A Common Fixed Point Theorem for Single-Valued and Set-Valued Mappings Satisfying A Generalized Contractive Condition

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Abstract: In this paper, we prove a common fixed point theorem for two pairs of single-valued and set-valued mappings satisfying a generalized contractive condition and a property using the concept of weak compatibility in metric spaces which generalizes Theorem 1 of [1] and Corollary 3 of [2].

Keywords: Weakly compatible mappings; Common fixed point; Set-valued mappings; Property (E.A).

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1 Introduction

Let \( S \) and \( T \) be self-mappings of a metric space \((X,d)\). \( S \) and \( T \) are commuting if \( STx = TSx \) for all \( x \in X \). Sessa [13] defined \( S \) and \( T \) to be weakly commuting if for all \( x \in X \)

\[
d(STx, TSx) \leq d(Tx, Sx). \tag{1.1}
\]

Jungck [9] defined \( S \) and \( T \) to be compatible as a generalization of weakly commuting if

\[
\lim_{n \to \infty} d(STx_n, TSx_n) = 0 \tag{1.2}
\]

whenever \( \{x_n\} \) is a sequence in \( X \) such that \( \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t \) for some \( t \in X \).

It is easy to show that commuting implies weakly commuting implies compatible and there are examples in the literature verifying that the inclusions are proper, see [9] and [13].

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Jungck [10] defined $S$ and $T$ to be weakly compatible if they commute at their coincidence points. It was proved that if $S$ and $T$ are compatible, then they are weakly compatible and the converse is not true in general.

Let $B(X)$ be the set of all nonempty bounded subsets of $X$.

As in [7] and [8], let $\delta(A, B)$ and $D(A, B)$ be the functions defined by

$$
\delta(A, B) = \sup \{d(a, b) : a \in A, b \in B\},
$$

$$
d(A, B) = \inf \{d(a, b) : a \in A, b \in B\}, \text{ for all } A, B \in B(X).
$$

If $A$ consists of a single point $a$, we write $\delta(A, B) = \delta(a, B)$. If $B$ consists also of a single point $b$, we write $\delta(A, B) = d(a, b)$.

It follows immediately from the definition of $\delta$ that for all $A, B, C \in B(X)$.

$$
\delta(A, B) = \delta(B, A) \geq 0,
$$

$$
\delta(A, B) \leq \delta(A, C) + \delta(C, B),
$$

$$
\delta(A, B) = 0 \text{ iff } A = B = \{a\},
$$

$$
\delta(A, A) = \text{diam}A.
$$

Sessa [14] extended (1.1) to single-valued and set-valued mappings as follows:

The mappings $F : X \to B(X)$ and $I : X \to X$ are said to be weakly commuting on $X$ if $IFx \in B(X)$ and

$$
\delta(FIx, IFx) \leq \max\{\delta(Ix, Fx), \text{diam}(IFx)\} \text{ for all } x \in X. \quad (1.3)
$$

Note that if $F$ is a single-valued mapping, then the set $IFx$ consists of a single point. Therefore,

$$
\text{diam}(IFx) = 0 \text{ for all } x \in X \text{ and condition (1.3) reduces to the condition (1.1)}.
$$

Two commuting mappings $F$ and $I$ clearly weakly commute but two weakly commuting $F$ and $I$ do not necessarily commute as it was shown in [14].

Jungck and Rhoades [11] defined the concepts of $\delta$–compatible and weakly compatible mappings which extend the concept of compatible mappings and weakly compatible in the single-valued setting to set-valued mappings as follows:

The mappings $I : X \to X$ and $F : X \to B(X)$ are said to be $\delta$–compatible if

$$
\lim_{n \to \infty} \delta(FIx_n, IFx_n) = 0
$$

whenever $\{x_n\}$ is a sequence in $X$ such that $IFx_n \in B(X)$, $Fx_n \to \{t\}$ and $Ix_n \to t$ as $n \to \infty$ for some $t$ in $X$.

The mappings $I : X \to X$ and $F : X \to B(X)$ are weakly compatible if they commute at their coincidence points, i.e., for each point $x$ in $X$ such that $Fu = \{Iu\}$, we have $FIu = IFu$.

Note that the equation $Fx = \{Ix\}$ implies that $Fx$ is a singleton.

It can be seen that any $\delta$–compatible pair $(F, I)$ is weakly compatible. Examples of weakly compatible pairs which are not $\delta$–compatible were given in [11].
**Definition 1.1.** [7] A sequence \(\{A_n\}\) of subsets of \(X\) is said to be convergent to a subset \(A\) of \(X\) if:

(i) given \(a \in A\), there is a sequence \(\{a_n\}\) in \(X\) such that \(a_n \in A_n\) for \(n = 1, 2, \ldots\) and \(\{a_n\}\) converges to \(a\).

(ii) given \(\epsilon > 0\), there exists a positive integer \(N\) such that \(A_n \subset A_\epsilon\) for \(n > N\) where \(A_\epsilon\) is the union of all open spheres with centers in \(A\) and radius \(\epsilon\).

**Lemma 1.2.** [7] If \(\{A_n\}\) and \(\{B_n\}\) are sequences in \(B(X)\) converging to \(A\) and \(B\) in \(B(X)\), respectively, then the sequence \(\{\delta(A_n, B_n)\}\) converges to \(\delta(A, B)\).

**Lemma 1.3.** [8] Let \(\{A_n\}\) be a sequence in \(B(X)\) and \(y\) a point in \(X\) such that \(\delta(A_n, y) \to 0\). Then, the sequence \(\{A_n\}\) converges to the set \(\{y\}\) in \(B(X)\).

**Lemma 1.4.** [8] A set-valued mapping \(F\) of \(X\) into \(B(X)\) is said to be continuous at \(x \in X\) if the sequence \(\{Fx_n\}\) in \(B(X)\) converges to \(Fx\) whenever \(\{x_n\}\) is a sequence in \(X\) converging to \(x\) in \(X\).

\(F\) is said to be continuous on \(X\) if it is continuous at every point in \(X\).

**Lemma 1.5.** [8] Let \(\{A_n\}\) be a sequence of nonempty subsets of \(X\) and \(z\) in \(X\) such that \(\lim_{n \to \infty} a_n = z\), \(z\) independent of the particular choice of each \(a_n \in A_n\). If a self-map \(I\) of \(X\) is continuous, then \(\{Iz\}\) is the limit of the sequence \(\{IA_n\}\).

**Definition 1.6.** [1] Two self-mappings \(S\) and \(T\) of a metric space \((X, d)\) satisfy the property \((E.A)\) if there exists a sequence \(\{x_n\}\) in \(X\) such that

\[
\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = z \text{ for some } z \in X.
\]

It is clear from the definition of compatibility that \(S\) and \(T\) are incompatible if there exists at least one sequence \(\{x_n\}\) in \(X\) such that \(\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = z\) for some \(z \in X\), but, \(\lim_{n \to \infty} d(STx_n, TSx_n)\) is either non-zero or does not exist. Therefore, two incompatible self-mappings of a metric space \((X, d)\) satisfy the property \((E.A)\).

## 2 Preliminaries

**Definition 2.1.** Two mappings \(f : X \to X\) and \(T : X \to B(X)\) satisfy the property \((E.A)\) if there exists a sequence \(\{x_n\}\) in \(X\) such that

\[
\lim_{n \to \infty} f x_n = z \quad \text{and} \quad \lim_{n \to \infty} T x_n = \{z\} \text{ for some } z \in X.
\]

**Example 2.2.** Let \(X = [1, \infty)\) with the usual metric. Define \(f : X \to X\) and \(T : X \to B(X)\) by \(f(x) = x + 1\) and \(Tx = [2, x + 1]\). Consider the sequence \(\{x_n\}\) such that \(x_n = 1 + \frac{1}{n}, n = 1, 2, \ldots\). Clearly, \(\lim_{n \to \infty} f x_n = 2\) and \(\lim_{n \to \infty} T x_n = \{2\}\).

Therefore, \(f\) and \(T\) satisfy the property \((E.A)\).
Example 2.3. Let $X = [1, \infty)$ with the usual metric. Define $f : X \to X$ and $T : X \to B(X)$ by $f(x) = x + 1$ and $Tx = \{x + 2\}$. Suppose that $f$ and $T$ satisfy property (E.A). Then, there exists a sequence $\{x_n\}$ in $X$, such that $\lim_{n \to \infty} f x_n = z$ and $\lim_{n \to \infty} T x_n = \{z\}$. Therefore, $\lim_{n \to \infty} x_n = z - 1 = z + 2$ for some $z \in X$ which is a contradiction. Hence, $f$ and $T$ do not satisfy the property (E.A).

It is clear from the definition of $\delta$-compatibility that $f$ and $T$ are noncompatible if there exists at least one sequence $\{x_n\}$ in $X$ such that $\lim_{n \to \infty} f x_n = z$ and $\lim_{n \to \infty} T x_n = \{z\}$ for some $z \in X$ but, $\lim_{n \to \infty} \delta(f T x_n, T f x_n)$ is either non-zero or does not exist. Therefore, two mappings $f : X \to X$ and $T : X \to B(X)$ of a metric space $(X, d)$ which are not $\delta$-compatible satisfy the property (E.A).

Several authors proved fixed point theorems and common fixed point theorems for mappings satisfying contractive conditions of integral type, see [2, 3, 4, 5, 6, 12, 15]. Recently, Zhang [16] and Aliouche [3] proved common fixed point theorems using generalized contractive conditions in metric spaces. These theorems extend well-known results in [4], [5], [12] and [15]. Let $A \in (0, \infty], R_A^+ = [0, A)$ and $F : R_A^+ \to \mathbb{R}$ satisfying

(i) $F(0) = 0$ and $F(t) > 0$ for each $t \in (0, A)$,
(ii) $F$ is increasing on $R_A^+$,
(iii) $F$ is continuous.

Define $f[0, A) = \{F : F$ satisfies (i)-(iii)$\}$. The following examples were given by [16].

1) Let $F(t) = t$, then $F \in f[0, A)$ for each $A \in (0, +\infty]$.  
2) Suppose that $\varphi$ is nonnegative, Lebesgue integrable on $[0, A)$ and satisfies 
\[
\int_0^\epsilon \varphi(t) dt > 0 \quad \text{for each } \epsilon \in (0, A).
\]

Let $F(t) = \int_0^t \varphi(s) ds$, then $F \in f[0, A)$.  
3) Suppose that $\psi$ is nonnegative, Lebesgue integrable on $[0, A)$ and satisfies 
\[
\int_0^\epsilon \psi(t) dt > 0 \quad \text{for each } \epsilon \in (0, A)
\]

and $\varphi$ is nonnegative, Lebesgue integrable on $[0, \int_0^A \psi(s) ds)$ and satisfies 
\[
\int_0^\epsilon \varphi(t) dt > 0 \quad \text{for each } \epsilon \in (0, \int_0^A \psi(s) ds).
\]

Let $F(t) = \int_0^t \varphi(u) du$, then $F \in f[0, A)$.  

\[
\int_0^\epsilon \varphi(t) dt > 0 \quad \text{for each } \epsilon \in (0, \int_0^A \psi(s) ds).
\]
4) If $G \in [0, A)$ and $F \in F[0, G(A - 0))$, then a composition mapping $F \circ G \in F[0, A)$. For instance, let $H(t) = \int_0^{F(t)} \varphi(s)ds$, then $H \in F[0, A)$ whenever $F \in F[0, A)$ and $\varphi$ is nonnegative, Lebesgue integrable on $F[0, F(A - 0))$ and satisfies

$$\int_0^\infty \varphi(t)dt > 0 \text{ for each } \epsilon \in (0, F(A - 0)).$$

**Lemma 2.4.** [16] Let $A \in (0, +\infty]$ and $F \in F[0, A)$. If $\lim_{n \to \infty} F(\epsilon_n) = 0$ for $\epsilon_n \in R_+^*$, then $\lim_{n \to \infty} \epsilon_n = 0$.

Let $A \in (0, +\infty]$, $\psi : R_+^* \to R_+$ satisfying

(i) $\psi(t) < t$ for each $t \in (0, A)$,

(ii) $\psi$ is nondecreasing and upper semi-continuous.

Define $\Psi[0, A] = \{\psi : \psi$ satisfies (i) and (ii) above$\}.$

**Lemma 2.5.** [16] If $\psi \in \Psi[0, A]$, then $\psi(0) = 0$.

The following Theorem was proved by [1].

**Theorem 2.6.** Let $A, B, S$ and $T$ be self-mappings of a metric space $(X, d)$ such that

$$d(Ax, By) \leq \phi(\max\{d(Sx, Ty), d(Sx, By), d(By, Ty)\})$$

for all $x, y \in X$. Suppose that $A(X) \subset T(X)$, $B(X) \subset S(X)$ and the $(A, S)$ or $(B, T)$ satisfies the property (E.A). If the range of one of the mappings $A, B, S$ and $T$ is a complete subspace of $X$, then $A, B, S$ and $T$ have a unique common fixed point in $X$.

It is our purpose in this paper to extend Theorem 2.6 for two pairs of single-valued and set-valued mappings and prove a common fixed point theorem using a generalized contractive condition and a property (E.A).

# Main Results

Let $D = \sup\{d(x, y) : x, y \in X\}$. Set $A = D$ if $D = \infty$ and $A > D$ if $D \in (0, \infty)$. $d(x, y) = 0$ if $x = y$.

**Theorem 3.1.** Let $f$ and $g$ be self-mappings of a metric space $(X, d)$ and $S$ and $T$ be mappings from $X$ into $B(X)$ satisfying

$$\cup S(X) \subset g(X) \quad \text{and} \quad \cup T(X) \subset f(X) \quad (3.1)$$

$$F(\delta(Sx, Ty)) \leq \psi(F(\max\{d(fx, gy), \delta(fx, Sx), \phi(gy, Ty), d(fx, Ty), d(Sx, gy)\})) \quad (3.2)$$

for all $x, y \in X$, where $F \in [0, A)$ and $\psi \in \Psi[0, F(A - 0))$. Suppose that the pair $(f, S)$ or $(g, T)$ satisfies the property (E.A), $(f, S)$ and $(g, T)$ are weakly compatible and $f(X)$ or $g(X)$ or $S(X)$ or $T(X)$ is a closed subset of $X$. Then, $f, g, S$ and $T$ have a unique common fixed point in $X$. 


Proof. Suppose that the pair \((g,T)\) satisfies the property (E.A). Then, there exists a sequence \(\{x_n\}\) in \(X\) such that \(\lim_{n \to \infty} gx_n = z\) and \(\lim_{n \to \infty} Tx_n = \{z\}\) for some \(z \in X\). Therefore, we have \(\lim_{n \to \infty} \delta(gx_n,Tx_n) = 0\). Since \(T(X) \subset f(X)\), there exists in \(X\) a sequence \(\{y_n\}\) such that \(fy_n \in Tx_n\). Assume that \(\limsup_{n \to \infty} \delta(Sy_n, Tx_n) = l > 0\). Using (3.2) we have

\[
F(\delta(Sy_n, Tx_n)) \leq \psi(F(\max\{d(fy_n, gx_n), \delta(fy_n, Sy_n), d(gx_n, Tx_n), d(fy_n, Tx_n), d(Sy_n, gx_n)\})
\]

\[
\leq \psi(F(\max\{\delta(gx_n, Tx_n), \delta(Sy_n, Tx_n), d(gx_n, Tx_n)
\delta(Sy_n Tx_n) + \delta(gx_n, Tx_n)\})
\]

Letting \(n \to \infty\) we get \(F(l) \leq \psi(F(l)) < F(l)\).

Which is a contradiction. Hence, \(\lim_{n \to \infty} F(\delta(Sy_n, Tx_n)) = 0\) and Lemma 2.4 implies that \(\lim_{n \to \infty} \delta(Sy_n, Tx_n) = 0\); i.e., \(\lim_{n \to \infty} Sy_n = \{z\}\).

Suppose that \(f(X)\) is a complete subspace of \(X\). Then, \(z = fu\) for some \(u \in X\).

If \(Su \neq \{z\}\), applying (3.2) we get

\[
F(\delta(Su, Tx_n)) \leq \psi(F(\max\{d(fu, gx_n), \delta(fu, Su), \delta(gx_n, Tx_n), d(fu, Tx_n), d(Su, gx_n)\})
\]

Letting \(n \to \infty\) we obtain

\[
F(\delta(Su, z)) \leq \psi(F(\delta(Su, z))
\]

and so \(Su = \{fu\} = \{z\}\). Since \(S(X) \subset g(X)\), there exists \(v \in X\) such that \(Su = \{gv\} = \{z\}\).

If \(Tv \neq \{z\}\), using (3.2) we have

\[
F(\delta(z, Tv)) = F(\delta(Su, Tv))
\]

\[
\leq \psi(F(\delta(z, Tv)))
\]

\[
< F(\delta(z, Tv))
\]

which implies that \(Tv = \{gv\} = \{z\}\). As the pairs \((f,S)\) and \((B,T)\) are weakly compatible, we get \(Sz = \{fz\}\) and \(Tz = \{gz\}\). If \(Sz \neq \{z\}\), using (3.2) we obtain

\[
F(\delta(Sz, z)) = F(\delta(Sz, Tv))
\]

\[
\leq \psi(F(\delta(Sz, z)))
\]

\[
< F(\delta(Sz, z))
\]

and so \(Sz = \{fz\} = \{z\}\). Similarly, we can prove that \(Tz = \{gz\} = \{z\}\).

The proof is similar when \(g(X)\) is assumed to be a closed subset of \(X\). By (3.1), the cases in which \(S(X)\) or \(T(X)\) is a closed subset of \(X\) are similar to the cases in which \(f(X)\) or \(g(X)\) is a closed subset of \(X\). The uniqueness of \(z\) follows from (3.2). \(\square\)
If $S$ and $T$ are single-valued mappings in Theorem 3.1 and $F(t) = \int_{0}^{t} \varphi(s)\,ds$, where $\varphi$ is nonnegative, Lebesgue integrable on $[0, A)$ and satisfies $\int_{0}^{A} \varphi(t)\,dt > 0$ for each $\epsilon \in (0, A)$, we get a generalization of Corollary 3 of [2].

If $S$ and $T$ are single-valued mappings and $F(t) = t$ in Theorem 3.1, we get a generalization of Theorem 2.5. If $S = T$ and $g = f$ in Theorem 3.1, we get the following Corollary.

**Corollary 3.2.** Let $f$ be a self-mapping of a metric space $(X, d)$ and $T$ be a mapping from $X$ into $B(X)$ such that

$$\bigcup T(X) \subset f(X)$$

$$F(\delta(Tx, Ty)) \leq \psi(F(\max\{d(fx, fy), \delta(fx, Tx), \delta(fy, Ty), d(fx, Ty), d(Tx, fy)\}))$$

for all $x, y \in X$, where $F \in [0, A)$ and $\psi \in \Psi[0, f(A - 0))$. Suppose that the pair $(f, T)$ satisfies the property (E.A), $(f, T)$ is weakly compatible and $f(X)$ or $T(X)$ is a closed subset of $X$. Then, $f$ and $T$ have a unique common fixed point in $X$.

If $F(t) = t$ in Theorem 3.1, we get the following Corollary.

**Corollary 3.3.** Let $f$ and $g$ be self-mappings of a metric space $(X, d)$ and $S, T$ be mappings from $X$ into $B(X)$ satisfying (3.1) and

$$\delta(Sx, Ty) \leq \psi(\max\{d(fx, fy), \delta(fx, Tx), \delta(fy, Ty), d(fx, Ty), d(Tx, fy)\})$$

for all $x, y \in X$. Suppose that the pair $(f, S)$ or $(g, T)$ satisfies the property (E.A), $(f, S)$ and $(g, T)$ are weakly compatible and one of $f(X)$ or $g(X)$ or $S(X)$ or $T(X)$ is a closed subset of $X$. Then, $f, g, S$ and $T$ have a unique common fixed point in $X$.

**Example 3.4.** Let $X = [0, 1]$ endowed with the Euclidean metric $d$. Define $S, T : X \to B(X)$ and $f, g : X \to X$ by

$$fx = \begin{cases} \frac{1}{x+1} & \text{if } x \in \left[0, \frac{1}{2}\right] \\ \frac{1}{2} & \text{if } x \in \left(\frac{1}{2}, 1\right) \end{cases}, \quad gx = \begin{cases} 1 - x & \text{if } x \in \left[0, \frac{1}{2}\right] \\ 0 & \text{if } x \in \left(\frac{1}{2}, 1\right) \end{cases}$$

$$Sx = \begin{cases} \frac{1}{2} & \text{for all } x \in [0, 1] \text{ and } Tx = \begin{cases} \frac{1}{2} & \text{if } x \in \left[0, \frac{1}{2}\right] \\ \frac{3}{2} & \text{if } x \in \left(\frac{1}{2}, 1\right) \end{cases} \end{cases}$$

let $F(s) = s^{\frac{1}{2}}$ and $\psi(t) = \frac{1}{8^\frac{3}{2}} \cdot 3^\frac{t}{2}$. Then, $F \in [0, A)$ and $\psi \in \Psi[0, e^\frac{3}{2}]$, where $A = e > D$.

We have

$$\cup S(X) = \left\{ \frac{1}{2} \right\} \subset g(X) = \left[\frac{1}{2}, 1\right] \cup \{0\} \quad \text{and}$$

$$\cup T(X) = \left(\frac{1}{2}, \frac{3}{2}\right] = f(X).$$
On the other hand, if \( x \in X \) and \( y \in [0, \frac{1}{2}] \), then

\[
F(\delta(Sx, Ty)) = 0 \\
\leq \psi(\max\{d(fx, gy), \delta(fx, Sx), \delta(gy, Ty), d(fx, Ty), d(Sx, gy)\}).
\]

If \( x \in X \) and \( y \in (\frac{1}{2}, 1] \), then

\[
\delta(Sx, Ty) \leq \frac{1}{8} \quad \text{and} \quad d(fx, gy) \geq \frac{3}{8}.
\]

Hence

\[
F(\delta(Sx, Ty)) \leq (\frac{1}{8})^8 \quad \text{and} \quad F(d(fx, gy)) \geq (\frac{3}{8})^8
\]

and so

\[
F(\delta(Sx, Ty)) \leq \frac{1}{8^{19}} \cdot 3^8 F(d(fx, gy)) \\
= \psi(F(d(fx, gy))) \\
\leq \psi(\max\{d(fx, gy), \delta(fx, Sx), \delta(gy, Ty), d(fx, Ty), d(Sx, gy)\}).
\]

\( g(X) \) is a closed subset of \( X \) and the pairs \( (f, S) \) and \( (g, T) \) are weakly compatible. Taking \( x_n = \frac{1}{2} - \frac{1}{2n} \), the pair \( (f, S) \) satisfies the property \( (E.A) \) with \( z = \frac{1}{2} \). Consequently, by Theorem 3.1, \( \frac{1}{2} \) is the unique common fixed point of \( f, g, S \) and \( T \).

References


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