as required. \( \square \)

**Corollary 2.2.** Let \( \alpha, \beta \in T_{SE}(X) \). Then \( (\alpha, \beta) \in \mathcal{L} \) if and only if \( A\alpha = A\beta \) for all \( A \in X/E \).

**Theorem 2.3.** Let \( \alpha, \beta \in T_{SE}(X) \). Then \( \alpha \in \beta T_{SE}(X) \) if and only if \( \pi(\beta) \) refines \( \pi(\alpha) \).

**Proof.** Assume that \( \alpha \in \beta T_{SE}(X) \). Then \( \alpha = \beta \delta \) for some \( \delta \in T_{SE}(X) \). Using the fact that \( \pi(\alpha) \) is a partition of \( X \), we have \( \cup \pi(\alpha) = \cup \pi(\beta) \). Let \( P \in \pi(\beta) \). Hence \( P\beta \ast = y \) for some \( y \in X\beta \). Thus \( P\alpha = P\beta \delta = \{ y\delta \} \) which implies that \( P \subseteq y\delta \alpha^{-1} \in \pi(\alpha) \). We conclude that \( \pi(\beta) \) refines \( \pi(\alpha) \).

Conversely, assume that \( \pi(\beta) \) refines \( \pi(\alpha) \). For each \( x \in X\beta \), there exists a unique \( P_x \in \pi(\beta) \) such that \( P_x\beta \ast = x \). By assumption, there exists a unique \( Q_x \in \pi(\alpha) \) such that \( P_x \subseteq Q_x \). Define \( \delta : X \to X \) by
\[
x\delta = \begin{cases} Q_x \alpha \ast & \text{if } x \in X\beta; \\
x & \text{otherwise.}
\end{cases}
\]
Clearly, $\delta$ is well-defined. To show that $\delta \in T_{SE}(X)$, let $x \in X$. If $x \notin X\beta$, then $(x, x\delta) = (x, x) \in E$. If $x \in X\beta$, then by the definition of $\delta$, $x\delta = Q_{x}\alpha_{x}$ where $P_{x}\beta_{x} = x$ and $P_{x} \subseteq Q_{x}$ for some $P_{x} \in \pi(\beta)$ and $Q_{x} \in \pi(\alpha)$. By Proposition 1.1, $\pi(\alpha)$ and $\pi(\beta)$ refine $X/E$, we then have $P_{x} \subseteq A$ and $Q_{x} \subseteq B$ for some $A, B \in X/E$. Since $\beta \in T_{SE}(X)$, it follows that $x \in A$. Since $P_{x} \subseteq Q_{x}$ and $X/E$ is a partition of $X$, we have that $A = B$, so $Q_{x} \subseteq A$. It follows from Proposition 1.2 that $Q_{x}\alpha_{x} \in A\alpha \subseteq A$. We deduce that $(x, x\delta) = (x, Q_{x}\alpha_{x}) \in E$. Therefore $\delta \in T_{SE}(X)$. Moreover, for $x \in X$,

$$x\beta\delta = (x\beta)\delta = Q_{x}\beta\alpha_{x} = x\alpha$$

since $x \in P_{x}\beta \subseteq Q_{x}\beta$ where $P_{x}\beta \in \pi(\beta)$ and $Q_{x}\beta \in \pi(\alpha)$. Therefore $\alpha = \beta\delta$, hence the theorem is thereby proved.

**Corollary 2.4.** Let $\alpha, \beta \in T_{SE}(X)$. Then $(\alpha, \beta) \in \mathcal{R}$ if and only if $\pi(\alpha) = \pi(\beta)$.

Since $\mathcal{H} = \mathcal{R} \cap \mathcal{L}$, the following corollary follows immediately from Corollary 2.2 and Corollary 2.4.

**Corollary 2.5.** Let $\alpha, \beta \in T_{SE}(X)$. Then $(\alpha, \beta) \in \mathcal{H}$ if and only if $\pi(\alpha) = \pi(\beta)$ and $A\alpha = A\beta$ for all $A \in X/E$.

The next lemma is verified to consider the relation $\mathcal{J}$.

**Lemma 2.6.** Let $\alpha, \beta, \delta, \gamma \in T_{X}$. If $\alpha = \delta\beta\gamma$, then the set $A = \{\cup A_{Q} : Q \in \pi(\beta)$ and $Q \cap X\delta \neq \emptyset\}$ is a refinement of $\pi(\alpha)$ where $A_{Q} = \{P \in \pi(\delta) : P\delta_{x} \subseteq Q\}$.

**Proof.** Assume that $\alpha = \delta\beta\gamma$. By the first part of the proof Theorem 2.3, we have $\pi(\delta)$ refines $\pi(\alpha)$. Claim that $\cup A = X$. Let $x \in X$. Then $x \in P$ for some $P \in \pi(\delta)$. We note that $x\delta\beta \in X\beta$, so $x\delta\beta = Q\beta_{x}$ for some $Q \in \pi(\beta)$. Then $P\delta_{x} = x\delta\beta \subseteq Q\beta_{x}$ and hence $Q \cap X\delta \neq \emptyset$. Thus $P \subseteq A_{Q}$ and $x \in P \subseteq \cup A_{Q} \subseteq \cup A$. Hence we have the claim. This means that $\cup A = \cup \pi(\alpha)$. Let $Q \in \pi(\pi(\beta))$ be such that $Q \cap X\delta \neq \emptyset$. To show that there exists $P \subseteq \pi(\alpha)$ such that $\cup A_{Q} \subseteq P$, let $x \in Q \cap X\delta$. Then there exists an element $x' \in X$ such that $x'\delta = x$. Since $\pi(\delta)$ is a partition of $X$, $x' \subseteq P$ for some $P \in \pi(\delta)$ and $P\delta_{x} = x'\delta$. Since $\pi(\delta)$ refines $\pi(\alpha)$, $P \subseteq \bar{P}$ for some $P \subseteq \pi(\alpha)$. Let $y \in \cup A_{Q}$. Then $y \in P'$ for some $P' \in A_{Q}$. By the definition of $A_{Q}$, $P'O_{x} \subseteq Q$. Hence $y\delta\beta = P'O_{x}\beta = Q\beta_{x} = x\beta = x'\delta\beta$. Since $x' \subseteq P \subseteq \bar{P}$, $x'\alpha = \bar{P}\alpha_{x}$. Thus

$$y\alpha = y\delta\beta\gamma = x'\delta\beta\gamma = x'\alpha = \bar{P}\alpha_{x}$$

which implies that $y \in \bar{P}$, hence $\cup A_{Q} \subseteq \bar{P}$. This proves that $A$ refines $\pi(\alpha)$, as required.

**Theorem 2.7.** Let $\alpha, \beta \in T_{SE}(X)$. Then $\alpha \in T_{SE}(X)\beta T_{SE}(X)$ if and only if there exists a refinement $A$ of $\pi(\alpha)$ and $\varphi : A \to \pi(\beta)$ such that $\varphi$ is an injection and for every $P \subseteq A$, $P, P\varphi \subseteq A$ for some $A \in X/E$. 
Proof. Assume that \( \alpha \in T_{SE}(X) \beta T_{SE}(X) \). Then \( \alpha = \delta \beta \gamma \) for some \( \delta, \gamma \in T_{SE}(X) \). Let \( \mathcal{A} = \{ \cup \mathcal{A}_Q : Q \in \pi(\beta) \) and \( Q \cap X \delta \neq \emptyset \} \) where \( \mathcal{A}_Q = \{ P \in \pi(\alpha) : P \delta_x \in Q \} \). Then by Lemma [2.6], \( \mathcal{A} \) is a refinement of \( \pi(\alpha) \). Define \( \varphi : \mathcal{A} \to \pi(\beta) \) by \((\cup \mathcal{A}_Q) \varphi = Q \). It is clear that \( \varphi \) is well-defined. Suppose that \((\cup \mathcal{A}_Q) \varphi = (\cup \mathcal{A}_Q) \varphi' \). By the definition of \( \varphi, Q = Q' \). Thus \( \mathcal{A}_Q = \mathcal{A}_{Q'} \) and so \( \varphi \) is an injection. 

Let \( \cup \mathcal{A}_Q \in \mathcal{A} \) where \( Q \in \pi(\beta) \). Since \( \pi(\beta) \) refines \( X/E \), \( Q \subseteq A \) for some \( A \in X/E \). For each \( P \in \mathcal{A}_Q \), we have \( P \delta_x \in Q \) and so \( P \delta_x \in A \). Hence \( P \subseteq A \) because \( \delta \in T_{SE}(X) \). Thus \( \cup \mathcal{A}_Q \subseteq A \), hence \((\cup \mathcal{A}_Q) \varphi = Q \subseteq A \).

Conversely, suppose that \( \varphi : \mathcal{A} \to \pi(\beta) \) is an injection where \( \mathcal{A} \) is a refinement of \( \pi(\alpha) \) and for every \( P \in \mathcal{A} \), \( P \beta \subseteq A \) for some \( A \in X/E \). Let \( x \in X \). Then \( x \in P \) for some \( P \in \mathcal{A} \), we choose and fix an element \( \tilde{x} \in P \varphi \). We define \( \delta : X \to X \) by \( x \delta = \tilde{x} \) for all \( x \in X \). By assumption, there exists \( A \in X/E \) such that \( P \beta \subseteq A \) which implies that \( \delta \in T_{SE}(X) \). Since \( \beta : \pi(\beta) \to X \beta \) is an injection and by assumption, \( \varphi \beta : \mathcal{A} \to X \beta \) is an injection. For each \( x \in \mathcal{A} \beta \), there exists a unique \( P_x \in \mathcal{A} \) such that \( x = P_x \varphi \beta_x \). We fix \( x' \in P_x \). Define \( \gamma : X \to X \) by

\[
x \gamma = \begin{cases} 
x' \alpha & \text{if } x \in \mathcal{A} \varphi \beta_x; \\
x & \text{otherwise.}
\end{cases}
\]

Let \( x \in X \). If \( x \notin \mathcal{A} \varphi \beta_x \), then \( (x, x \gamma) = (x, x) \in E \). If \( x \in \mathcal{A} \varphi \beta_x \), then \( x = P_x \varphi \beta_x \) for some \( P_x \in \mathcal{A} \). By Proposition 1.1, there is \( A \in X/E \) such that \( P_x \beta \subseteq A \). From \( (x', x' \alpha) \in E \) and \( x' \in A \), we then have \( x' \alpha \in A \). Thus \( (x, x \gamma) = (x, x' \alpha) \in E \). This shows that \( \gamma \in T_{SE}(X) \). We show that \( \alpha = \delta \beta \gamma \).

Let \( x \in X \). Then \( x \in P \) for some \( P \in \mathcal{A} \). Since \( \tilde{x} \beta_x = P \beta_x \subseteq A \beta \subseteq A \), we conclude that \( P \beta \beta_x = \tilde{x} \beta_x = P_{\tilde{x} \beta_x} \beta_x \). It follows from \( \varphi \beta_x \) is injective that \( P = P_{\tilde{x} \beta} \). Hence \( x, (\tilde{x} \beta)' \in P \). Thus \( x, (\tilde{x} \beta)' \in P' \) which implies that \( x \alpha = (\tilde{x} \beta)' \alpha \).

Therefore, the theorem is completely proved.

**Corollary 2.8.** Let \( \alpha, \beta \in T_{SE}(X) \). Then \( \alpha \in T_{SE}(X) \beta T_{SE}(X) \) if and only if there exists an injection \( \varphi : \pi(\alpha) \to \pi(\beta) \) such that for every \( P \in \pi(\alpha) \), \( P \varphi \subseteq A \) for some \( A \in X/E \).

**Proof.** Assume that \( \alpha \in T_{SE}(X) \beta T_{SE}(X) \). By Theorem 2.7, there exist a refinement \( \mathcal{A} \) of \( \pi(\alpha) \) and \( \varphi : \mathcal{A} \to \pi(\beta) \) such that \( \varphi \) is an injection and for every \( P \in \mathcal{A} \), \( P \varphi \subseteq A \) for some \( A \subseteq X/E \). For each \( P \in \pi(\alpha) \), we choose and fix \( P' \in \mathcal{A} \) such that \( P' \subseteq P \). Define \( \varphi' : \pi(\alpha) \to \pi(\beta) \) by

\[
P \varphi' = P' \varphi \quad \text{for all } P \in \pi(\alpha).
\]

It is easy to see that \( \varphi \) is well-defined. Let \( P, Q \in \pi(\alpha) \) be such that \( P \varphi = Q \varphi' \). Hence \( P' \varphi = Q' \varphi' \). By \( \varphi \) is injective, \( P' = Q' \). Since \( \pi(\alpha) \) is a partition of \( X \), \( P = Q \). Thus \( \varphi' \) is injective. Let \( P \in \pi(\alpha) \). We then have \( P', P \varphi \subseteq A \) for some
Let \( \alpha, \beta \in T_{SE}(X) \). Then \( (\alpha, \beta) \in \mathcal{D} \) if and only if there exist injections \( \psi : \pi(\alpha) \to \pi(\beta) \) and \( \psi' : \pi(\beta) \to \pi(\alpha) \) such that for every \( P \in \pi(\alpha) \) and \( Q \subseteq \pi(\beta) \), \( P, P\psi \subseteq A \) and \( Q, Q\psi' \subseteq A' \) for some \( A, A' \in X/E \).

The remaining matter is only to consider the relation \( \mathcal{D} \).

**Lemma 2.10.** Let \( \alpha, \beta \in T_{SE}(X) \) and \( A \in X/E \). If \( \varphi : \pi(\alpha) \to \pi(\alpha) \) is a bijection, then there exists \( \delta_A : A \to X \) satisfies \( \pi(\delta_A) = \pi(\beta) \) and \( A\delta_A = A\alpha \).

**Proof.** Assume that \( \varphi : \pi(\alpha) \to \pi(\alpha) \) is bijective. Let \( x \in A \). By Proposition 1.3, there exists \( P_x \in \pi(\beta) \) such that \( x \in P_x \). We define \( \delta_A : A \to X \) by

\[
x \delta_A = (P_x \varphi)x_{\alpha}
\]

for all \( x \in A \).

It is clear that \( \delta_A \) is well-defined. We then have \( \cup \pi(\alpha) = A = \cup \pi(\delta_A) \) by Proposition 1.3. Claim that \( \pi(\alpha) = \pi(\delta_A) \). From the definition of \( \delta_A \), we note that for every \( P \in \pi(\alpha) \), \( P\delta_A = \{P\varphi_\alpha\} \), hence \( P \subseteq Q \) for some \( Q \subseteq \pi(\delta_A) \). For each \( Q \subseteq \pi(\delta_A) \), \( Q\delta_{A*} = P\varphi_\alpha \) for some \( P \subseteq \pi(\alpha) \). Next, to show \( Q \subseteq P \), let \( x \in Q \). Then there exists \( P_x \in \pi(\beta) \) such that \( x \in P_x \). Hence \( Q\delta_{A*} = x\delta_A = P_x \varphi_\alpha \). We deduce that \( P\varphi_\alpha = P_x \varphi_\alpha \). Since \( \varphi_\alpha \) is a bijection, \( P = P_x \). Hence \( x \in P \), so \( Q \subseteq P \). Therefore we conclude that \( \pi(\alpha) = \pi(\delta_A) \). To show that \( A\delta_A = A\alpha \), let \( x \in A \). Then \( x \in P_{\pi(\alpha)} \) for some \( P_{\pi(\alpha)} \in \pi(\beta) \), whence \( x\delta_A = P_x \varphi_\alpha \in A\alpha \). Thus \( A\delta_A \subseteq A\alpha \). For the reverse inclusion, let \( y \in A \). Since \( y\alpha \in A \alpha \), \( y\alpha = P\alpha \) for some \( P \subseteq \pi(\alpha) \). Since \( \varphi \) is bijective, there exists a unique \( P' \in \pi(\alpha) \) such that \( P' \varphi = P \). Choose \( z \in P' \); we have that \( z\delta_A = P' \varphi_\alpha = P\alpha \) which implies that \( y\alpha \in A\delta_A \). Hence \( A\alpha \subseteq A\delta_A \) as we wished to show.

**Theorem 2.11.** Let \( \alpha, \beta \in T_{SE}(X) \). Then \( (\alpha, \beta) \in \mathcal{D} \) if and only if for every \( A \in X/E \), there exists a bijection \( \varphi_A : \pi(\beta) \to \pi(\alpha) \).

**Proof.** Suppose that \( (\alpha, \beta) \in \mathcal{D} \). Then there exists \( \delta \in T_{SE}(X) \) such that \( (\alpha, \delta) \in \mathcal{L} \) and \( (\delta, \beta) \in \mathcal{R} \). Let \( A \in X/E \). For each \( P \in \pi(\alpha) \), we then have \( P \subseteq \pi(\beta) \) and \( P \cap A \neq \emptyset \). By Corollary 2.2, we have \( \pi(\delta) = \pi(\beta) \), so \( P \subseteq \pi(\delta) \). Since \( \pi(\delta) \) refines \( X/E \) and \( P \cap A \neq \emptyset \), we deduce that \( P \subseteq A \), hence \( P\delta_A \in A\delta \). From Corollary 2.2, we obtain that \( A\alpha = A\delta \), that is \( P\delta_A \subseteq A\alpha \subseteq X\alpha \). Then there exists \( Q_P \subseteq \pi(\alpha) \) such that \( Q_P \alpha = P\delta_A \). Since \( Q_P \alpha \subseteq A \) and \( \alpha \in T_{SE}(X) \), \( Q_P \subseteq \pi(\beta) \). Then \( Q_P \in \pi(\alpha) \). Define \( \varphi_A : \pi(\beta) \to \pi(\alpha) \) by

\[
P\varphi_A = Q_P \quad \text{for all } P \in \pi(\beta).
\]

If \( Q' \in \pi(\alpha) \) is such that \( Q' \alpha = P\delta_A \), then \( Q_P = Q' \) because \( Q_P \alpha = Q' \alpha \). This shows that \( \varphi_A \) is well-defined. Suppose that \( P\varphi_A = P'\varphi_A \). Then \( Q_P = Q' \), hence \( P\delta_A = Q_P \alpha = Q'P\alpha = P'\delta_A \) which implies that \( P = P' \). Therefore \( \varphi \) is an injection. Claim that \( \varphi_A \) is onto, let \( Q \subseteq \pi(\alpha) \). Then \( Q\alpha \subseteq A \alpha = A\delta \).
Thus there exists $P \in \pi(\delta)$ such that $P\delta = Q\alpha$. Since $\delta \in T_{SE}(X)$, $P \subseteq A$. By $\pi(\delta) = \pi(\beta)$, we have $P \in \pi(\beta)$ and so $P \in \pi_A(\beta)$. Thus $P\varphi_A = Q$, so we have the claim. Therefore $\varphi_A$ is bijective, as required.

Conversely, for every $A \in X/E$, there exists a bijection $\varphi_A : \pi_A(\beta) \to \pi_A(\alpha)$. It follows from Lemma 2.10 that there exists $\delta_A : A \to X$ corresponding to $A \in X/E$ such that $\pi_A(\beta) = \pi(\delta_A)$ and $A\delta_A = A\alpha$. Thus we define $\delta : X \to X$ by $\delta|_A = \delta_A$ for all $A \in X/E$. Since $X/E$ is a partition of $X$, $\delta$ is well-defined. We note that for each $A \in X/E$, $A\delta = A\delta_A = A\alpha \subseteq A$ by $\alpha \in T_{SE}(X)$, hence $\delta \in T_{SE}(X)$. Finally, we can see that

$$\pi(\beta) = \bigcup_{A \in X/E} \pi_A(\beta) = \bigcup_{A \in X/E} \pi(\delta_A) = \bigcup_{A \in X/E} \pi_A(\delta) = \pi(\delta),$$

it follows by Corollary 2.4 that $(\delta, \beta) \in R$. Since $A\delta = A\delta_A = A\alpha$, we deduce that $(\alpha, \delta) \in L$ by Corollary 2.2. Therefore $(\alpha, \beta) \in L \circ R = D$.

Hence the theorem is completely proved. \hfill \Box

## 3 Regularity for the Self-E-Order Preserving Transformation Semigroups

### Proposition 3.1

The semigroup $T_{SE}(X)$ is a regular semigroup.

**Proof.** Let $\alpha \in T_{SE}(X)$. Then for each $x \in X\alpha$, there exists $P_x \in \pi(\alpha)$ such that $P_x \alpha = x$. We choose an element $x' \in P_x$, whence $x' \alpha = x$ for all $x \in X\alpha$. Define $\beta : X \to X$ by

$$x\beta = \begin{cases} x' & \text{if } x \in X\alpha; \\ x & \text{otherwise.} \end{cases}$$

Let $x \in X$. If $x \not\in X\alpha$, then $(x, x\beta) = (x, x) \in E$. If $x \in X\alpha$, then $(x, x\beta) = (x, x') = (x'\alpha, x') \in E$. We also have that

$$x\alpha \beta \alpha = (x\alpha) \beta \alpha = (x\alpha)' \alpha = x\alpha.$$

These show that $\beta \in T_{SE}(X)$ and $\alpha = \alpha \beta \alpha$, respectively. Therefore $\alpha$ is a regular element of $T_{SE}(X)$.

**Theorem 3.2.** Let $\alpha \in T_{SE}(X)$. Then $\alpha$ is left regular if and only if for every $P \in \pi(\alpha)$, $P \cap X\alpha \neq \emptyset$.

**Proof.** Assume that $\alpha$ is a left regular element of $T_{SE}(X)$. Then $\alpha = \beta \alpha^2$ for some $\beta \in T_{SE}(X)$. Let $P \in \pi(\alpha)$. Then $P\alpha = y$ for some $y \in X\alpha$. For each $x \in P$, we have that $x\alpha = y$. Hence

$$y = x\alpha = x\beta \alpha^2 = (x\beta \alpha) \alpha.$$

This implies that $x\beta \alpha \in y^{-1} = P$ and $x\beta \alpha \in X\alpha$. Therefore $P \cap X\alpha \neq \emptyset$. 


Conversely, suppose that \( P \cap X \alpha \neq \emptyset \) for all \( P \in \pi(\alpha) \). For each \( x \in X \), there exists \( P \in \pi(\alpha) \) such that \( x \in P \). By assumption, we choose and fix \( y_x \in P \cap X \alpha \). So \( x \alpha = y_x \alpha \). Since \( y_x \in X \alpha \), we can fix \( y'_x \in X \) such that \( y'_x \alpha = y_x \). Define \( \beta : X \to X \) by
\[
x\beta = y'_x \quad \text{for all } x \in X.
\]
Since \( \pi(\alpha) \) is a partition of \( X \), \( \beta \) is well-defined. To show that \( \beta \in T_{SE}(X) \), let \( x \in X \). Since \( (x, x\alpha), (y_x, y_x \alpha) \in E \) and \( E \) is transitive, \( (x, y_x) \in E \). Since \( (y'_x, y_x) = (y'_x, y'_x \alpha) \in E \), it then follows by transitive of \( E \) that, \( (x, x\beta) = (x, y'_x) \in E \). Hence \( \beta \in T_{SE}(X) \). And
\[
x\beta \alpha^2 = y'_x \alpha^2 = (y'_x \alpha) \alpha = y_x \alpha = x\alpha.
\]
Thus \( \alpha \) is a left regular element of \( T_{SE}(X) \).

**Theorem 3.3.** Let \( \alpha \in T_{SE}(X) \). Then \( \alpha \) is right regular if and only if \( \alpha|_{X \alpha} \) is an injection.

**Proof.** Assume that \( \alpha \) is a right regular element of \( T_{SE}(X) \). Then \( \alpha = \alpha^2 \beta \) for some \( \beta \in T_{SE}(X) \). Let \( x, y \in X \alpha \) be such that \( x \alpha = y \alpha \). Since \( x, y \in X \alpha \), \( x = x' \alpha \) and \( y = y' \alpha \) for some \( x', y' \in X \). Hence
\[
x = x' \alpha = x' \alpha^2 \beta = (x' \alpha) \alpha \beta = (x \alpha) \beta = (y \alpha) \alpha \beta = y' \alpha \beta = y' \alpha = y.
\]
This proves that \( \alpha|_{X \alpha} \) is injective.

For the converse, assume that \( \alpha|_{X \alpha} \) is an injection. We construct \( \beta \in T_{SE}(X) \) such that \( \alpha = \alpha^2 \beta \). For any \( x \in X \alpha^2 \), we choose \( x' \in X \alpha \) such that \( x = x' \alpha \). Define \( \beta : X \to X \) by
\[
x\beta = \begin{cases} 
x' & \text{if } x \in X \alpha^2; 
\end{cases} x \quad \text{otherwise}.
\]
It is easy to verify that \( \beta \in T_{SE}(X) \). Let \( x \in X \). Then \( x \alpha = y \) for some \( y \in X \alpha \) and \( y \in X \alpha^2 \). Then there exists \( (y \alpha)' \in X \alpha \) such that \( (y \alpha)' \alpha = y \alpha \). It follows by assumption that \( (y \alpha)' = y \). Thus \( x \alpha^2 \beta = (x \alpha) \alpha \beta = (y \alpha) \beta = y \alpha = x \alpha \). Hence \( \alpha \) is a right regular element of \( T_{SE}(X) \).

**Theorem 3.4.** Let \( \alpha \in T_{SE}(X) \). Then \( \alpha \) is completely regular if and only if for every \( P \in \pi(\alpha), |P \cap X \alpha| = 1 \).

**Proof.** Suppose that \( \alpha \) is a completely regular element of \( T_{SE}(X) \). Then \( \alpha \) is left regular and right regular. By Theorem 3.2, \( P \cap X \alpha \neq \emptyset \). It follows from Theorem 3.3 that \( |P \cap X \alpha| = 1 \).

Conversely, assume that \( |P \cap X \alpha| = 1 \) for all \( P \in \pi(\alpha) \). For each \( P \in \pi(\alpha) \), by assumption, there exists a unique \( x_P \in P \cap X \alpha \). Since \( x_P \in X \alpha \), \( P' \alpha \ast = x_P \) for some \( P' \in \pi(\alpha) \). Similarly, there is a unique \( x_{P'} \in P' \cap X \alpha \) and hence \( x_{P' \alpha} = x_P \). Define \( \beta : X \to X \) by
\[
x\beta = x_{P'} \quad \text{for all } x \in P \text{ and each } P \in \pi(\alpha).
\]
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Then \( \beta \in T_X \). To show that \( \beta \in T_{SE}(X) \), let \( x \in X \). Thus \( x \in P \) for some \( P \in \pi(\alpha) \). By assumption \( x\alpha = x_P\alpha \) where \( x_P \in P \cap X\alpha \), so \( (x, x_P\alpha) = (x, x\alpha) \in E \). Since \( (x, x_P\alpha), (x_P, x_P\alpha) \in E \), \( (x, x_P) \in E \) by transitive of \( E \). Then we obtain that \( (x, x_P) = (x, x_P\alpha) \). Since \( (x_P, x_P\alpha) \in E \), it follows that \( (x, x_P) \in E \). We deduce that \( (x, x\beta) = (x, x_P\alpha) \in E \). Thus \( \beta \in T_{SE}(X) \). Finally, to show that \( \alpha = \alpha\beta\alpha \) and \( \alpha\beta = \beta\alpha \), let \( x \in X \). Hence \( x\alpha = x_P \) for some \( x_P \in P \cap X\alpha \) where \( P \in \pi(\alpha) \). Then \( x\alpha\beta\alpha = x_P\beta\alpha = x_P\alpha = x_P = x\alpha \). Moreover, we get \( x \in P' \) where \( P' \in \pi(\alpha) \) and \( P'\alpha = x_P \). Then there exists a unique \( x_{P''} \in P'' \cap X\alpha \) where \( P'' \in \pi(\alpha) \) and \( x_{P''}\alpha = x_P \). By the definition of \( \beta \), we have \( x\beta = x_{P''} \). Also, we have that \( x\alpha\beta = x_P\beta = x_{P''} = x_{P''}\alpha = x\beta\alpha \). These mean that \( \alpha = \alpha\beta\alpha \) and \( \alpha\beta = \beta\alpha \). Therefore \( \alpha \) is completely regular of \( T_{SE}(X) \). □

Acknowledgement : The author would like to thank the referees for valuable comments and suggestions which improved the paper.

References


(Received 2 April 2017)
(Accepted 8 July 2017)